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Dynamic Cooperation of the O2O Supply Chain Based on Time Delays and Bidirectional Free-Riding

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Abstract: Advertising and service investment can enhance brand goodwill to increase the sales of branded goods. However, the impact of advertising and services on brand goodwill is not immediate but delayed. At the same time, due to the different service characteristics provided by various channels, the phenomenon of bidirectional free-riding occurs. Therefore, this paper studies the dynamic cooperation between service and advertising in the O2O (online to offline) supply chain dominated by brand owners and explores the impacts of advertising, service delay and service free-riding among channels on the dynamic cooperation decisions of the O2O supply chain. A differential game model between brands and retailers is constructed by incorporating the delay effect and the bidirectional free-riding phenomenon. The optimal advertising and service strategies and performance problems of O2O supply chain enterprises under a centralized decision, brand cost-sharing decision and bilateral cost-sharing decision are compared and analyzed. The influence of delay time, showrooming and webrooming effects on the profit of each firm is investigated by example. The results show that the service strategy, advertising strategy and brand goodwill of the O2O supply chain members are optimal under a centralized decision. Still, the supply chain profit is not necessarily optimal under the delay time, showrooming and webrooming effect coefficients. Bilateral cost-sharing contracts can achieve Pareto improvement of supply chain performance. Appropriate setting of a bilateral cost-sharing ratio can adjust the adverse effects of delay and bidirectional free-riding. The long-term strategies to deal with the delay and bidirectional free-riding phenomena are as follows: the bilateral cost-sharing contract can improve corporate profits. Setting the wholesale price, online direct-selling price and service-sharing ratio by brand owners can effectively promote retailers' investment in service, achieving a win-win situation. Retailers maintain high pricing and service levels to enhance the brand premium ability of physical stores and achieve long-term development.

Keywords: time delay; bidirectional free-riding; differential game; bilateral cost-sharing decisions

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1. Introduction

The rapid development of mobile technology and the Internet has enriched the shopping channels of consumers [1]. Channel services have their characteristics, the convenience of online channels and the experience of offline channels, so channel integration has become a development trend [2,3]. It is known as the showrooming phenomenon when consumers experience the product and then select the online channel to make a purchase [4]. When consumers search for product information (model, price, reviews, etc.) on an e-commerce platform and then choose an offline channel to make a purchase, it is called the webrooming phenomenon [5]. The combination of online and offline shopping has become common practice in omnichannel consumer behavior [6]. Van Baal and Dach [7] showed that 24.6% of consumers visited offline brick-and-mortar stores before completing transactions online, while 20.4% of consumers collected product information on the Internet before making offline purchases. Thus, this shows that the different service characteristics of the online and offline channels have made the bidirectional free-riding phenomenon (i.e., showrooming and webrooming) inevitable. To alleviate service competition between channels, a new business strategy, the O2O (online to offline) model has been introduced. The concept of O2O was first proposed by Alex Rampell, the founder of TrialPay in the United States, in August 2010. The earliest form of O2O is the "online purchase, offline consumption" model represented by online travel, catering, leisure and entertainment group purchase websites. In addition to the channel cooperation model in the service industry, the O2O model has also gradually attached importance in the retail industry. For example, retail brands such as Uniqlo and New Look have begun implementing this business strategy.

Meanwhile, studies have shown that advertising investment can alleviate channel conflict to a certain extent while increasing market demand [8]. The brand owners conduct national advertising through social media or mobile advertising to enhance the brand goodwill and promote the market demands of the brands [9]. At the same time, the retailers will improve the regional market demands of the brands by means of traditional promotional advertising. This vertical advertising cooperation among supply chain members is considered effective in enhancing the market demands [10–14]. The cost-sharing mechanism of cooperative advertising, in which manufacturers share part of the advertising costs of retailers, is a vital tool to mitigate competition and expand demand [15,16]. For example, in 2015, the manufacturer's retailer provided USD 36 billion in cooperative advertising, accounting for 12% of its total advertising costs [9]. Without cooperative advertising, the retailer's advertising efforts are often lower than the manufacturer's expectations [17]. Therefore, the optimal design of advertising cooperation in the O2O supply chain has attracted much attention from scholars [18–20]. As a result, service and advertising cooperation become the focal point of channel integration in the O2O model. This paper studies the O2O supply chain, which consists of the brands with online channels and the offline retailers and discusses the issue of advertising and service cooperation between channels.

In the O2O retail market, there are company-owned stores and franchised stores. For example, most of Zara's offline stores are owned by the company. Some small and medium-sized brands are more inclined to join the form, such as the Chinese clothing brand Metersbonwe, whose offline stores are mainly franchised. In addition to selling authorized brand products, franchise stores sell products as independent retailers and compete with the online channels of brand vendors [21]. Therefore, in the O2O model, when channels belong to different subjects, there are channel conflicts among channels.

In practice, traditional supply chain service and advertising cooperation are generally achieved by the upstream brand owners of the supply chain by sharing the cost and incentivizing the downstream retailers to invest in service and advertising levels, thus realizing vertical supply chain cooperation [9,15,21,22]. Chinese clothing brand INMAN, for example, shares some of the costs of servicing its offline franchises. However, in the O2O supply chain, in addition to the above vertical collaboration, there is horizontal competition among the supply chain members. The brand owners will share part of the service and advertising costs of the retailers, which, while improving the offline sales profits, will be horizontal competition to the online channel of the brand owners. Therefore, the question is raised: how do we coordinate the conflicts between the O2O supply chain channels through service and advertising cooperation? How do we design the service and advertising cost-sharing mechanism to achieve coordination to increase profits? There is little literature on both service and advertising cooperation, and the design of a service and cost-sharing mechanism to affect the decision-making and profit of the O2O supply chain has not been fully discussed.

The concept of SCM (supply chain management) first emerged in the 1980s; SCM has now become a hot issue in business management and practical application. Fang et al. [23] found that sustainable supply chains have become a hot research direction in supply chain management since 2010, but particularly in the last three years. Salas-Navarro et al. [24] found that developing an inventory model with environmental protection variables and parameters helps companies create a complex and solid inventory structure for sustainable development. Acevedo-Chedid et al. [25] identified the benefits of collaborative planning for production systems that can be used to make decisions for profit maximization for all participants in the supply chain. Related studies mainly focus on two areas: the bidirectional free-riding phenomenon of service and the dynamic cooperation of supply chain advertising.

The free-rider phenomenon was first proposed by Telser [26], who pointed out that it was not conducive to retailers' sales and would prevent them from providing pre-sale information services. Singley and Williams [27] proposed that free-riding reduces the profits of retailers providing comprehensive services. However, Wu et al. [28], Guan et al. [29] and Shin et al. [30] put forward different views, believing that retailers providing information services may benefit even if there is a free-riding phenomenon. The above research mainly focuses on free-riding between offline physical stores. E-commerce has emerged as a new sales method, and the emergence of homogeneous products in different or dual channels leads to free-riding among consumers [7]. In the dual-channel supply chain, most studies mainly focus on channel conflicts among supply chain members [31–34]. In contrast, Aubrey et al. [34] argued that with the increasing popularity of online shopping, online and offline channels can complement each other rather than compete. Zhang et al. [35] also studied how to manage the conflict between online and offline channels, pointing out that retailers can achieve the best returns by coordinating the two channels and adopting appropriate pricing strategies and channel combinations. To address the channel conflict, retailers should design coordination mechanisms to weaken the hitchhiking problem to alleviate channel conflicts. Common coordination mechanisms are cost-sharing contracts [35,36], the repurchase contract and sales rebate contract [37] and inventory subsidy method [38].

The above studies mainly focus on the showrooming phenomenon, while research on the webrooming phenomenon has also increased in recent years. For example, Wang et al. [39], Yuan et al. [40], and Ma et al. [41] studied the impact of the webrooming phenomenon on the supply chain sales model, pricing decision-making and the relationship with quality expectations. The above literature, whether in respect of the showrooming phenomenon or the webrooming phenomenon, studied the one-way free-riding phenomenon between channels, and research on the directional free-riding phenomenon is rare. Liu et al. [42] looked at the impact of free-riding behavior on dual-channel supply chain pricing. Luo et al. [43] studied the impact of bidirectional free-riding behavior based on information services on manufacturers' introduction of direct sales network channels. Li et al. [44] and Gong et al. [45] studied the bidirectional free-riding behavior in the dual-channel supply chain. The above research on the phenomenon of bidirectional free-riding takes the static supply chain system as the research object. It rarely considers the impact of bidirectional free-riding on the supply chain operation from the long-term dynamic perspective. These studies suggest that the integration of online and offline channels is preferred to competition between channels.

Unlike channel competition in the dual-channel supply chain, the O2O supply chain emphasizes the cooperation between online and offline channels, which can take advantage of the advantages of both channels [46]. The bidirectional free-riding phenomenon in this study, including showrooming and webrooming phenomena, is consistent with cross-channel shopping [7,47–49] research. However, the current research on showrooming and webrooming focuses more on the shopping experience and driving factors from consumers' perspectives. Sahu et al. [1] found that showroom or webrooming factors can be divided into consumer-led, company-led and situational. The company's dominant factors include price, quality, customer service, channel integration and media richness. The above studies all studied the service cooperation of the O2O supply chain from a static perspective. In contrast, the service cooperation among members of the O2O supply chain is long-term, complex and dynamic [21]. Therefore, this paper studies the impact of the bidirectional free-riding phenomenon on the optimal decision of service cooperation and supply chain profits in the O2O supply chain from a long-term dynamic perspective.

Another related field of this paper is the research on the dynamic cooperation of supply chain advertising. Nerlove and Arrow [50] developed the classic Nerlove–Arrow (N–A) advertising model, in which the goodwill is a dynamic equation that evolves. Jørgensen and Zaccour [16] adapted the N–A model and introduced the concept of advertising cooperation into supply chain decision-making. Zhang and Zhang [51] analyzed the cooperative advertising problem of the supply chain members by using the differential game to build a dynamic model. The above research is based on the assumption of the immediacy of the advertising effects. However, there is a time difference between consumers' contact with advertising and their actual demands, which is called the time delay. Gao et al. [52] present the current status of research on the application of complex dynamical systems in the modeling and control of systems with time delays. Zheng et al. [53] constructed a three-stage supply chain model using a system of differential equations to reveal the interplay among producers, distributors and end customers. Gao et al. [54] outlined emerging research and application directions in condition measurement, optimization and advanced control of complex industrial processes. Berkowitz et al. [55] and Baack et al. [56] proposed the dynamic advertising model, and Chen et al. [57] applied the delayed dynamic advertising model to the advertising cooperation among the supply chain members. Yu et al. [58] studied the effects of both delay and memory effects on the advertising decisions of supply chain members. In terms of the dual-channel supply chain, Chen et al. [59] and Cao et al. [60] considered the advertising cooperation strategy when both manufacturer and retailer advertising have delay effects. The above research mainly focuses on the advertising dynamic cooperation model of the dual-channel supply chain, but there are few studies on the O2O supply chain advertising dynamic cooperation. For the literature related to O2O supply chain advertising cooperation in this study, please refer to Table 1. The above literature discussed the advertising cooperation strategy of the O2O supply chain but did not consider the impact of advertising delay time on cooperation. Therefore, this study evaluates the effect of advertising delay on dynamic collaboration in the O2O supply chain. In addition, we consider the simultaneous service cooperation situation.

Table 1. Literature related to advertising cooperation in the O2O supply chain.

Sources	Broad Theme	Research Questions
Li et al. [11]	Bilateral participation in advertising cooperation	• How cooperative advertising is applied in the O2O business model • Who can improve O2O supply chain performance more with unilateral or bilateral advertising cooperation • Practical application of the O2O supply chain bilateral cooperation advertising strategy in the industry
Li et al. [22]	BOPS (buy-online-and-pick-up-in-store)	• The effect of BOPS on advertising cooperation between manufacturers and retailers • The effect of BOPS on the optimal strategy of the O2O supply chain • The effectiveness of advertising cooperation in the O2O supply chain
Wang and Shu [61]	Fairness concerns	• The online retail platforms' Media Resource Replacement Plans (MRRPs) impact manufacturer and platform advertising partnerships • The optimal decisions of advertising efforts and participation rates between the supply chain members • Conditions for manufacturers to participate in the MRRP of online retailer platforms • The impact of manufacturers' participation in MRRP on the profitability of O2O supply chain members

In contrast to the above studies, the main contributions of this paper are as follows: (1) introducing time delay and bidirectional free-riding into the O2O supply chain dynamic model at the same time; (2) considering the time delay of brand goodwill caused by national advertising of the brand owners and offline experience service of the retailers; (3) designing the centralized decision, the brand cost-sharing decision and the bilateral cost-sharing decision to respond to three types of cooperative relationships among the O2O supply chain members: full cooperation, one-way participation cooperation and two-way participation cooperation. We consider the impact of the time delay phenomenon and bidirectional free-riding phenomenon on the cooperative decision choices of the O2O supply chain members.

The following is the structure of this paper. Section 2 describes the research questions and related hypotheses in detail. Optimization models are developed and explored in Section 3. Section 4 provides a comparative analysis of the various models. Finally, Section 5 summarizes the results and their practical implications and proposes directions for future research.

2. Model Summary and Relevant Assumptions

This paper considers the O2O supply chain system composed of a single brand owner M and a single retailer R. The brand owner sells the products through direct online sales and offline retailers. The brand owner and the retailer cooperate in advertising and service. The brand owner invests in national advertising and provides the online information service, and the retailer supports regional promotional advertising and provides the offline experience service. Among them, the national advertising of the brand owner and the offline experience service of the retailer has an impact on the brand goodwill, which is the time delay phenomenon. The game process of O2O supply chain is shown in Figure 1.

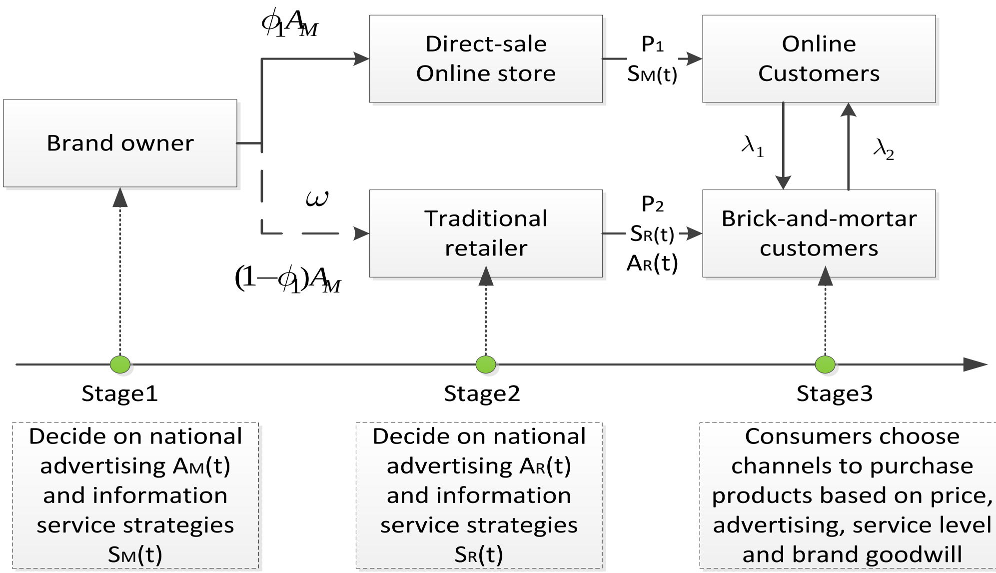

Figure 1. Game process of O2O supply chain members.

Descriptions of the model symbols in the paper are as follows:

$G(t), G(0)$: the brand goodwill at the moment t and the original goodwill level, respectively, and $G(0) = G_0$;

$A_M(t), A_R(t)$: the national advertising level of the brand owner and the regional advertising level of the retailer at the moment t, respectively;

$S_M(t), S_R(t)$: the information service level of the brand owner and the experience service level of the retailer at the moment t, respectively;

$d_1(t), d_2(t)$: the advertising delay time of the brand owner and the service delay time of the retailer, respectively;

$D_E(\mathrm{t}), D_R(\mathrm{t})$: the demand functions of the online channel and offline channel at the moment t, respectively;

$C(A_M(t)), C(A_R(t))$: the advertising cost functions of the brand owner and the retailer at the moment t, respectively;

$C(S_M(t)), C(S_R(t))$: the service cost functions of the brand owner and the retailer at the moment t, respectively;

$\omega(t), P_1(t), P_2(t)$: the wholesale price, online retail price, and offline retail price of the product at the moment t, respectively, and $\omega(t) < P_1(t) < P_2(t)$;

δ, ρ: the decay rate and discount factor of the brand goodwill, respectively, and $\delta > 0, \rho > 0$;

γ_1, γ_2: the influence coefficients of the brand owner's advertising level and retailer's service level to the brand goodwill, respectively, and $\gamma_1 > 0, \gamma_2 > 0$;

α, θ: the market capacity and consumers' brand preference, respectively, and $\alpha > 0, \theta > 0$;

β_1, β_2: the price demand flexibility coefficients of the online and offline channels, respectively, and $0 < \beta_1 < 1, 0 < \beta_2 < 1$;

ϕ_1, ϕ_2: the conversion rate of the brand owner's advertising to the online channel demand and the conversion rate of the retailer's advertising to the offline channel demand, respectively, and $0 < \phi_1 < 1, 0 < \phi_2 < 1$;

λ_1, λ_2: webrooming effect coefficient and showrooming effect coefficient, respectively, and $0 < \lambda_1 < 1, 0 < \lambda_2 < 1$;

$k_i, \eta_i\ (i = 1, 2)$: the correlation coefficient between the advertising level and advertising cost and the correlation coefficient between the service level and service cost, respectively, and $k_i > 0, \eta_i > 0$.

Assumption 1. *By referring to the literature* [59,60], *to further consider the impact of the offline experience service of the retailer on the promotion of the brand goodwill based on the N–A advertising model and to describe the dynamic change of the brand goodwill at the moment t by using the delay differential as follows*:

$$\dot{G}(t) = \gamma_1 A_M(t - d_1(t)) + \gamma_2 S_R(t - d_2(t)) - \delta G(t), G(0) = G_0 \tag{1}$$

Assumption 2. *The purchase behavior of the consumers is comprehensively affected by the price, brand goodwill, advertising, service and other factors. Therefore, drawing on the literature* [44,45], *the demand functions of the online channel and offline channel are given as follows*:

$$D_E(t) = \alpha - \beta_1 P_1(t) + \theta G(t) + \phi_1 A_M(t) + (1 - \lambda_1) S_M(t) + \lambda_2 S_R(t) \tag{2}$$

$$D_R(t) = \alpha - \beta_2 P_2(t) + \theta G(t) + (1 - \phi_1) A_M(t) + \phi_2 A_R(t) + (1 - \lambda_2) S_R(t) + \lambda_1 S_M(t) \tag{3}$$

Assumption 3. *The cost function of the advertising and service shows a quadratic curve relationship, and relevant literature on this assumption is more common.*

$$C(A_M(t)) = \frac{1}{2} k_1 {A_M}^2(t), C(A_R(t)) = \frac{1}{2} k_2 {A_R}^2(t) \tag{4}$$

$$C(S_M(t)) = \frac{1}{2} \eta_1 {S_M}^2(t), C(S_R(t)) = \frac{1}{2} \eta_2 {S_R}^2(t) \tag{5}$$

The brand owner and the retailer have the same profit discount factor ρ at any time, and the objective is to maximize the net discounted profit within an infinite time horizon. To simplify the solution process, assume that the delay time is constant, i.e., $d_1(t) = d_1,\ d_2(t) = d_2$. Considering that the product price information is relatively transparent, assume that the wholesale price of the product and the retail prices of online and offline channels are also constant, i.e., $\omega(t) = \omega, P_1(t) = P_1, P_2(t) = P_2$.

3. Model Solution and Analysis

3.1. Centralized Decision

Under this decision, the brand owner and the retailer are in a fully cooperative relationship. Therefore, the supply chain members make optimal advertising and service decisions based on maximizing the supply chain profits. This decision is represented by superscript "C", and the objective function of the supply chain profits is as follows:

$$J_{MR}^{C} = \int_{0}^{\infty} e^{-\rho t}[P_1(\mathrm{t})D_E(\mathrm{t}) + P_2(\mathrm{t})D_R(\mathrm{t}) - \frac{k_1}{2}A_{\mathrm{M}}{}^2(\mathrm{t}) - \frac{\eta_1}{2}S_{\mathrm{M}}{}^2(\mathrm{t}) - \frac{k_2}{2}A_{\mathrm{R}}{}^2(\mathrm{t}) - \frac{\eta_2}{2}S_{\mathrm{R}}{}^2(\mathrm{t})]\mathrm{dt}$$

Proposition 1. *Under centralized decision-making, the optimal advertising and service strategies of the brand owner and the retailer are as follows*:

$$A_M^{C*}(\mathrm{t}) = \frac{\theta\gamma_1 e^{\delta d_1}(P_1 + P_2) + (\delta+\rho)[P_2(1-\phi_1) + \phi_1 P_1]}{k_1(\delta+\rho)}, A_R^{C*}(\mathrm{t}) = \frac{\phi_1 P_2}{k_2},$$

$$S_M^{C*}(\mathrm{t}) = \frac{P_1 + \lambda_1(P_2 - P_1)}{\eta_1}, S_R^{C*}(\mathrm{t}) = \frac{\theta\gamma_2 e^{\delta d_2}(P_1 + P_2) + (\delta+\rho)[\lambda_2(P_1 - P_2) + P_2]}{\eta_2(\delta+\rho)}.$$

Brand goodwill level is as follows:

$$\begin{aligned} G^{C*}(t) = \quad & e^{-\delta t}G_0 + \tfrac{\theta\gamma_1{}^2 e^{\delta d_1}(P_1+P_2) + \gamma_1(\delta+\rho)[P_2(1-\phi_1)+\phi_1 P_1]}{k_1\delta(\delta+\rho)}(1-e^{-\delta t}) \\ & + \tfrac{\theta\gamma_2{}^2 e^{\delta d_2}(P_1+P_2) + \gamma_2(\delta+\rho)[\lambda_2(P_1-P_2)+P_2]}{\eta_2\delta(\delta+\rho)}(1-e^{-\delta t}) \end{aligned}$$

The supply chain profits are as follows:

$$\begin{aligned} J_{MR}^{C} = \quad & \tfrac{1}{\rho}\big(P_1[\alpha - \beta_1 P_1 + \phi_1 A_M^{C*} + (1-\lambda_1)S_M^{C*} + \lambda_2 S_R^{C*}] + P_2[\alpha - \beta_2 P_2 + (1-\phi_1)A_M^{C*}) + \phi_2 A_R^{C*} + (1-\lambda_2)S_R^{C*} + \lambda_1 S_M^{C*}] \\ & - \tfrac{k_1}{2}A_M^{C*2} - \tfrac{\eta_1}{2}S_M^{C*2} - \tfrac{k_2}{2}A_R^{C*2} - \tfrac{\eta_2}{2}S_R^{C*2}) + \theta(P_1+P_2)[\tfrac{G_0}{\rho+\delta} + \tfrac{\gamma_1 A_M^{C*}}{\rho(\rho+\delta)} + \tfrac{\gamma_2 S_R^{C*}}{\rho(\rho+\delta)}] \end{aligned}$$

Proving: the optimal decision problem is characterized as the optimal control problem.

$$\max_{A_M > 0, S_R > 0} J_{MR}^{C}$$

$$\text{s.t. } \dot{G}(t) = \gamma_1 A_M(t - d_1(t)) + \gamma_2 S_R(t - d_2(t)) - \delta G(t)$$

Construct the Hamiltonian function such that

$$\begin{aligned} H_{MR}^{C} = \quad & e^{-\rho t}\left\{\begin{array}{l} P_1[\alpha - \beta_1 P_1 + \theta G + \phi_1 A_{\mathrm{M}} + (1-\lambda_1)S_M + \lambda_2 S_R] + P_2[\alpha - \beta_2 P_2 + \theta G + (1-\phi_1)A_{\mathrm{M}} \\ +\phi_2 A_{\mathrm{R}} + (1-\lambda_2)S_R + \lambda_1 S_M] - \frac{k_1}{2}A_{\mathrm{M}}{}^2 - \frac{\eta_1}{2}S_{\mathrm{M}}{}^2 - \frac{k_2}{2}A_{\mathrm{R}}{}^2 - \frac{\eta_2}{2}S_{\mathrm{R}}{}^2 \end{array}\right\} \\ & + q(t)[\gamma_1 A_M(t-d_1) + \gamma_2 S_R(t-d_2) - \delta G(t)] \end{aligned} \quad (6)$$

$$\frac{dH_{MR}}{dA_{\mathrm{M}}(t)} = e^{-\rho t}[\phi_1 P_1 + P_2(1-\phi_1) - k_1 A_{\mathrm{M}}(t)] + q(t)\gamma_1 \frac{dA_{\mathrm{M}}(t-d_1)}{dA_{\mathrm{M}}(t)} = 0; \quad (7)$$

$$\frac{dH_{MR}}{dS_{\mathrm{R}}(t)} = e^{-\rho t}[\lambda_2 P_1 + P_2(1-\lambda_2) - \eta_2 S_{\mathrm{R}}(t)] + q(t)\gamma_2 \frac{dS_{\mathrm{R}}(t-d_2)}{dS_{\mathrm{R}}(t)} = 0; \quad (8)$$

$$\frac{dq(t)}{dt} = -\frac{dH_{MR}}{dG} = \delta q(t) - \theta(P_1 + P_2)e^{-\rho t}$$

so

$$q(t) = C_0 e^{\delta t} + \frac{\theta(P_1 + P_2)}{\rho + \delta}e^{-\rho t}, C_0 \in R \quad (9)$$

According to the literature [57], substitute $\frac{dA_M(t-d_1)}{dA_M(t)} = e^{\delta d_1}$, $\frac{dS_R(t-d_2)}{dS_R(t)} = e^{\delta d_2}$ and Equation (9) into Equations (7) and (8), receiving the results as follows:

$$A_M^{C*}(\mathrm{t}) = \frac{P_2(1-\phi_1)+\phi_1 P_1}{k_1} + \frac{\gamma_1}{k_1} C_0 e^{(\delta+\rho)t+\delta d_1} + \frac{\theta\gamma_1(P_1+P_2)}{k_1(\delta+\rho)} e^{\delta d_1}$$

$$S_R^{C*}(\mathrm{t}) = \frac{\lambda_2(P_1-P_2)+P_2}{\eta_2} + \frac{\gamma_2}{\eta_2} C_0 e^{(\delta+\rho)t+\delta d_1} + \frac{\theta\gamma_2(P_1+P_2)}{\eta_2(\delta+\rho)} e^{\delta d_2}$$

Considering that the national advertising of the brand owner and the experience service of the retailer cannot be unlimited, i.e.,

$\lim_{t\to\infty} A_M^{C*}(t) < \infty$, $\lim_{t\to\infty} S_R^{C*}(t) < \infty$, thus receiving the result $C_0 = 0$.

At this time, the optimal level of investment for the brand's national advertising and the retailer's experiential services are as follows:

$$A_M^{C*}(\mathrm{t}) = \frac{\theta\gamma_1 e^{\delta d_1}(P_1+P_2)+(\delta+\rho)[P_2(1-\phi_1)+\phi_1 P_1]}{k_1(\delta+\rho)}$$

$$S_R^{C*}(\mathrm{t}) = \frac{\theta\gamma_2 e^{\delta d_2}(P_1+P_2)+(\delta+\rho)[\lambda_2(P_1-P_2)+P_2]}{\eta_2(\delta+\rho)}$$

Similarly, assuming that

$$\frac{\partial H_{MR}}{\partial A_R(t)} = e^{-\rho t}[\phi_2 P_2 - k_2 A_R(t)] = 0, \frac{\partial H_{MR}}{\partial S_M(t)} = e^{-\rho t}[(1-\lambda_1)P_1 + \lambda_1 P_2 - \eta_1 S_M(t)] = 0$$

The following conclusions were drawn: $A_R^{C*}(\mathrm{t}) = \frac{\phi_1 P_2}{k_2}, S_M^{C*}(\mathrm{t}) = \frac{P_1+\lambda_1(P_2-P_1)}{\eta_1}$.

Substituting $A_M^{C*}(\mathrm{t})$ and $S_R^{C*}(\mathrm{t})$ into Equation (1), we receive as follows:

$$\begin{aligned} G^{C*}(t) = \quad & e^{-\delta t} G_0 + \tfrac{\theta{\gamma_1}^2 e^{\delta d_1}(P_1+P_2)+\gamma_1(\delta+\rho)[P_2(1-\phi_1)+\phi_1 P_1]}{k_1\delta(\delta+\rho)}(1-e^{-\delta t}) \\ & + \tfrac{\theta{\gamma_2}^2 e^{\delta d_2}(P_1+P_2)+\gamma_2(\delta+\rho)[\lambda_2(P_1-P_2)+P_2]}{\eta_2\delta(\delta+\rho)}(1-e^{-\delta t}) \end{aligned}$$

Substituting the optimal decision value and the brand goodwill into the equation, we can receive the optimal profit ${J_{MR}}^{C*}$ of the whole supply chain, and the evidentiary process ends.

Further, we can analyze the impact of the delay time d_1 and d_2, webrooming effect and showrooming effect coefficients λ_1 and λ_2, the conversion rates ϕ_1 and ϕ_2 of the advertising investment levels on the online channel and offline channel demand, and the online and offline retail prices P_1 and P_2 on the optimal strategies of the supply chain members.

Deduction 1. *The static analysis results of the optimal strategies of the brand owner and retailer concerning relevant parameters are as follows*:

$$\frac{\partial A_M^{C*}(\mathrm{t})}{\partial d_1} > 0, \frac{\partial A_M^{C*}(\mathrm{t})}{\partial \phi_1} < 0, \frac{\partial A_M^{C*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial A_M^{C*}(\mathrm{t})}{\partial P_2} > 0;$$

$$\frac{\partial S_M^{C*}(\mathrm{t})}{\partial \lambda_1} > 0, \frac{\partial S_M^{C*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_M^{C*}(\mathrm{t})}{\partial P_2} > 0;$$

$$\frac{\partial A_R^{C*}(\mathrm{t})}{\partial \phi_2} > 0, \frac{\partial A_R^{C*}(\mathrm{t})}{\partial P_2} > 0;$$

$$\frac{\partial S_R^{C*}(\mathrm{t})}{\partial d_2} > 0, \frac{\partial S_R^{C*}(\mathrm{t})}{\partial \lambda_2} < 0, \frac{\partial S_R^{C*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_R^{C*}(\mathrm{t})}{\partial P_2} > 0.$$

Deduction 1 indicates that under this decision: (1) the optimal advertising strategy $A_M^{C*}(\mathrm{t})$ of the brand owner and the optimal service strategy $S_R^{C*}(\mathrm{t})$ of the retailer are an increasing function of the delay time d_1 and d_2. With the growth of the delay time d_1 and d_2, the brand owner's advertising and the retailer's service investment are increasing. In contrast, the brand goodwill level cannot reach the ideal state immediately, thus affecting the market demand. At this time, if the enterprise blindly increases the investment in advertising and service, the investment cost increases while the investment return period extends, which will undoubtedly increase the business risk of the enterprise; (2) the optimal service strategy $S_M^{C*}(\mathrm{t})$ of the brand owner is an increasing function of the webrooming effect λ_1, and the optimal service strategy $S_R^{C*}(\mathrm{t})$ of the retailer is a decreasing function of the showrooming effect λ_2. Due to the fact $P_2 > P_1$, the increase in the webrooming effect λ_1 has promoted the overall profit of the supply chain on the contrary, and the brand owner shall actively improve the service level $S_M^{C*}(\mathrm{t})$. Similarly, when the showrooming effect λ_2 intensifies, some consumers will choose the relatively cheap online channel to complete the purchase, which will reduce the overall profit of the supply chain, and the retailer shall reduce the service level $S_R^{C*}(\mathrm{t})$; (3) the brand owner advertising strategy $A_M^{C*}(\mathrm{t})$ is a decreasing function of ϕ_1, and the retailer advertising strategy $A_R^{C*}(\mathrm{t})$ is an increasing function of ϕ_2. Changes in national advertising $A_M^{C*}(\mathrm{t})$ investment by the brand owner will impact online and offline channel demand. The lower the conversion rate ϕ_1 of online channel demand, the higher the conversion rate $1-\phi_1$ of offline channel demand, which is better for the promotion of the overall profit of the supply chain. A change in the investment level of the regional promotional advertising $A_R^{C*}(\mathrm{t})$ will only have an impact on the offline channel demand. The higher the conversion rate ϕ_2 of the offline channel demand, the higher the advertising investment that should be made by the retailer; (4) the advertising strategy $A_M^{C*}(\mathrm{t})$ and the service strategy $S_M^{C*}(\mathrm{t})$ of the brand owner, and the retailer's service strategy $S_R^{C*}(\mathrm{t})$ are the increasing function of the retail prices P_1 and P_2 of both online and offline channels. The retailer's advertising strategy $A_{\mathrm{R}}{}^{C*}(\mathrm{t})$ of is the increasing function of P_2. Equations (2) and (3) show that channel demand is negatively related to the channel retail price. The enterprises should mitigate the impact of price on consumers' purchase behavior by improving the brand goodwill, advertising and service.

3.2. Brand Cost-Sharing Decision

Under this decision, as the supply chain leader, the brand owner participates in the cooperation unilaterally. The brand owner will bear part of the advertising and service costs to encourage the retailer to improve the advertising and service level. The supply chain members all aim to maximize their profits. First, the brand owner decides to bear the proportion ξ_1 and ξ_2 of advertising and service costs of the retailer. Then the brand owner and retailer select their advertising and service strategies independently, using the inverse induction method to solve the problem. Expressed by superscript "M", the profit functions of the brand owner and the retailer are as follows:

$$J_M^M = \int_0^\infty e^{-\rho t}[P_1(\mathrm{t})D_E(\mathrm{t}) + \omega D_R(\mathrm{t}) - \frac{k_1}{2}A_{\mathrm{M}}{}^2(\mathrm{t}) - \frac{\eta_1}{2}S_{\mathrm{M}}{}^2(\mathrm{t}) - \frac{\xi_1 k_2}{2}A_{\mathrm{R}}{}^2(\mathrm{t}) - \frac{\xi_2\eta_2}{2}S_{\mathrm{R}}{}^2(\mathrm{t})]\mathrm{dt} \tag{10}$$

$$J_R^M = \int_0^\infty e^{-\rho t}[(P_2(\mathrm{t}) - \omega)D_R(\mathrm{t}) - \frac{(1-\xi_1)k_2}{2}A_{\mathrm{R}}{}^2(\mathrm{t}) - \frac{(1-\xi_2)\eta_2}{2}S_{\mathrm{R}}{}^2(\mathrm{t})]\mathrm{dt} \tag{11}$$

Proposition 2. *Under this decision, the optimal strategies of the brand owner and the retailer are as follows:*

$$A_M^{M*}(\mathrm{t}) = \frac{\theta\gamma_1 e^{\delta d_1}(P_1+\omega) + (\delta+\rho)[(1-\phi_1)\omega + \phi_1 P_1]}{k_1(\delta+\rho)}, A_R^{M*}(\mathrm{t}) = \frac{\phi_2(\omega+P_2)}{2k_2}$$

$$S_M^{M*}(\mathrm{t}) = \frac{(1-\lambda_1)P_1 + \lambda_1\omega}{\eta_1}, S_R^{M*}(\mathrm{t}) = \frac{\theta\gamma_2[e^{\delta d_2}(P_2-\omega) + 2(P_1+\omega)] + (\delta+\rho)[(1-\lambda_2)(\omega+P_2) + 2\lambda_2 P_1]}{2\eta_2(\delta+\rho)} \tag{12}$$

Cost-sharing ratio of the brand owner:

$$\xi_1{}^{M*} = \begin{cases} \frac{3\omega - P_2}{\omega + P_2}, & 3\omega > P_2 \\ 0, & 3\omega \le P_2 \end{cases}$$

$$\xi_2{}^{M*} = \begin{cases} \frac{X_1\omega + Y_1P_1 - Z_1P_2}{X_2\omega + Y_1P_1 + Z_1P_2}, & X_1\omega + Y_1P_1 > Z_1P_2 \\ 0, & X_1\omega + Y_1P_1 \le Z_1P_2 \end{cases}$$

Brand goodwill level is as follows:

$$\begin{aligned} G^{\mathrm{M}*}(t) = \quad & e^{-\delta t}G_0 + \frac{\theta\gamma_1{}^2 e^{\delta d_1}(P_1+\omega) + \gamma_1(\delta+\rho)[(1-\phi_1)\omega + \phi_1 P_1]}{k_1\delta(\delta+\rho)}(1-e^{-\delta t}) \\ & + \frac{\theta\gamma_2{}^2[e^{\delta d_2}(P_2-\omega) + 2(P_1+\omega)] + \gamma_2(\delta+\rho)[(1-\lambda_2)(\omega+P_2) + 2\lambda_2 P_1]}{2\eta_2\delta(\delta+\rho)}(1-e^{-\delta t}) \end{aligned}$$

In the equation:

$$X_1 = \theta\gamma_2(2 + e^{\delta d_2}) + 3(\delta+\rho)(1-\lambda_2)$$

$$X_2 = \theta\gamma_2(2 - e^{\delta d_2}) + (\delta+\rho)(1-\lambda_2)$$

$$Y_1 = 2\theta\gamma_2 + 2\lambda_2(\delta+\rho)$$

$$Z_1 = \theta\gamma_2 e^{\delta d_2} + (\delta+\rho)(1-\lambda_2)$$

Proving: First, confirm the cost-sharing ratios ξ_1^M and ξ_2^M of the brand owner, and determine the optimal strategies of the supply chain members. The retailer's optimal decision problem is characterized as the optimal control problem, i.e.,

$$\max_{A_M>0, S_R>0} J_R^M$$

$$\text{s.t. } \dot{G}(t) = \gamma_1 A_M(t - d_1(t)) + \gamma_2 S_R(t - d_2(t)) - \delta G(t)$$

Similar to the evidentiary process of the Proposition 1, build the Hamilton function as follows:

$$\begin{aligned} H_R^M = e^{-\rho t}&\left\{ \begin{array}{l} (P_2-\omega)[\alpha - \beta_2 P_2 + \theta G + (1-\phi_1)A_{\mathrm{M}} + \phi_2 A_{\mathrm{R}} + (1-\lambda_2)S_R \\ +\lambda_1 S_M] - \frac{(1-\xi_1)k_2}{2}A_{\mathrm{R}}{}^2 - \frac{(1-\xi_2)\eta_2}{2}S_{\mathrm{R}}{}^2] \end{array} \right\} \\ &+q(t)[\gamma_1 A_M(t-d_1) + \gamma_2 S_R(t-d_2) - \delta G(t)] \end{aligned}$$

Assuming

$$\frac{dH_R}{dA_{\mathrm{R}}(t)} = e^{-\rho t}[\phi_2(P_2-\omega) - k_2(1-\xi_1)A_{\mathrm{M}}(t)] = 0$$

Receiving

$$A_{\mathrm{R}}{}^{\mathrm{M}*}(\mathrm{t}) = \frac{\phi_2(P_2-\omega)}{k_2(1-\xi_1)} \tag{13}$$

Assuming

$$\frac{dH_R}{dS_R(t)} = e^{-\rho t}[(1-\lambda_2)(P_2-\omega) - \eta_2(1-\xi_2)S_R(t)] + q(t)\gamma_2\frac{dS_R(t-d_2)}{dS_R(t)} = 0 \tag{14}$$

$$\frac{dq(t)}{dt} = -\frac{dH_R}{dG} = \delta q(t) - \theta(P_2-\omega)e^{-\rho t}$$

$$q(t) = C_1e^{\delta t} + \frac{\theta(P_2-\omega)}{\rho+\delta}e^{-\rho t},\ C_1 \in R \tag{15}$$

Substituting $\frac{dS_R(t-d_2)}{dS_R(t)} = e^{\delta d_2}$ and Equation (15) into Equation (14).
Receiving

$$S_R^{M*}(\mathrm{t}) = \frac{[(1-\lambda_2)(\delta+\rho)+\theta\gamma_2e^{\delta d_2}](P_2-\omega)}{\eta_2(1-\xi_2)(\delta+\rho)} + \frac{\gamma_2}{\eta_2(1-\xi_2)}C_1e^{(\delta+\rho)t+\delta d_2}$$

As $\lim_{t\to\infty} S_R^{M*}(\mathrm{t}) < \infty$, so $C_1 = 0$. Therefore:

$$S_R^{M*}(\mathrm{t}) = \frac{[(1-\lambda_2)(\delta+\rho)+\theta\gamma_2e^{\delta d_2}](P_2-\omega)}{\eta_2(1-\xi_2)(\delta+\rho)} \tag{16}$$

The optimal decision problem of the brand owner is characterized as the optimal control problem, i.e.,

$$\max_{A_M>0,S_R>0} J_M^M$$

$$\text{s.t. } \dot{G}(t) = \gamma_1A_M(t-d_1(t)) + \gamma_2S_R(t-d_2(t)) - \delta G(t)$$

Hamilton function is as follows:

$$\begin{aligned} H_M^M = \quad & e^{-\rho t}\left\{\begin{array}{l} P_1[\alpha-\beta_1P_1+\theta G+\phi_1A_M+(1-\lambda_1)S_M+\lambda_2S_R]+\omega[\alpha-\beta_2P_2+\theta G+(1-\phi_1)A_M \\ +\phi_2A_R+(1-\lambda_2)S_R+\lambda_1S_M]-\frac{k_1}{2}A_M{}^2-\frac{\eta_1}{2}S_M{}^2-\frac{\xi_1k_2}{2}A_R{}^2-\frac{\xi_2\eta_2}{2}S_R{}^2\Big] \end{array}\right\} \\ & +q(t)[\gamma_1A_M(t-d_1)+\gamma_2S_R(t-d_2)-\delta G(t)] \end{aligned}$$

Similar to the evidentiary process of the optimal decision of the retailer, receiving:

$$A_M^{M*}(\mathrm{t}) = \frac{\theta\gamma_1e^{\delta d_1}(P_1+\omega)+(\delta+\rho)[(1-\phi_1)\omega+\phi_1P_1]}{k_1(\delta+\rho)} \tag{17}$$

$$S_M^{M*}(\mathrm{t}) = \frac{(1-\lambda_1)P_1+\lambda_1\omega}{\eta_1} \tag{18}$$

Substituting Equations (16) and (17) into Equation (1), receiving the brand goodwill as follows:

$$\begin{aligned} G^{M*}(t) = \quad & e^{-\delta t}G_0 + \tfrac{\theta\gamma_1{}^2e^{\delta d_1}(P_1+\omega)+\gamma_1(\delta+\rho)[(1-\phi_1)\omega+\phi_1P_1]}{k_1\delta(\delta+\rho)}(1-e^{-\delta t}) \\ & +\tfrac{\gamma_2[(1-\lambda_2)(\delta+\rho)+\theta\gamma_2e^{\delta d_2}](P_2-\omega)}{\eta_2\delta(1-\xi_2)(\delta+\rho)}(1-e^{-\delta t}) \end{aligned} \tag{19}$$

The brand owner decides the optimal cost-sharing ratio and substitutes Equations (13), (16)–(19) into the Equation (10), receiving the profit function J_M^{M*} of the brand owner. Find the maximums of $\xi_1{}^M$ and $\xi_2{}^M$ by J_M^{M*}, i.e.,

Assume $\frac{\partial J_M^{M*}}{\partial\xi_1} = 0$ and $\frac{\partial J_M^{M*}}{\partial\xi_2} = 0$, the conclusion of ξ_1^{M*} and ξ_2^{M*} will obtained. The evidentiary process ends.

Further, we can analyze the impact of the delay time d_1 and d_2, webrooming effect and showrooming effect coefficients λ_1 and λ_2, the conversion rates ϕ_1 and ϕ_2 of the advertising investment levels on the online channel and offline channel demand, and the impacts of the online and offline retail prices P_1 and P_2, and the wholesale price ω on the optimal strategies of the brand owner and retailer.

Deduction 2. *The static analysis results of the optimal strategies of the brand owner and retailer concerning relevant parameters are as follows:*

$$\frac{\partial A_M^{M*}(\mathrm{t})}{\partial d_1} > 0, \frac{\partial A_M^{M*}(\mathrm{t})}{\partial \phi_1} > 0, \frac{\partial A_M^{M*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial A_M^{M*}(\mathrm{t})}{\partial \omega} > 0;$$

$$\frac{\partial S_M^{M*}(\mathrm{t})}{\partial \lambda_1} < 0, \frac{\partial S_M^{M*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_M^{M*}(\mathrm{t})}{\partial \omega} > 0;$$

$$\frac{\partial A_R^{M*}(\mathrm{t})}{\partial \phi_2} > 0, \frac{\partial A_R^{M*}(\mathrm{t})}{\partial P_2} > 0, \frac{\partial A_R^{M*}(\mathrm{t})}{\partial \omega} > 0;$$

$$\frac{\partial S_R^{M*}(\mathrm{t})}{\partial d_2} > 0, \frac{\partial S_R^{M*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_R^{M*}(\mathrm{t})}{\partial P_2} > 0,$$

$$\begin{cases} \frac{\partial S_R^{M*}(\mathrm{t})}{\partial \lambda_2} > 0, & P_1 < (\omega + P_2)/2 \\ \frac{\partial S_R^{M*}(\mathrm{t})}{\partial \lambda_2} < 0, & P_1 > (\omega + P_2)/2 \end{cases}, \begin{cases} \frac{\partial S_R^{M*}(\mathrm{t})}{\partial \omega} > 0, & d_2 < \mathrm{L}_2 \\ \frac{\partial S_R^{M*}(\mathrm{t})}{\partial \omega} < 0, & d_2 > \mathrm{L}_2 \end{cases};$$

$$\frac{\partial \xi_1^{M*}}{\partial P_2} < 0, \frac{\partial \xi_1^{M*}}{\partial \omega} > 0; \frac{\partial \xi_2^{M*}}{\partial d_2} < 0, \frac{\partial \xi_2^{M*}}{\partial \lambda_2} > 0, \frac{\partial \xi_2^{M*}}{\partial P_1} > 0, \frac{\partial \xi_2^{M*}}{\partial P_2} < 0, \frac{\partial \xi_2^{M*}}{\partial \omega} > 0;$$

Among which $\mathrm{L}_2 = \frac{1}{\delta} \ln \frac{(\delta+\rho)(1-\lambda_2)+2\theta\gamma_2}{\theta\gamma_2}$.

Deduction 1 indicates that under this decision:

(1) The optimal advertising strategy $A_M^{M*}(\mathrm{t})$ of the brand owner and the optimal service strategy $S_R^{M*}(\mathrm{t})$ of the retailer are an increasing function of the delay time d_1 and d_2, which is consistent with the centralized decision. The cost-sharing ratio ξ_2^{M*} of the brand owner is a decreasing function of the delay time d_2; as the delay time d_2 increases, the brand owner is willing to share a smaller share of the retailer's service cost;

(2) The brand owner's optimal service strategy $S_M^{M*}(\mathrm{t})$ is a decreasing function of the webrooming effect λ_1, while the brand owner's cost-sharing ratio ξ_2^{M*} is an increasing function of the showrooming effect λ_2 The impact of the showrooming effect λ_2 on the retailer's optimal service strategy $S_R^{M*}(\mathrm{t})$ is influenced by the brand owner's cost-sharing ratio ξ_2^{M*} and the retail price P_1 of the online channel. Under the condition of $P_1 < (\omega + P_2)/2$, the price gap between the channels decreases and the showrooming effect λ_2 is relieved, while the service $S_R^{M*}(\mathrm{t})$ of the retailer is promoted. This shows that if the online channel price of the brand owner is properly decided, ξ_2^{M*} can relieve the impact of the showrooming effect on the retailer's service strategy;

(3) The brand owner's advertising strategy $A_M^{M*}(\mathrm{t})$ is an increasing function of ϕ_1 and the lower the online channel demand conversion rate ϕ_1, the lower the brand owner's advertising level should be. A change in the level of the regional promotional advertising $A_R^{M*}(\mathrm{t})$ by the retailer will only affect the offline channel demand and will not be affected by the decision-making; thus, the retailer shall improve the advertising level to promote the conversion rate ϕ_2 of the offline channel demand;

(4) The wholesale price ω is an important factor influencing the optimal service and advertising strategies of the supply chain members. The higher the wholesale price ω, the higher the revenue the brand owner receives from the offline channel, while the retailer's product cost will be increased. To motivate the retailer to actively promote the advertising and service levels, the brand owner shall increase the cost-sharing ratios ξ_1^{M*} and ξ_2^{M*} with the retailer. The retailer shall promote the advertising level

$A_R^{M*}(\mathrm{t})$, but the delay time d_2 also affects whether to increase the level of service. Under the condition of $d_2 > \mathrm{L}_2$, ξ_2^{M*} cannot motivate the retailer to promote its service. Thus, under the condition of $d_2 < \mathrm{L}_2$, that the brand owner can promote the optimal advertising and service strategies of the supply chain members if it sets a comparatively high wholesale price ω.

3.3. Bilateral Cost-Sharing Decision

Under this decision, by referring to the literature [51], the brand owner and the retailer are participating in cooperation together and share the cost in a bilateral way, namely, the retailer shares the national advertising cost ratio ξ_1^D of the brand owner, and the brand owner shares the experience service cost ratio ξ_2^D of the retailer. Presented by the superscript "D", the objective profit functions of the brand owner and retailer are as follows:

$$J_M^D = \int_0^\infty e^{-\rho t}[P_1(\mathrm{t})D_E(\mathrm{t}) + \omega D_R(\mathrm{t}) - \frac{(1-\xi_1)k_1}{2}A_{\mathrm{M}}{}^2 - \frac{\eta_1}{2}S_{\mathrm{M}}{}^2 - \frac{\xi_2\eta_2}{2}S_{\mathrm{R}}]\mathrm{dt} \tag{20}$$

$$J_M^D = \int_0^\infty e^{-\rho t}[P_1(\mathrm{t})D_E(\mathrm{t}) + \omega D_R(\mathrm{t}) - \frac{(1-\xi_1)k_1}{2}A_{\mathrm{M}}{}^2 - \frac{\eta_1}{2}S_{\mathrm{M}}{}^2 - \frac{\xi_2\eta_2}{2}S_{\mathrm{R}}]\mathrm{dt} \tag{21}$$

Proposition 3. *Under this decision, the optimal strategies of the brand owner and the retailer are as follows:*

$$A_M^{D*}(\mathrm{t}) = \frac{\theta\gamma_1[e^{\delta d_1}(P_1+\omega) + 2(P_2-\omega)] + (\delta+\rho)[(1-\phi_1)(P_2-\omega) + \phi_1 P_1]}{2k_1(\delta+\rho)}, A_R^{D*}(\mathrm{t}) = \frac{\phi_2(P_2-\omega)}{k_2},$$

$$S_M^{D*}(\mathrm{t}) = \frac{(1-\lambda_1)P_1 + \lambda_1\omega}{\eta_1}, S_R^{D*}(\mathrm{t}) = \frac{\theta\gamma_2[e^{\delta d_2}(P_2-\omega) + 2(P_1+\omega)] + (\delta+\rho)[(1-\lambda_2)(\omega+P_2) + 2\lambda_2 P_1]}{2\eta_2(\delta+\rho)}(\mathrm{t}).$$

$$\xi_1^{D*} = \begin{cases} \frac{Z_2P_2 - Y_2P_1 - X_3\omega}{Z_2P_2 + Y_2P_1 - X_4\omega}, & X_3\omega + Y_2P_1 < Z_2P_2 \\ 0, & X_3\omega + Y_2P_1 \ge Z_2P_2 \end{cases}, \xi_2^{D*} = \begin{cases} \frac{X_1\omega + Y_1P_1 - Z_1P_2}{X_2\omega + Y_1P_1 + Z_1P_2}, & X_1\omega + Y_1P_1 > Z_1P_2 \\ 0, & X_1\omega + Y_1P_1 \le Z_1P_2 \end{cases}.$$

Bilateral cost-sharing ratios are as follows:
In the equation:

$$X_3 = \theta\gamma_1(2 + e^{\delta d_1}) + 3(\delta+\rho)(1-\phi_1)$$
$$X_4 = \theta\gamma_1(2 - e^{\delta d_1}) + (\delta+\rho)(1-\phi_1)$$
$$Y_2 = \theta\gamma_1 e^{\delta d_1} + \phi_1(\delta+\rho)$$
$$Z_2 = 2\theta\gamma_1 + 2(\delta+\rho)(1-\phi_1)$$

The brand goodwill:

$$\begin{aligned} G^{\mathrm{D}*}(t) = \quad & e^{-\delta t}G_0 + \frac{\theta\gamma_1{}^2[e^{\delta d_1}(P_1+\omega) + 2(P_2-\omega)] + \gamma_1(\delta+\rho)[(1-\phi_1)(P_2-\omega) + \phi_1P_1]}{2k_1\delta(\delta+\rho)}(1-e^{-\delta t}) \\ & + \frac{\theta\gamma_2{}^2[e^{\delta d_2}(P_2-\omega) + 2(P_1+\omega)] + \gamma_2(\delta+\rho)[(1-\lambda_2)(\omega+P_2) + 2\lambda_2P_1]}{2\eta_2\delta(\delta+\rho)}(1-e^{-\delta t}) \end{aligned}$$

Proving: under this decision, after the brand owner decides its service cost-sharing ratio ξ_2^D for the retailer and the retailer decides its advertising cost ratio ξ_1^D for the brand owner, the brand owner and retailer decide their optimal service and advertising strategies, respectively. The evidentiary process is similar to Proposition 2 and will not be repeated here.

$S_M^{M*}(t) = S_M^{D*}(t)$, $S_R^{M*}(t) = S_R^{D*}(t)$ follows from Propositions 2 and 3. The impacts of the webrooming effect and showrooming effect coefficients λ_1 and λ_2 on the supply chain members' service strategies are the same as Proposition 2. The impacts of the delay time d_1 and d_2, the wholesale price ω, and the brand owner's advertising conversion rate ϕ_1 on the supply chain members' optimal strategies are further analyzed.

Deduction 3. *The static analysis results of the optimal strategies of the brand owner and retailer concerning relevant parameters are as follows:*

$$\frac{\partial A_M^{D*}(\mathrm{t})}{\partial d_1} > 0, \frac{\partial A_M^{D*}(\mathrm{t})}{\partial \phi_1} < 0, \frac{\partial A_M^{D*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial A_M^{D*}(\mathrm{t})}{\partial P_2} > 0,$$

$$\begin{cases} \frac{\partial A_M^{D*}(\mathrm{t})}{\partial \omega} > 0, & \phi_1 > L_1 \\ \frac{\partial A_M^{D*}(\mathrm{t})}{\partial \omega} < 0, & \phi_1 < L_1 \end{cases};$$

$$\frac{\partial S_M^{D*}(\mathrm{t})}{\partial \lambda_1} < 0, \frac{\partial S_M^{D*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_M^{D*}(\mathrm{t})}{\partial \omega} > 0;$$

$$\frac{\partial A_R^{D*}(\mathrm{t})}{\partial \phi_2} > 0, \frac{\partial A_R^{D*}(\mathrm{t})}{\partial P_2} > 0, \frac{\partial A_R^{D*}(\mathrm{t})}{\partial \omega} < 0;$$

$$\frac{\partial S_R^{D*}(\mathrm{t})}{\partial d_2} > 0, \frac{\partial S_R^{D*}(\mathrm{t})}{\partial P_1} > 0, \frac{\partial S_R^{D*}(\mathrm{t})}{\partial P_2} > 0,$$

$$\begin{cases} \frac{\partial S_R^{D*}(\mathrm{t})}{\partial \lambda_2} > 0, & \omega - 2P_1 + P_2 < 0 \\ \frac{\partial S_R^{D*}(\mathrm{t})}{\partial \lambda_2} < 0, & \omega - 2P_1 + P_2 > 0 \end{cases}, \begin{cases} \frac{\partial S_R^{D*}(\mathrm{t})}{\partial \omega} > 0, & d_2 < \mathrm{L}_2 \\ \frac{\partial S_R^{D*}(\mathrm{t})}{\partial \omega} < 0, & d_2 > \mathrm{L}_2 \end{cases};$$

$$\frac{\partial \xi_1^{D*}}{\partial d_1} < 0, \frac{\partial \xi_1^{D*}}{\partial \phi_1} < 0, \frac{\partial \xi_1^{D*}}{\partial P_1} < 0, \frac{\partial \xi_1^{D*}}{\partial P_2} > 0, \frac{\partial \xi_1^{D*}}{\partial \omega} < 0;$$

$$\frac{\partial \xi_2^{D*}}{\partial d_2} < 0, \frac{\partial \xi_2^{D*}}{\partial \lambda_2} > 0, \frac{\partial \xi_2^{D*}}{\partial P_1} > 0, \frac{\partial \xi_2^{D*}}{\partial P_2} < 0, \frac{\partial \xi_2^{D*}}{\partial \omega} > 0.$$

Among which, $\mathrm{L}_1 = \frac{1}{\delta} \ln \frac{(\delta+\rho)(1-\phi_1)+2\theta\gamma_1}{\theta\gamma_1}$.

Deduction 3 indicates that under this decision: (1) ξ_1^{D*} and ξ_2^{D*} are a decreasing function of d_1 and d_2, respectively. The longer the delay in the brand advertising and the retailer service to the brand goodwill, the lower the rate of cost-sharing between the brand and the retailer participation in a bidirectional sense; (2) the brand owner's advertising strategy is a decreasing function of the conversion rate ϕ_1. As the conversion rate ϕ_1 decreases and the conversion rate $1 - \phi_1$ of the offline channel demand increases, the retailer increases its cost-sharing ratio ξ_1^{D*} to encourage the brand owner to invest in the advertising. Therefore, although ϕ_1 decreases, the brand owner shall also increase its advertising level; (3) with the wholesale price ω increase, the brand owner promotes its service level $S_M^{D*}(\mathrm{t})$ and cost-sharing ratio ξ_2^{D*}, and the retailer decreases its advertising level $A_R^{D*}(\mathrm{t})$ and cost-sharing ratio ξ_1^{D*}. However, the delay time influences whether the cost-sharing ratios ξ_1^{D*} and ξ_2^{D*} can affect the brand owner's advertising strategy $A_M^{D*}(\mathrm{t})$ and the retailer's service strategy $S_R^{D*}(\mathrm{t})$. When $d_1 < L_1$, $d_2 < L_2$ controlled by ξ_1^{D*} and ξ_2^{D*}, the brand owner should reduce its advertising level and the retailer should improve its service level. Under the condition of $d_1 > L_1$, $d_2 > L_2$, with the rise in the wholesale price ω, the brand owner should improve its advertising level and the retailer should lower its service level.

4. Comparative Analysis

4.1. Comparison of the Optimal Strategies of the Supply Chain Members under Three Decisions

Deduction 4. *As compared under the three decisions, it is constant as follows:*

$$A_M^{C*}(\mathrm{t}) > A_M^{D*}(\mathrm{t}) > A_M^{M*}(\mathrm{t});\ A_R^{C*}(\mathrm{t}) > A_R^{M*}(\mathrm{t}) > A_R^{D*}(\mathrm{t}); \tag{22}$$

$$\begin{gathered} S_M^{C*}(t) > S_M^{M*}(t) = S_M^{D*}(t); S_R^{C*}(\mathrm{t}) > S_R^{M*}(\mathrm{t}) = S_R^{D*}(\mathrm{t}); \\ G^{\mathrm{C}*}(t) > G^{\mathrm{D}*}(t) > G^{\mathrm{M}*}(t);\ \xi_1^{M*} = \xi_2^{D*}. \end{gathered} \tag{23}$$

Deduction 4 indicates that: compared to the other two decisions, under the Centralized Decision "C", they are the optimal ones regardless of the service or advertising strategy of the supply chain members, or the goodwill of the brand owner. The brand owner's advertising strategy $A_M(\mathrm{t})$ is better under the Bilateral Cost-Sharing Decision "D", which indicates that the bilateral cost-sharing decision can motivate the brand owner to improve its advertising level. The retailer's advertising strategy $A_{\mathrm{R}}(\mathrm{t})$ is better under the brand cost-sharing decision "M", which can encourage the retailer to improve its advertising level. No matter whether it is the brand cost-sharing decision or the bilateral cost-sharing decision, the brand owner's cost-sharing ratios on the retailer's service are consistent, i.e., $\xi_1^{M*} = \xi_2^{D*}$. Meanwhile, the service investment levels of the brand owner and the retailer are also compatible, i.e., $S_M^{M*}(t) = S_M^{D*}(t), S_R^{M*}(\mathrm{t}) = S_R^{D*}(\mathrm{t})$.

4.2. Profit Comparison under Three Decisions

The impact of time delay and bidirectional free-riding phenomenon on the profit of the supply chain and the supply chain members is observed through analysis of the arithmetic examples. It is assumed that the parameters included in the above model are as follows: $\delta = 0.01, \rho = 0.05, \theta = 0.4, \gamma_1 = 0.5, \gamma_2 = 0.6, \phi_1 = 0.5, \phi_2 = 0.6, \lambda_1 = 0.3, \lambda_2 = 0.4,$ $\mathrm{P}_1 = 3, \mathrm{P}_2 = 5.5, \omega = 2, \mathrm{G}_0 = 0, \alpha = 1, \beta_1 = 0.6, \beta_2 = 0.4, k_1 = k_2 = 1, \eta_1 = \eta_2 = 1.$

4.2.1. Impact of Time Delay on the Profit of the Supply Chain and Supply Chain Members

Under the above parameter settings, the value range of time delay takes values in the range $d_1,\ d_2 \in (0, 30)$. This paper analyzes the impact of time delay on the supply chain, the brand owner and the retailer profit under different decisions.

Figure 2 indicates the impact of the change of delay time $d_1,\ d_2$ on the profit of the supply chain under different decisions. With the increase in the delay time $d_1,\ d_2$, the supply chain profit under the centralized decision shows a downward trend, while it shows an upward trend under the other two decisions. Therefore, when the delay time $d_1,\ d_2$ is comparatively short, the supply chain profit under the centralized decision is the highest. When the delay time $d_1,\ d_2$ is comparatively long, the supply chain profit under the bilateral cost-sharing decision becomes the optimal value. In the delay time $d_1,\ d_2$ range, the supply chain profit under a bilateral cost-sharing decision is always higher than that under the brand cost-sharing decision.

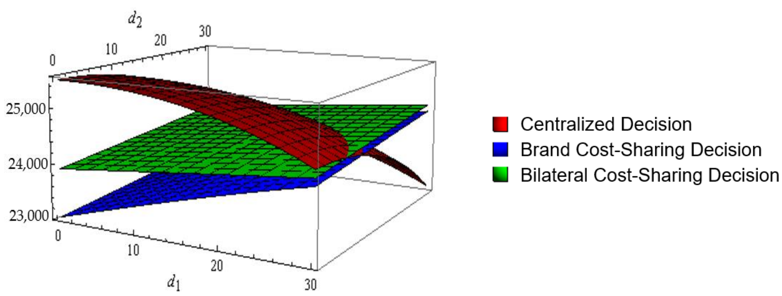

Figure 2. Impact of Time Delay on the Supply Chain Profit.

As can be seen in Figure 3, the increase in delay time d_1, d_2 will have different impacts on the profit of the brand owner. With the growth of the delay time d_1 for national advertising, the brand owners' advertising investment level is increasing. In contrast, the impact on brand goodwill and market demand is not immediately apparent, resulting in a decreasing trend in the brand owners' profits.

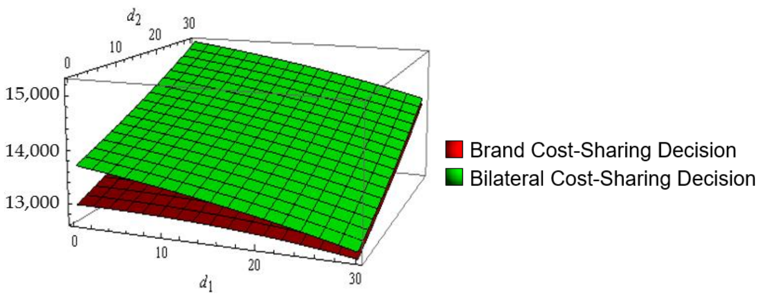

Figure 3. Impact of Time Delay on the Brand Owner's Profit.

As the service delay time d_2 of the retailer increases, the brand owner benefits from the improvement of the retailer's service level. According to Deductions 2 and 3, the brands' share of the retailer service cost ratio ξ_2 decreases at this moment, so the profit of the brand owner increases continuously.

As can be seen in Figure 4, similar to the case of the brand owner, the increase in the delay time d_1, d_2 will have different impacts on the retailer's profit. What is different is that with the rise in the national advertising delay time d_1, the retailer benefits from the improvement of the advertising level of the brand owner. Meanwhile, according to Deduction 3, the brand owner's advertising cost ratio ξ_1 decreases at this moment, increasing the retailer's profit. As the retailer's service delay time d_2 increases, the retailer's cost of providing the service increases while the brand owner's share of the service cost ratio ξ_2 decreases, decreasing the retailer's profit.

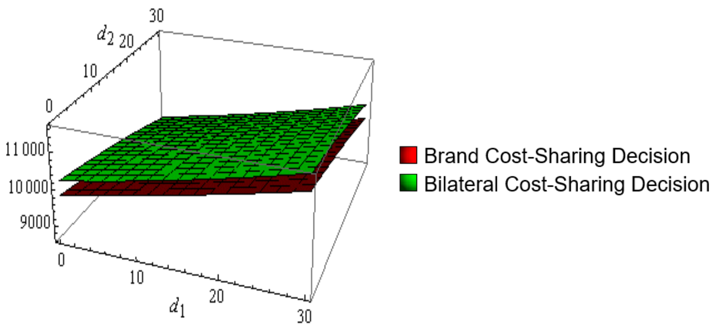

Figure 4. Impact of Time Delay on the Retailer's Profit.

Comparing Figure 3 with Figure 4, the bilateral cost-sharing decision can promote the profits of the brand owner and retailer at the same time and realize the Pareto improvement. Therefore, when there is a time delay, the supply chain members should prefer to implement the bilateral cost-sharing decision.

4.2.2. Impact of Bidirectional Free-Riding on the Profit of Supply Chain and Supply Chain Members

The webrooming effect and showrooming effect coefficients λ_1, λ_2 will impact the service decisions of the brand owner and the retailer, respectively, affecting the supply chain's profit. From Deduction 2, under the condition of $P_1 > (\omega + P_2)/2$, that the showrooming effect λ_2 is weakening, the brand owner reduces its cost-sharing ratio ξ_2 and the retailer reduces its service level. Therefore, on the premise that other parameters are unchanged, we have chosen the two different conditions of $P_1 = 3 < (\omega + P_2)/2$ and

$P_1 = 4 > (\omega + P_2)/2$ to analyze the impact of the changes of the webrooming effect and showrooming effect coefficients λ_1, λ_2 on the profits of the supply chain, the brand owner and the retailer. Set $d_1 = d_2 = 20$ and $P^* = (\omega + P_2)/2$.

From Figures 5 and 6, the changes in the webrooming effect and showrooming effect coefficients λ_1, λ_2 will have different impacts on the supply chain profit. Regardless of $P_1 < P^*$ or $P_1 > P^*$, the supply chain profit is about the increasing function of the webrooming effect coefficient λ_1 and the decreasing function of the showrooming effect coefficient λ_2; the change in the showrooming effect coefficient λ_2 will have a more significant impact. With the decrease in λ_1 and the increase in λ_2, the supply chain profit under the centralized decision tends to decrease gradually than under the other two decisions. Under the condition of $P_1 < P^*$, the supply chain profit under the bilateral cost-sharing decision is higher than that under the brand cost-sharing decision; and under the condition of $P_1 > P^*$, the supply chain profit under the brand cost-sharing decision is higher than that under the bilateral cost-sharing decision.

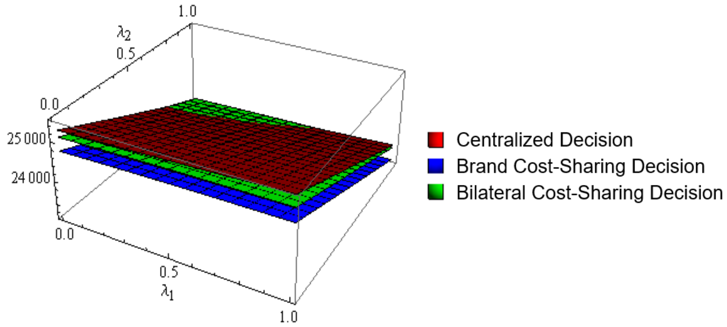

Figure 5. Impact of λ_i on the Supply Chain Profit When $P_1 < P^*$.

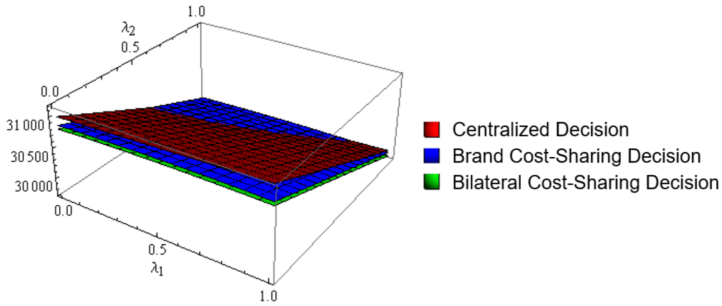

Figure 6. Impact of λ_i on the Supply Chain Profit When $P_1 > P^*$.

Figures 7 and 8 have reflected the impact of the change in the webrooming effect and showrooming effect coefficients λ_1, λ_2 on the brand owner's profit under different value ranges of P_1. Under the condition of $P_1 < P^*$, the difference in the coefficient λ_1, λ_2 on the brand owner's profit is the same when the profit of the brand owner is a decreasing function in λ_1, λ_2 and the brand owner's profit will be higher under the bilateral cost-sharing decision.

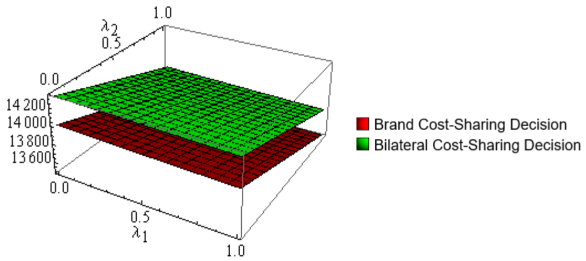

Figure 7. Impact of λ_i on the Brand Owner's Profit When $P_1 < P^*$.

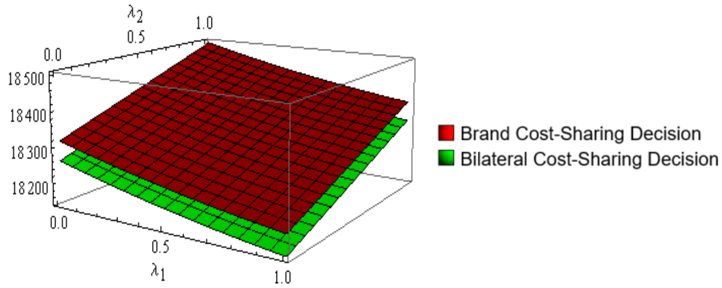

Figure 8. Impact of λ_i on the Brand Owner's Profit When $P_1 > P^*$.

Under the condition of $P_1 > P^*$, changes in λ_1, λ_2 will have different impacts on the profit of the brand owner. Under the condition that λ_1 remains unchanged, the profit of the brand owner will increase with the increase in λ_2; and under the condition that λ_2 remains unchanged, the profit of the brand owner will decrease with the increase in λ_1, and the profit of the brand owner will be higher under the brand cost-sharing decision.

Figures 9 and 10 illustrate the impacts of the changes of the webrooming effect and showrooming effect coefficients λ_1, λ_2 on the retailer's profit. It is clear from the figures that no matter whether the condition of $P_1 < P^*$ or $P_1 > P^*$, the result from the changes in the webrooming effect and showrooming effect coefficients λ_1, λ_2 on the retailer's profit are different. With the growth in λ_1, the retailer's profit shows a slight increase; while with the growth in λ_2, the retailer's profit shows a rapid decrease. Under both conditions, the retailer's profit under the bilateral cost-sharing decision is the higher one.

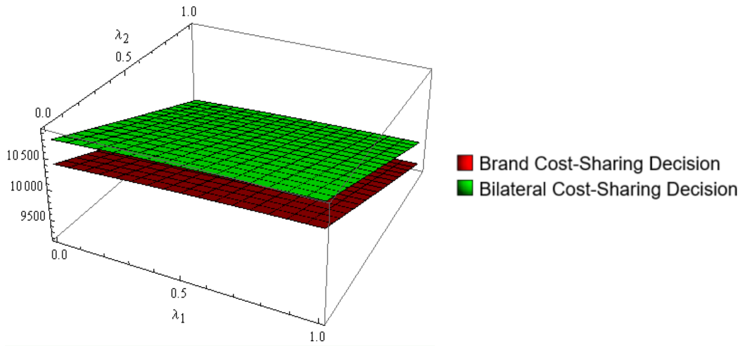

Figure 9. Impact of λ_i on the Retailer's Profit When $P_1 < P^*$.

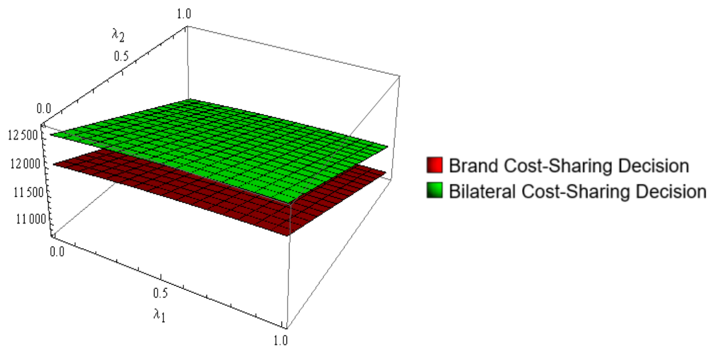

Figure 10. Impact of λ_i on the Retailer's Profit When $P_1 > P^*$.

5. Conclusions

5.1. Research Conclusions

Channel convergence is the focal issue of the O2O supply chain, and inter-channel advertising and service cooperation is the leading research area of channel convergence. This paper considered the impact of time delay and bidirectional free-riding on inter-channel service and advertising cooperation strategies. We adopted differential game theory to solve the optimal advertising and service decisions of the brand owners and retailers under a centralized decision, the brand owners' cost-sharing decision, and bilateral cost-sharing decision. We also explored the effects of delay time, the bidirectional free-riding coefficient, price and other factors on cooperative advertising and service decisions. The analysis leads to the following conclusions:

(1) Among these three decisions, the supply chain members' advertising strategy, service strategy and brand goodwill are optimal under a centralized decision. However, the supply chain profit is not necessarily optimal due to the time delay and bidirectional free-riding phenomenon. The supply chain profit depends on the delay time length and the bidirectional free-riding coefficient size.
(2) Under the two kinds of cost-sharing decisions, the supply chain members and the brand owner's service decision and the brand owner's cost-sharing of the retailer's service are the same. The brand cost-sharing decision can promote the advertising level of the retailer, while the bilateral cost-sharing decision can promote the brand owner's national advertising and the brand goodwill. From the perspective of the supply chain members' profit, the bilateral cost-sharing decision can promote the profits of the brand owner and the retailer at the same time; regardless of the length of delay time. Irrespective of the bidirectional free-riding coefficient size, the bilateral cost-sharing decision can still promote the profit of the retailer; however, whether this decision can promote the profit of the brand owner is affected by the retail price of the online channel.
(3) As the delay time grows, the brand owners' national advertising level and the retailer's experience service level continue to improve, which causes the brand owner's and retailer's investment costs to continue to increase, while it is difficult to achieve the expected effect in the short term, raising the enterprise's operational risks. The bilateral cost-sharing ratio is a decreasing function of the delay time, which can coordinate the impact of time delay on the brand owner's advertising and retailer's service decisions to a certain extent.
(4) Under a centralized decision, brands are willing to cooperate fully and improve their service levels even if the webrooming phenomenon is intensified to increase the profitability of the whole supply chain. Under the two cost-sharing decisions, the brand owner is willing to cooperate partially. The brand owner can moderate the impact of the showrooming phenomenon on retailers' service strategies by setting retail prices and sharing retail service costs proportionally in the online channel.
(5) The supply chain members' optimal service and advertising strategy is an increasing function of the retail prices of the online and offline channels. In contrast, the impact of the wholesale price on the optimal strategy of the supply chain members is affected by the decision mode. Under the brand cost-sharing decision, by setting a comparatively high wholesale price, the brand owner can adjust the impact of the time delay on the service strategy of the retailer and promote the service and advertising levels of the supply chain members. Under the bilateral cost-sharing decision, a higher wholesale price enhances the service level of supply chain members; however, it reduces the advertising level of supply chain members and exacerbates the impact of the delay phenomenon on their national advertising strategies.

This study can further explore the following aspects of the following elements. First, this study only considers the self-run online channels of brand owners and does not consider the cooperative relationship with e-commerce platforms. At present, the cooperation between brands and e-commerce platforms mainly includes two modes: resale and commission, which will become the direction for the improvement of cooperation subjects in the future of this study. Secondly, this paper only considers the pre-sale advertising and service cooperation of the O2O supply chain and can further evaluate the collaboration of after-sale delivery and return.

5.2. Practical Implications

The above research results provide some management suggestions for the service and advertising cooperation of O2O supply chain members.

(1) The research in this paper shows that the bilateral cost-sharing contract is a compelling long-term strategy to solve the adverse effects of delay and two-way free-riding. The bilateral cost-sharing contract can increase the profits of each node enterprise in the O2O supply chain, thus improving the performance of the whole Pareto.
(2) Brands should reasonably set wholesale prices, online direct-selling prices and cost-sharing ratio, which can effectively promote retailers' investment in offline experience services, to achieve a win-win situation and improve the entire supply chain performance.
(3) From the perspective of long-term development, the key for retailers to deal with showrooming behavior is to improve their brand premium ability, highlight the experience services provided by physical stores to consumers, and create the uniqueness of physical sales.

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References

1. Sahu, K.C.; Khan, M.N.; Das Gupta, K. Determinants of Webrooming and Showrooming Behavior: A Systematic Literature Review. *J. Internet Commer.* **2021**, *20*, 137–166. [CrossRef]
2. Brynjolfsson, E.; Hu, Y.J.; Rahman, M.S. Competing in the age of omnichannel retailing. *MIT Sloan Manag. Rev.* **2013**, *54*, 23–29.
3. Bell, D.R.; Gallino, S.; Moreno, A. How to win in an omnichannel world. *MIT Sloan Manag. Rev.* **2014**, *56*, 45–53.
4. Neslin, S.A.; Jerath, K.; Bodapati, A.; Bradlow, E.T.; Deighton, J.; Gensler, S.; Lee, L.; Montaguti, E.; Telang, R.; Venkatesan, R.; et al. The interrelationships between brand and channel choice. *Mark. Lett.* **2014**, *25*, 319–330. [CrossRef]
5. Flavián, C.; Gurrea, R.; Orús, C. Choice confidence in the webrooming purchase process: The impact of online positive reviews and the motivation to touch. *J. Consum. Behav.* **2016**, *15*, 459–476. [CrossRef]
6. Flavián, C.; Gurrea, R.; Orús, C. Combining channels to make smart purchases: The role of webrooming and showrooming. *J. Retail. Consum. Serv.* **2020**, *52*, 101923. [CrossRef]
7. Van Baal, S.; Dach, C. Free riding and customer retention across retailers' channels. *J. Interact. Mark.* **2005**, *19*, 75–85. [CrossRef]
8. Jørgensen, S.; Sigué, S.P.; Zaccour, G. Dynamic cooperative advertising in a channel. *J. Retail.* **2000**, *76*, 71–92. [CrossRef]
9. Karray, S.; Martín-Herrán, G.; Sigué, S.P. Cooperative advertising for competing manufacturers: The impact of long-term promotional effects. *Int. J. Prod. Econ.* **2017**, *184*, 21–32. [CrossRef]
10. Aust, G.; Buscher, U. Vertical cooperative advertising and pricing decisions in a manufacturer–retailer supply chain: A game-theoretic approach. *Eur. J. Oper. Res.* **2012**, *223*, 473–482. [CrossRef]
11. Li, X.; Li, Y.; Cao, W. Cooperative advertising models in O2O supply chains. *Int. J. Prod. Econ.* **2019**, *215*, 144–152. [CrossRef]
12. Li, D.; Yu, H.; Tee, K.P.; Wu, Y.; Ge, S.S.; Lee, T.H. On time-synchronized stability and control. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *52*, 2450–2463. [CrossRef]
13. Thiruchelvam, S.; Ismail, M.F.; Ghazali, A.; Mustapha, K.N.; Norkhair, F.F.; Yahya, N.; Isa, A.A.M.; Muda, Z.C. Development of humanitraian supply chain performance conceptual framework in creating resilient logistics network. *Malays. J. Geosci.* **2018**, *2*, 30–33. [CrossRef]
14. Yan, L.; Yin-He, S.; Qian, Y.; Zhi-Yu, S.; Chun-Zi, W.; Zi-Yun, L. Method of reaching consensus on probability of food safety based on the integration of finite credible data on block chain. *IEEE Access* **2021**, *9*, 123764–123776. [CrossRef]

15. Aust, G.; Buscher, U. Cooperative advertising models in supply chain management: A review. *Eur. J. Oper. Res.* **2014**, *234*, 31–63. [CrossRef]
16. Jørgensen, S.; Zaccour, G. A survey of game-theoretic models of cooperative advertising. *Eur. J. Oper. Res.* **2014**, *237*, 1–14. [CrossRef]
17. Zhao, L.; Zhang, J.; Xie, J. Impact of demand price elasticity on advantages of cooperative advertising in a two-tier supply chain. *Int. J. Prod. Res.* **2015**, *54*, 2541–2551. [CrossRef]
18. Martín-Herrán, G.; Sigué, S.P. An integrative framework of cooperative advertising: Should manufacturers continuously support retailer advertising? *J. Bus. Res.* **2017**, *70*, 67–73. [CrossRef]
19. Chutani, A.; Sethi, S.P. Dynamic cooperative advertising under manufacturer and retailer level competition. *Eur. J. Oper. Res.* **2018**, *268*, 635–652. [CrossRef]
20. Chatterjee, P.; Karray, S.; Sigué, S.P. Cooperative advertising programs: Are accrual constraints necessary? *Int. Trans. Oper. Res.* **2017**, *26*, 2230–2247. [CrossRef]
21. Xu, Q.; Fu, G.; Fan, D. Service sharing, profit mode and coordination mechanism in the Online-to-Offline retail market. *Econ. Model.* **2020**, *91*, 659–669. [CrossRef]
22. Li, M.; Zhang, X.; Dan, B. Cooperative advertising and pricing in an O2O supply chain with buy-online-and-pick-up-in-store. *Int. Trans. Oper. Res.* **2020**, *28*, 2033–2054. [CrossRef]
23. Fang, H.; Fang, F.; Hu, Q.; Wan, Y. Supply Chain Management: A Review and Bibliometric Analysis. *Processes* **2022**, *10*, 1681. [CrossRef]
24. Salas-Navarro, K.; Serrano-Pájaro, P.; Ospina-Mateus, H.; Zamora-Musa, R. Inventory Models in a Sustainable Supply Chain: A Bibliometric Analysis. *Sustainability* **2022**, *14*, 6003. [CrossRef]
25. Acevedo-Chedid, J.; Salas-Navarro, K.; Ospina-Mateus, H.; Villalobo, A.; Sana, S.S. Production System in a Collaborative Supply Chain Considering Deterioration. *Int. J. Appl. Comput. Math.* **2021**, *7*, 69. [CrossRef]
26. Telser, L.G. Why should manufacturers want fair trade? *J. Law Econ.* **1960**, *3*, 86–105. [CrossRef]
27. Singley, R.B.; Williams, M.R. Free Riding in Retail Stores: An Investigation of Its Perceived Prevalence and Costs. *J. Mark. Theory Pract.* **1995**, *3*, 64–74. [CrossRef]
28. Wu, D.; Ray, G.; Geng, X.; Whinston, A. Implications of reduced search cost and free riding in e-commerce. *Mark. Sci.* **2004**, *23*, 255–262. [CrossRef]
29. Guan, X.; Liu, B.; Chen, Y.; Wang, H. Inducing Supply Chain Transparency through Supplier Encroachment. *Prod. Oper. Manag.* **2019**, *29*, 725–749. [CrossRef]
30. Shin, J. How Does Free Riding on Customer Service Affect Competition? *Mark. Sci.* **2007**, *26*, 488–503. [CrossRef]
31. Carlton, D.W.; Chevalier, J.A. Free Riding and Sales Strategies for the Internet. *J. Ind. Econ.* **2001**, *49*, 441–461. [CrossRef]
32. Mittelstaedt, R.A. Sasquatch, the Abominable Snowman, Free Riders and other Elusive Beings. *J. Macromarketing* **1986**, *6*, 25–35. [CrossRef]
33. Umit Kucuk, S.; Maddux, R.C. The role of the Internet on free-riding: An exploratory study of the wallpaper industry. *J. Retail. Consum. Serv.* **2010**, *17*, 313–320. [CrossRef]
34. Aubrey, C.; Judge, D. Re-imagine retail: Why store innovation is key to a brand's growth in the 'new normal', digitally-connected and transparent world. *J. Brand Strategy* **2012**, *1*, 31–39.
35. Zhang, L.; Wang, J. Coordination of the traditional and the online channels for a short-life-cycle product. *Eur. J. Oper. Res.* **2017**, *258*, 639–651. [CrossRef]
36. Basak, S.; Basu, P.; Avittathur, B.; Sikdar, S. A game theoretic analysis of multichannel retail in the context of "showrooming". *Decis. Support Syst.* **2017**, *103*, 34–45. [CrossRef]
37. Mehra, A.; Kumar, S.; Raju, J.S. Competitive strategies for brick-and-mortar stores to counter "showrooming". *Manag. Sci.* **2018**, *64*, 3076–3090. [CrossRef]
38. Wang, Y.; Gerchak, Y. Supply chain coordination when demand is shelf-space dependent. *Manuf. Serv. Oper. Manag.* **2001**, *3*, 82–87. [CrossRef]
39. Wang, Z.Q.; Yang, D.F.; Ran, L. Relationship between webrooming and consumer quality expectations. *J. Manag. Sci. China* **2021**, *24*, 71–88.
40. Yuan, W.W. Pricing decision of dual-channel competition in presence of webrooming effect. *Sci.-Technol. Manag.* **2021**, *23*, 70–80.
41. Ma, D.Q.; Wang, X.Q.; Hu J, S. Sales format selection for platform-based supply chain considering consumers' webrooming phenomenon in multi-channel retailing. *Chin. J. Manag. Sci.* **2022**, 1–12. [CrossRef]
42. Liu, C.; Dan, Y.; Dan, B.; Xu, G. Cooperative strategy for a dual-channel supply chain with the influence of free-riding customers. *Electron. Commer. Res. Appl.* **2020**, *43*, 101001. [CrossRef]
43. Luo, M.L.; Li, G.; Zhang, W.J. The bidirectional free-riding in a dual-channel supply chain. *J. Syst. Manag.* **2014**, *23*, 314–323+338.
44. Li, J.B.; Zhu, M.P.; Dai, B. Optimal pricing and sales effort decisions in a dual-channel supply chain in case of bidirectional free riding. *Syst. Eng.-Theory Pract.* **2016**, *36*, 3046–3058.
45. Gong, Y.H.; He, Y.; Zhang, M. Decision-making strategy for service effort of dual-channel retailers under condition of bidirectional free riding. *Stat. Decis.* **2020**, *36*, 162–166.
46. Herhausen, D.; Binder, J.; Schoegel, M.; Herrmann, A. Integrating Bricks with Clicks: Retailer-Level and Channel-Level Outcomes of Online–Offline Channel Integration. *J. Retail.* **2015**, *91*, 309–325. [CrossRef]

47. Heitz-Spahn, S. Cross-channel free-riding consumer behavior in a multichannel environment: An investigation of shopping motives, sociodemographics and product categories. *J. Retail. Consum. Serv.* **2013**, *20*, 570–578. [CrossRef]
48. Huré, E.; Picot-Coupey, K.; Ackermann, C.-L. Understanding omni-channel shopping value: A mixed-method study. *J. Retail. Consum. Serv.* **2017**, *39*, 314–330. [CrossRef]
49. Zheng, J.; Xu, Q. Service Sharing Decisions between Channels considering Bidirectional Free Riding Based on a Dual-Equilibrium Linkage Algorithm. *Comput. Intell. Neurosci.* **2022**, *2022*, 1540820. [CrossRef]
50. Nerlove, M.; Arrow, K.J. Optimal advertising policy under dynamic conditions. *Economica* **1962**, *29*, 129. [CrossRef]
51. Zhang, S.P.; Zhang, S.Y. Dynamic cooperative advertising strategies based on differential games in a supply chain. *Control. Decis.* **2006**, *21*, 153–157+162.
52. Gao, Z.; Kong, D.; Gao, C. Modeling and control of complex dynamic systems: Applied mathematical aspects. *J. Appl. Math.* **2012**, *2012*, 869792. [CrossRef]
53. Zheng, J.; Zhang, Q.; Xu, Q.; Xu, F.; Shi, V. Synchronization of a supply chain model with four chaotic attractors. *Discret. Dyn. Nat. Soc.* **2022**, *2022*, 6390456. [CrossRef]
54. Gao, Z.; Chen, M.Z.Q.; Zhang, D. Special issue on "Advances in Condition Monitoring, Optimization and Control for Complex Industrial Processes". *Processes* **2021**, *9*, 664. [CrossRef]
55. Berkowitz, D.; Allaway, A.; D'Souza, G. Estimating differential lag effects for multiple media across multiple stores. *J. Advert.* **2001**, *30*, 59–65. [CrossRef]
56. Baack, D.W.; Wilson, R.T.; Till, B.D. Creativity and memory effects: Recall, recognition, and an exploration of nontraditional media. *J. Advert.* **2008**, *37*, 85–94. [CrossRef]
57. Chen, D.Y.; Yu, H.; Hou, L. Dynamic cooperative advertising strategies in a supply chain with lagged effect. *J. Manag. Sci. China* **2017**, *20*, 25–35.
58. Yu, H.; Chen, D.Y.; Huang, C.L. Advertising strategies for supply chain with lagged and memory effects. *Control Decis.* **2018**, *33*, 1871–1878.
59. Chen, G.P.; Zhang, X.M.; Xiao, J. Coordination model for cooperative advertising between manufacturers and retailers in dual-channel supply chain. *J. Syst. Manag.* **2017**, *26*, 1168–1175.
60. Cao, D.Y.; Xiao, J.; Zhang, X.M. Cooperative advertising strategy in dual-channel supply chain with lagged effect. *Comput. Integr. Manuf. Syst.* **2020**, *26*, 2253–2265.
61. Wang, Y.B.; Shu, L.Y. Cooperative advertising models in O2O supply chains with fairness concerns. In Proceedings of the IEEE 2019 16th International Conference on Service Systems and Service Management (ICSSSM), Shenzhen, China, 13–15 July 2019; pp. 1–6. [CrossRef]