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Event-Triggered Filtering for Delayed Markov Jump Nonlinear Systems with Unknown Probabilities

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Abstract: This paper focuses on the problem of event-triggered H_∞ asynchronous filtering for Markov jump nonlinear systems with varying delay and unknown probabilities. An event-triggered scheduling scheme is adopted to decrease the transmission rate of measured outputs. The devised filter is mode dependent and asynchronous with the original system, which is represented by a hidden Markov model (HMM). Both the probability information involved in the original system and the filter are assumed to be only partly available. Under this framework, via employing the Lyapunov–Krasovskii functional and matrix inequality transformation techniques, a sufficient condition is given and the filter is further devised to ensure that the resulting filtering error dynamic system is stochastically stable with a desired H_∞ disturbance attenuation performance. Lastly, the validity of the presented filter design scheme is verified through a numerical example.

Keywords: event-triggered scheduling; Markov jump nonlinear systems(MJNSs); error threshold; partly unknown probabilities; asynchronous filtering

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1. Introduction

Markov jump systems (MJSs), as a kind of significant hybrid stochastic systems, have attracted immense attention in recent decades owing to their wide range of applications in aerospace, electric power systems, communication, economic, traffic and other areas [1–4]. Scholars have put a lot of effort into research on MJSs since they were first proposed by Krasovskii and Lidskii [5] in the 1960s, and many results for MJSs have been released in the literature (see [6–13] and the references therein). Additionally, it is a fact that the nonlinearity in MJSs, which makes the system more complex, is ubiquitous in many real-world applications. Therefore, the research on Markov jump nonlinear systems (MJNSs) has great theoretical significance and practical application value and has been widely examined [14–18]. Among this research, neural network (NN) [16–18] is one of the most popular approaches to deal with nonlinearity. For instance, the exponential stability problem was discussed for multiple-delayed Markov jump NNs (MJNNs) in [16].

Moreover, the filtering or estimation is a very essential issue in the field of cybernetics and has received strong interest from scholars [19], mainly for the reason that it is often a difficult job to obtain the accurate values of system states in engineering practice, and thus, a high-quality filter is essential for state estimation. The problem of filtering or estimation for MJSs has been investigated in [12–14,20–25]. To mention a few such studies, the H_∞ filtering and the dissipative asynchronous filtering for periodic MJSs were investigated in [12,13], respectively. The state estimation problem for a class of MJNSs was explored in [14], which put forward a moving horizon estimation algorithm, and the optimal estimate was obtained by minimizing a quadratic estimation cost function.

On the other hand, due to the increasing complexity of networks, communication constraint is also a serious problem for networked control systems (NCSs), which have been extensively used in real systems in recent years [26]. To the best of our knowledge, the event-triggered (ET) scheduling scheme is a useful and emerging approach to deal with this trouble and has become one of the current research hotspots [27–30]. Event-triggered scheduling means that the information transmission of nodes in the system determine whether to execute or not according to the preset event-triggered conditions. Based on this scheme, the measured outputs are transmitted only when the ET condition holds. Compared with the traditional periodic transmission scheme, it has the advantages of reducing redundant communication and saving energy, and so on. In recent years, some useful results for MJSs on this topic have been reported [20–22,31–33]. Specially, the filtering or estimation problem was addressed in [20–22,33]. For example, the event-based state estimation problem was explored for MJSs considering quantization and stochastic nonlinearity simultaneously in [20], in which both the ET and quantization schemes were introduced into the model of MJSs, then an estimator was devised to ensure that the filtering error system was randomly bounded and satisfied a desired H_∞ performance. The event-based dissipative filtering issue was studied for delayed MJSs [22], by means of the Lyapunov and Wirtinger inequality techniques, the stochastic stability with strict dissipativity of the error system and the co-design design scheme of the ET matrices and filter parameters were presented.

In NCSs, the plant, filter or controller are always geographically scattered and connected through communication network, which will inevitably cause some issues, e.g., network-induced delay and data dropout, and lead to incomplete data transmission among different nodes, thus causing asynchronous problems in MJSs [25]. However, in many of the existing works, this problem is ignored by assuming that the filter/controller is mode independent [34,35] or synchronous [36,37] with the original system. Mode independence implies that there is no use of available mode information, which will bring about more conservatism, and the assumption of synchronization means that the modes of the filter/controller are completely consistent with those of the plant, which is too rigorous. Due to realizing the irrationality of these assumptions, recently, scholars have paid increasing attention to the investigation of asynchronous techniques [23–25,38]. In [23,24], the asynchronous phenomenon was described as a piecewise homogeneous Markov process. In [13,25,38], a hidden Markov model (HMM) was proposed to address the asynchronous issue, which related the filter/controller to the plant with a conditional probability matrix (CPM). Based on this, the asynchronous filtering problem for MJNNs in [25] and the asynchronous control problem for MJSs in [38] were investigated, respectively.

Based on the foregoing discussions, we know that some important results have been released about ET scheduling schemes or asynchronous techniques in MJSs. These works have important theoretical and practical significance. Nevertheless, there are few results concerning the ET asynchronous filtering/control for MJSs or MJNSs, which is one of the motivations for our work. In addition, it should be noted that the above asynchronous results are restricted due to the assumption that both the probability information of the original system and the filter are considered to be fully accessible. However, it is difficult or costly to fulfill in many engineering applications. However, some results on the partly unknown transition probabilities (TPs) case [39,40] have been reported recently, in which it is assumed that the modes of the plant and the filter/controller are synchronous, and the unknown entries only exist in the transition probability matrix (TPM), so they are not suitable for asynchronous cases. For instance, in the HMM-based asynchronous case, there may be unknown entries in both TPM and CPM, which is more complex and challenging. This is another motivation for our work.

This paper will concentrate on the issue of ET asynchronous filter design for discrete-time MJNSs based on a NN model with varying delay and unknown probabilities. An ET scheduling scheme is introduced to decrease the transmission rate of measured outputs. The modes of the devised filter are dependent on and asynchronous with those of the original system, represented by an HMM. It is assumed that both the probability information involved in the original system and the filter are only partly available. By utilizing the

Lyapunov–Krasovskii functional (LKF) and matrix inequality transformation techniques, an asynchronous filter is devised to ensure the stochastic stability and a desired H_∞ performance of the error system. The slack matrix technique and Projection lemma are introduced to facilitate the filter design. Lastly, a numerical example is offered to demonstrate the validity of the obtained results. The major contributions of this work are stated as follows:

(1) A more practical scenario is considered, which includes not only the varying delay, partly unknown probabilities and nonlinearity of the original system, but also the network-induced communication constraint and asynchronous problem.

(2) The ET asynchronous filtering problem based on HMM is first explored for discrete-time delayed MJNNs, in which both the TPM of the original system and the CPM of the filter are assumed to be only partly accessible.

(3) The filtering scheme proposed in this paper has strong versatility since the asynchronous strategy based on HMM contains two special cases: mode independence and synchronization, and the case with partly unknown probabilities considered in this paper covers both fully known and fully unknown cases.

2. Preliminaries

In this work, the physical plant, which is a discrete-time MJNN with varying delay, is addressed as below:

$$\mathbb{S}_0 : \begin{cases} x(k+1) = A(\alpha_k)x(k) + A_d(\alpha_k)x(k-d(k)) \\ \qquad\qquad +E(\alpha_k)g(x(k)) + E_d(\alpha_k)g(x(k-d(k))) \\ \qquad\qquad +B(\alpha_k)w(k) \\ y(k) = C_1(\alpha_k)x(k) + C_d(\alpha_k)x(k-d(k)) \\ \qquad\quad +D_1(\alpha_k)w(k) \\ z(k) = C_2(\alpha_k)x(k) + D_2(\alpha_k)w(k) \\ x(k_0) = \chi(k_0), k_0 = -\tau_2, -\tau_2+1, \cdots, -1, 0 \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the system state with the initial value $\chi(k_0)$, $y(k) \in R^p$ is the output signal, $z(k) \in R^q$ is the target value to be estimated, and $w(k) \in R^r$ is referring to the disturbance with $w(k) \in l_2[0, \infty)$. $g(x(k)) \in R^n$ denotes a nonlinear function. $d(k) \in N^+$ means the system delay with lower bound τ_1 and upper bound τ_2 . $A(\alpha_k)$, $A_d(\alpha_k)$, $E(\alpha_k)$, $E_d(\alpha_k)$, $B(\alpha_k)$, $C_1(\alpha_k)$, $C_d(\alpha_k)$, $D_1(\alpha_k)$, $C_2(\alpha_k)$ and $D_2(\alpha_k)$ are known constant matrices with proper dimensions. α_k refers to a Markov chain which regulates the jumps of system($\mathbb{S}_0$) in a set of modes $\mathscr{S}_1 = \{1, 2, \cdots, s_1\}$ with a TPM $\Phi = \{\phi_{\rm ij}\}$, and its TP $\phi_{\rm ij}$ is defined as

$$\Pr\{\alpha_{k+1} = j | \alpha_k = i\} = \phi_{\rm ij} \tag{2}$$

in which $\phi_{\rm ij} \geq 0$ and $\sum_{j=1}^{s_1} \phi_{\rm ij} = 1$ for $\forall i, j \in \mathscr{S}_1$.

Next, a filter will be devised for estimating $z(k)$ according to measured outputs. Nevertheless, due to the introduction of an ET scheduler, the output signal will be transmitted only when the ET condition holds(see Figure 1). While the deviation between the current measured output and the last transmission signal is bigger than its relative error, the output signal will be transmitted (i.e., $\rho(k) = 1$), otherwise it will not be transmitted (i.e., $\rho(k) = 0$).

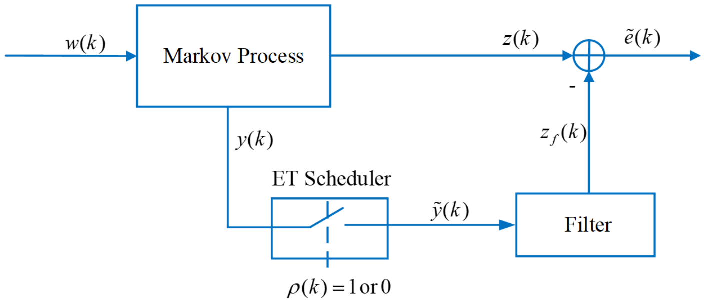


Figure 1. Block diagram of ET asynchronous filtering

Therefore, at the sampling instant k, if the ET condition holds, the filter will receive the latest measured output, otherwise it will keep the last transmission value by zero order holder (ZOH). Based on this scheme, the input of the filter during the period k is addressed as:

$$\tilde{y}_i(k) = \begin{cases} y_i(k) & |\tilde{y}_i(k-1) - y_i(k)| > \delta_i |y_i(k)| \\ \tilde{y}_i(k-1) & |\tilde{y}_i(k-1) - y_i(k)| \le \delta_i |y_i(k)| \end{cases} \tag{3}$$

where $i = 1, 2, \cdots, p$; $\delta_i \in [0, 1]$ is the error threshold.

Setting $H(k) = diag\{\nabla_1(k), \nabla_2(k), \cdots, \nabla_p(k)\}$, $\nabla_i(k) \in [-\delta_i, \delta_i]$, $i = 1, 2, \cdots, p$, then in accordance with (3) , we can obtain

$$\tilde{y}(k) = (I + H(k))y(k) \tag{4}$$

Remark 1. *Thanks to the introduction of the ET scheduler into MJNSs, the measured outputs need not be transmitted in each sampling period, thus achieving the aim of reducing the data transmission rate. In the following, we introduce a communication performance index of* $MTR = \bar{n}_{sent}/n_{total}$*, which denotes the mean transmission rate (* $\bar{n}_{sent}$ *and* n_{total} *denote the average number of measured output* $y(k)$ *transmitted with and without the ET scheduler in the simulation time, respectively.). The smaller* MTR *means better communication performance.*

Based on the ET outputs (4), we will adopt a mode-dependent filter to estimate $z(k)$:

$$\mathbb{S}_f : \begin{cases} x_f(k+1) = A_f(\beta_k)x(k) + B_f(\beta_k)\tilde{y}(k) \\ z_f(k) = C_f(\beta_k)x(k) + D_f(\beta_k)\tilde{y}(k) \end{cases} \tag{5}$$

where $x_f(k) \in R^n$ refers to the filter state, $z_f(k) \in R^q$ denotes the estimated value of $z(k)$. $A_f(\beta_k)$, $B_f(\beta_k)$, $C_f(\beta_k)$ and $D_f(\beta_k)$ are parameters of the filter to be obtained, which are dependent on the filter mode β_k, $\beta_k \in \mathscr{S}_2 = \{1, 2, \cdots, s_2\}$.

In this paper, filter($\mathbb{S}_f$) is mode dependent, and its mode β_k is influenced by the mode α_k of system($\mathbb{S}_0$) via a CPM $\Omega = \{\sigma_{im}\}$, where the conditional probability(CP) σ_{im} is given by

$$\Pr\{\beta_k = m | \alpha_k = i\} = \sigma_{im} \tag{6}$$

which denotes the probability that filter($\mathbb{S}_f$) is in the m-th mode while the plant works in the i-th mode. Obviously, $\sigma_{im} \ge 0$ and $\sum\limits_{m=1}^{s_2} \sigma_{im} = 1$ for $\forall i \in \mathscr{S}_1, m \in \mathscr{S}_2$.

Remark 2. *Notice that the devised filter acts asynchronously with the original system as their jumping processes are controlled by different Markov parameters,* β_k *and* α_k*, respectively. However, the parameter* β_k *is affected by* α_k *through the CP (6). Thus, the set* $(\alpha_k, \beta_k, \Phi, \Omega)$ *is addressed as an HMM, linking filter(*$\mathbb{S}_f$*) and system(*$\mathbb{S}_0$*) tightly with a CPM which can reflect the asynchronous degree between them. We should mention that the devised asynchronous filter under this scheme is more general because it includes the synchronous and mode-independent cases [38].*

Considering the complexity of practical systems, in this paper, we assume that the entries of TPM Φ and CPM Ω are partly inaccessible; namely, Φ and Ω may take the forms as follows:

$$\Phi=\begin{bmatrix} \phi_{11} & ? & ? \\ ? & ? & \phi_{23} \\ ? & \phi_{32} & ? \end{bmatrix}, \Omega=\begin{bmatrix} \sigma_{11} & ? & ? \\ ? & ? & \sigma_{23} \\ ? & \sigma_{32} & ? \end{bmatrix} \tag{7}$$

in which "?" refers to the unknown elements. For $\forall i \in \mathscr{S}_1$, define $\mathscr{S}_1=\mathscr{S}^i_{1K}+\mathscr{S}^i_{1U}$ and $\mathscr{S}_2=\mathscr{S}^i_{2K}+\mathscr{S}^i_{2U}$, where

$$\begin{cases} \mathscr{S}^i_{1K}=\{j:\phi_{ij} \text{ is known}\} \\ \mathscr{S}^i_{1U}=\{j:\phi_{ij} \text{ is unknown}\} \end{cases} \quad \begin{cases} \mathscr{S}^i_{2K}=\{m:\sigma_{im} \text{ is known}\} \\ \mathscr{S}^i_{2U}=\{m:\sigma_{im} \text{ is unknown}\} \end{cases} \tag{8}$$

Remark 3. *In recent years, there have been some research results on the HMM-based asynchronous filtering/control of MJSs, e.g., [25,38], in which all TPs in TPM and CPs in CPM are assumed to be known. Nevertheless, it is very arduous or costly to obtain all the information about TPM or CPM. Hence, a more complex and challenging case where both TPM and CPM are only partly accessible will be explored in this paper. It is worth pointing out that our result under this framework is more general because it contains two special cases: (1) the fully known case, i.e.,* $\mathscr{S}^i_{1U}=\emptyset$ *or* $\mathscr{S}^i_{2U}=\emptyset$*, which is the most studied case at present; (2) the fully unknown case, i.e.,* $\mathscr{S}^i_{1K}=\emptyset$ *or* $\mathscr{S}^i_{2K}=\emptyset$ *.*

For brevity of notation, in the following, parameters α_k, α_{k+1} and β_k are simplified to i, j and m of the subscript, for example, $A(\alpha_k)\triangleq A_i$, $A_f(\beta_k)\triangleq A_{fm}$.

Selecting the augmented vector $\tilde{x}(k)=\begin{bmatrix} x^T(k) & x_f^T(k) \end{bmatrix}^T$ and the estimated error $\tilde{e}(k)=z(k)-z_f(k)$, and synthesizing (1), (4) and (5), we derive the filtering error dynamic system as follows:

$$\mathbb{S}_e:\begin{cases} \tilde{x}(k+1)=\tilde{A}_{im}\tilde{x}(k)+\tilde{A}_{dim}x(k-d(k)) \\ \qquad +\tilde{B}_{im}w(k)+\bar{I}[E_ig(x(k)) \\ \qquad +E_{di}g(x(k-d(k)))] \\ \tilde{e}(k)=\tilde{C}_{im}\tilde{x}(k)+\tilde{C}_{dim}x(k-d(k)) \\ \qquad +\tilde{D}_{im}w(k) \end{cases} \tag{9}$$

where $\tilde{A}_{im}=\begin{bmatrix} A_i & 0 \\ B_{fm}(I+H(k))C_{1i} & A_{fm} \end{bmatrix}$, $\tilde{A}_{dim}=\begin{bmatrix} A_{di} \\ B_{fm}(I+H(k))C_{di} \end{bmatrix}$, $\tilde{B}_{im}=\begin{bmatrix} B_i \\ B_{fm}(I+H(k))D_{1i} \end{bmatrix}$, $\bar{I}=\begin{bmatrix} I \\ 0 \end{bmatrix}$, $\tilde{C}_{im}=\begin{bmatrix} C_{2i}-D_{fm}(I+H(k))C_{1i} & -C_{fm} \end{bmatrix}$, $\tilde{C}_{dm}=\begin{bmatrix} -D_{fm}(I+H(k))C_{di} \end{bmatrix}$, $\tilde{D}_{im}=\begin{bmatrix} D_{2i}-D_{fm}(I+H(k))D_{1i} \end{bmatrix}$.

Next, we will provide some important definitions, assumptions and lemmas that promote the work of this paper.

Definition 1 ([41])**.** *The filtering error system($\mathbb{S}_e$) with $w(k)=0$ is said to be stochastically stable if the following condition is satisfied for the arbitrary initial condition $(\tilde{x}(0),\alpha_0)$*

$$\mathbf{E}\left\{\sum_{k=0}^{\infty}\|\tilde{x}(k)\|^2|\tilde{x}(0),\alpha_0\right\}<\infty \tag{10}$$

Definition 2 ([41])**.** *The filtering error system($\mathbb{S}_e$) with $w(k)\in l_2[0,\infty)$ is said to have an H_∞ disturbance attenuation performance γ, if under the zero initial condition, the error $\tilde{e}(k)$ fulfills the condition as follows:*

$$\sum_{k=0}^{\infty} \mathbf{E}\left\{ \|\tilde{e}(k)\|^2 \right\} < \gamma^2 \sum_{k=0}^{\infty} \|w(k)\|^2 \tag{11}$$

where γ is a positive scalar.

Assumption 1 ([42]). *The continuous nonlinear function $g_i(\bullet)$ in system($\mathbb{S}_0$) is supposed to be bounded, and satisfies the following condition*

$$l_i \le \frac{g_i(x)}{x} \le h_i \; x \ne 0, x \in R$$

where l_i and h_i are constants, $i = 1, 2, \cdots, n$.

Lemma 1 ([42]). *Based on Assumption 1, there is a symmetric matrix $N > 0$, satisfying*

$$\begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^T \begin{bmatrix} Y_1 N & -Y_2 N \\ * & N \end{bmatrix} \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} < 0$$

where $Y_1 {=} diag\{l_1 h_1, l_2 h_2, \cdots, l_n h_n\}$, $Y_2 {=} diag\{(l_1{+}h_1)/2, (l_2{+}h_2)/2, \cdots, (l_n{+}h_n)/2\}$.

Lemma 2 (Projection lemma [43]). *For given matrices X, U and V, there exists a matrix Y such that*

$$X + U^T Y V + V^T Y^T U < 0$$

is satisfied, if and only if the inequalities listed below are true

$$U_\perp^T X U_\perp < 0, \; V_\perp^T X V_\perp < 0$$

where $U_\perp$ and U, $V_\perp$ and V are orthogonal complements, respectively.

Based on the above, the objective of this paper is to develop a feasible ET asynchronous filter($\mathbb{S}_f$) for discrete-time delayed MJNSs ($\mathbb{S}_0$) with unknown probabilities, such that the error system ($\mathbb{S}_e$) is stochastically stable and has a desired H_∞ performance γ.

3. Main Results

We will first provide a sufficient condition about the stochastic stability with an H_∞ performance γ of the error system ($\mathbb{S}_e$) in this section, then present a design scheme of a solvable filter.

For brevity, we first introduce the following notations:

$$\bar{A}_{im} = \begin{bmatrix} A_i & 0 \\ B_{fm} C_{1i} & A_{fm} \end{bmatrix}, \bar{A}_{dim} = \begin{bmatrix} A_{di} \\ B_{fm} C_{di} \end{bmatrix}, \bar{B}_{im} = \begin{bmatrix} B_i \\ B_{fm} D_{1i} \end{bmatrix}, \bar{C}_{im} = \begin{bmatrix} C_{2i} - D_{fm} C_{1i} & -C_{fm} \end{bmatrix},$$
$$\bar{C}_{dim} = \begin{bmatrix} -D_{fm} C_{di} \end{bmatrix}, \bar{D}_{im} = \begin{bmatrix} D_{2i} - D_{fm} D_{1i} \end{bmatrix}, \eta_1(k) = \begin{bmatrix} \tilde{x}^T(k) & g^T(x(k)) & x^T(k-d(k)) & g^T(x(k-d(k))) \end{bmatrix}^T,$$
$$\eta(k) = \begin{bmatrix} \eta_1^T(k) & w^T(k) \end{bmatrix}^T, \Lambda {=} diag\{\delta_1, \delta_2, \cdots, \delta_p\}, Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}, \tau {=} \tau_2 - \tau_1 + 1.$$

By use of LKF and H_∞ theory, we can obtain the following conclusions.

Theorem 1. *For a prescribed $\gamma > 0$, the filtering error dynamic system($\mathbb{S}_e$) based on Assumption 1 is stochastically stable with the H_∞ performance γ, if there are matrices A_{fm}, B_{fm}, C_{fm}, D_{fm}, $P_i > 0$, $F_{im} > 0$, and $Q > 0$, and diagonal matrices $N_1 > 0$, $N_2 > 0$, and $W_{im} > 0$, such that the following two conditions are fulfilled for $\forall i \in \mathscr{S}_1, m \in \mathscr{S}_{2U}^i$*

$$\mathcal{F}_i^K {+} (1 - \sigma_i^K) F_{im} < P_i \tag{12}$$

and for $\forall i \in \mathscr{S}_1, j \in \mathscr{S}_{1U}^i, m \in \mathscr{S}_2$

$$\Pi_{im} = \begin{bmatrix} \Pi_{im}^1 & U_m & Z_i^T \Lambda W_{im} \\ * & -W_{im} & 0 \\ * & * & -W_{im} \end{bmatrix} < 0 \tag{13}$$

where $\mathcal{F}_i^K = \sum_{m \in \mathscr{S}_{2K}^i} \sigma_{im} F_{im}$, $\sigma_i^K = \sum_{m \in \mathscr{S}_{2K}^i} \sigma_{im}$, $\bar{P}_i = \mathcal{P}_i^K + (1-\phi_i^K) P_{ij}$, $\mathcal{P}_i^K = \sum_{j \in \mathscr{S}_{1K}^i} \phi_{ij} P_j$, $\phi_i^K = \sum_{j \in \mathscr{S}_{1K}^i} \phi_{ij}$, $\Pi_{im}^1 = \begin{bmatrix} \Pi_i^{11} & \Pi_{im}^{12} & \Pi_{im}^{13} & \Pi_{im}^{14} \\ * & \tau\bar{Q} - \Pi^{22} - \bar{F}_{im} & 0 & 0 \\ * & * & -Q - \Pi^{33} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix}$, $\Pi_i^{11} = \begin{bmatrix} -\bar{P}_i^{-1} & 0 \\ 0 & -I \end{bmatrix}$, $\Pi_{im}^{12} = \begin{bmatrix} \bar{A}_{im} & \bar{I}E_i \\ \bar{C}_{im} & 0 \end{bmatrix}$, $\Pi_{im}^{13} = \begin{bmatrix} \bar{A}_{dim} & \bar{I}E_{di} \\ \bar{C}_{dim} & 0 \end{bmatrix}$, $\Pi_{im}^{14} = \begin{bmatrix} \bar{B}_{im} \\ \bar{D}_{im} \end{bmatrix}$, $\bar{Q} = \begin{bmatrix} \bar{I}Q_{11}\bar{I}^T & \bar{I}Q_{12} \\ * & Q_{22} \end{bmatrix}$, $\bar{F}_{im} = \begin{bmatrix} F_{im} & 0 \\ 0 & 0 \end{bmatrix}$, $\Pi^{22} = \begin{bmatrix} \bar{I}Y_1 N_1 \bar{I}^T & -\bar{I}Y_2 N_1 \\ * & N_1 \end{bmatrix}$, $\Pi^{33} = \begin{bmatrix} Y_1 N_2 & -Y_2 N_2 \\ * & N_2 \end{bmatrix}$, $U_m = \begin{bmatrix} \begin{bmatrix} 0 & B_{fm}^T \end{bmatrix} & -D_{fm}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $Z_i = \begin{bmatrix} 0 & 0 & \begin{bmatrix} C_{1i} & 0 \end{bmatrix} & 0 & C_{di} & 0 & D_{1i} \end{bmatrix}$.

Proof. First, we will derive some useful results according to (12) and (13). Equation (12) ensures that

$$\sum_{m=1}^{s_2} \sigma_{im} F_{im} - P_i < 0 \tag{14}$$

holds, because when $\sigma_i^K < 1$,

$$\begin{aligned} \sum_{m=1}^{s_2} \sigma_{im} F_{im} - P_i &= \mathcal{F}_i^K + (1-\sigma_i^K) \sum_{m \in \mathscr{S}_{2U}^i} \frac{\sigma_{im}}{1-\sigma_i^K} F_{im} - P_i \\ &= \sum_{m \in \mathscr{S}_{2U}^i} \frac{\sigma_{im}}{1-\sigma_i^K} \{\mathcal{F}_i^K + (1-\sigma_i^K) F_{im} - P_i\} \end{aligned} \tag{15}$$

and when $\sigma_i^K = 1$, obviously, (12) is equivalent to (14).

In terms of the Schur complement, (13) is equivalent to

$$\Pi_{im}^1 + U_m W_{im}^{-1} U_m^T + Z_i^T \Lambda W_{im} \Lambda Z_i < 0 \tag{16}$$

which obtains

$$\Pi_{im}^2 \triangleq \Pi_{im}^1 + U_m H(k) Z_i + Z_i^T H^T(k) U_m^T < 0 \tag{17}$$

and it is easy to derive that

$$\Pi_{im}^2 = \begin{bmatrix} \Pi_i^{11} & \tilde{\Pi}_{im}^{12} & \tilde{\Pi}_{im}^{13} & \tilde{\Pi}_{im}^{14} \\ * & \tau\bar{Q} - \Pi^{22} - \bar{F}_{im} & 0 & 0 \\ * & * & -Q - \Pi^{33} & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{18}$$

where $\tilde{\Pi}_{im}^{12} = \begin{bmatrix} \tilde{A}_{im} & \bar{I}E_i \\ \tilde{C}_{im} & 0 \end{bmatrix}$, $\tilde{\Pi}_{im}^{13} = \begin{bmatrix} \tilde{A}_{dim} & \bar{I}E_{di} \\ \tilde{C}_{dim} & 0 \end{bmatrix}$, $\tilde{\Pi}_{im}^{14} = \begin{bmatrix} \tilde{B}_{im} \\ \tilde{D}_{im} \end{bmatrix}$.

Then, based on the Schur complement and the analysis similar to (14) and (15), we can derive from (18) that

$$\begin{cases} \Pi_{im}^3 \triangleq \upsilon + \mu_{im}^T \tilde{P}_i \mu_{im} < \tilde{F}_{im} \\ \Pi_{im}^4 \triangleq \xi - \varsigma_{im}^T \tilde{\Pi}_i^{11} \varsigma_{im} < \hat{F}_{im} \end{cases} \tag{19}$$

where $\mu_{im} = \begin{bmatrix} \tilde{A}_{im} & \bar{I}E_i & \tilde{A}_{dim} & \bar{I}E_{di} \end{bmatrix}$, $\upsilon = \begin{bmatrix} \tau\bar{Q} - \Pi^{22} & 0 \\ * & -Q - \Pi^{33} \end{bmatrix}$, $\tilde{\Pi}_i^{11} = \begin{bmatrix} -\tilde{P}_i & 0 \\ 0 & -I \end{bmatrix}$, $\xi = diag\{\upsilon, -\gamma^2 I\}$, $\varsigma_{im} = \begin{bmatrix} \bar{\Pi}_{im}^{12} & \bar{\Pi}_{im}^{13} & \bar{\Pi}_{im}^{14} \end{bmatrix}$, $\tilde{P}_i = \sum_{j=1}^{s_1} \phi_{ij} P_j$, $\tilde{F}_{im} = \mathrm{diag}\{\bar{F}_{im}, 0\}$, $\hat{F}_{im} = \mathrm{diag}\{\bar{F}_{im}, 0, 0\}$.

Next, a mode-dependent LKF is introduced as follows:

$$V(k) = \sum_{l=1}^{2} V_l(k) \tag{20}$$

where $V_1(k) = \tilde{x}^T(k) P_{\alpha_k} \tilde{x}(k)$, $V_2(k) = \sum_{b=-\tau_2+1}^{-\tau_1+1} \sum_{a=k-1+b}^{k-1} \begin{bmatrix} x(a) \\ g(x(a)) \end{bmatrix}^T Q \begin{bmatrix} x(a) \\ g(x(a)) \end{bmatrix}$.

Then, we calculate $\nabla V(k)$ along the locus of the error system ($\mathbb{S}_e$) and take the expectation. It is easy to find that $\mathbf{E}\{\nabla V(k)\} = \mathbf{E}\{\nabla V_1(k)\} + \mathbf{E}\{\nabla V_2(k)\}$.

$$\begin{aligned}
&\mathbf{E}\{\nabla V_1(k)\} = \mathbf{E}\{V_1(k+1) - V_1(k) | \tilde{x}(k), \alpha_k = i\} \\
&= \mathbf{E}\{\tilde{x}^T(k+1) P_j \tilde{x}(k+1) - \tilde{x}^T(k) P_i \tilde{x}(k)\} \\
&= \mathbf{E}\left\{ \sum_{m=1}^{s_2} \sum_{j=1}^{s_1} \sigma_{im} \phi_{ij} \tilde{x}^T(k+1) P_j \tilde{x}(k+1) - \tilde{x}^T(k) P_i \tilde{x}(k) \right\} \\
&= \mathbf{E}\left\{ \sum_{m=1}^{s_2} \sigma_{im} \tilde{x}^T(k+1) \tilde{P}_i \tilde{x}(k+1) - \tilde{x}^T(k) P_i \tilde{x}(k) \right\} \\
&= \mathbf{E}\left\{ \sum_{m=1}^{s_2} \sigma_{im} \eta^T(k) \begin{bmatrix} \mu_{im}^T \\ \tilde{B}_{im}^T \end{bmatrix} \tilde{P}_i \begin{bmatrix} \mu_{im} & \tilde{B}_{im} \end{bmatrix} \eta(k) - \tilde{x}^T(k) P_i \tilde{x}(k) \right\}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathbf{E}\{\nabla V_2(k)\} = \mathbf{E}\{V_2(k+1) - V_2(k)\} \\
&= \mathbf{E}\left\{ \tau \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix}^T Q \begin{bmatrix} x(k) \\ g(x(k)) \end{bmatrix} - \right. \\
&\quad \left. \sum_{a=k-\tau_2}^{k-\tau_1} \begin{bmatrix} x(a) \\ g(x(a)) \end{bmatrix}^T Q \begin{bmatrix} x(a) \\ g(x(a)) \end{bmatrix} \right\} \\
&\leq \mathbf{E}\left\{ \begin{bmatrix} \tilde{x}(k) \\ g(x(k)) \end{bmatrix}^T \tau\bar{Q} \begin{bmatrix} \tilde{x}(k) \\ g(x(k)) \end{bmatrix} - \right. \\
&\quad \left. \begin{bmatrix} x(k-d(k)) \\ g(x(k-d(k))) \end{bmatrix}^T Q \begin{bmatrix} x(k-d(k)) \\ g(x(k-d(k))) \end{bmatrix} \right\}
\end{aligned} \tag{22}$$

According to Lemma 1, there are diagonal matrices $N_1 > 0$, $N_2 > 0$ such that (23) and (24) are satisfied

$$\begin{bmatrix} \tilde{x}(k) \\ g(x(k)) \end{bmatrix}^T \Pi^{22} \begin{bmatrix} \tilde{x}(k) \\ g(x(k)) \end{bmatrix} \leq 0 \tag{23}$$

$$\begin{bmatrix} x(k-d(k)) \\ g(x(k-d(k))) \end{bmatrix}^T \Pi^{33} \begin{bmatrix} x(k-d(k)) \\ g(x(k-d(k))) \end{bmatrix} \leq 0 \tag{24}$$

Synthesizing (22)–(24), we get that

$$\mathbf{E}\{\nabla V_2(k)\} \leq \mathbf{E}\left\{ \eta_1^T(k) \upsilon \eta_1(k) \right\} \tag{25}$$

Next, we will verify that $(\mathbb{S}_e)$ with $w(k) = 0$ is stochastically stable, and that

$$
\begin{aligned}
&\mathbf{E}\{\nabla V(k)\} = \mathbf{E}\{\nabla V_1(k)\} + \mathbf{E}\{\nabla V_2(k)\} \\
&\leq \mathbf{E}\left\{\sum_{m=1}^{s_2} \sigma_{im}\eta_1^T(k)\Pi_{im}^3\eta_1(k) - \tilde{x}^T(k)P_i\tilde{x}(k)\right\} \\
&< \mathbf{E}\left\{\eta_1^T(k)(\sum_{m=1}^{s_2} \sigma_{im}\tilde{F}_{im})\eta_1(k) - \tilde{x}^T(k)P_i\tilde{x}(k)\right\} \\
&= \mathbf{E}\left\{\tilde{x}^T(k)\left(\sum_{m=1}^{s_2} \sigma_{im}F_{im} - P_i\right)\tilde{x}(k)\right\} \\
&\leq \varepsilon\mathbf{E}\{\tilde{x}^T(k)\tilde{x}(k)\}
\end{aligned} \tag{26}
$$

where "$<$" is based on (19), $\varepsilon = \lambda_{\max\limits_{i\in\mathscr{S}_1}}(\sum_{m=1}^{s_2} \sigma_{im}F_{im} - P_i)$.

Notice that $\varepsilon < 0$ due to (12) and (14); then,

$$
\mathbf{E}\{\sum_0^\infty \nabla V(k)\} = \mathbf{E}\{V(\infty) - V(0)\} \leq \varepsilon\mathbf{E}\left\{\sum_0^\infty \tilde{x}^T(k)\tilde{x}(k)\right\} \tag{27}
$$

therefore,

$$
\mathbf{E}\left\{\sum_0^\infty \tilde{x}^T(k)\tilde{x}(k)\right\} < \infty \tag{28}
$$

which conforms to Definition 1, so we have verified the stochastic stability for $(\mathbb{S}_e)$ with $w(k) = 0$.

Next, we will verify that $(\mathbb{S}_e)$ with $w(k) \in l_2[0,\infty)$ has an H_∞ performance γ. Define the performance index as

$$
\begin{aligned}
J &= \sum_{k=0}^{\infty} \mathbf{E}\{\tilde{e}^T(k)\tilde{e}(k) - \gamma^2 w^T(k)w(k)\} \\
&= \sum_{k=0}^{\infty} \mathbf{E}\{\tilde{e}^T(k)\tilde{e}(k) - \gamma^2 w^T(k)w(k) + \nabla V(k)\} \\
&\quad + \mathbf{E}\{V(0)\} - \mathbf{E}\{V(\infty)\}
\end{aligned} \tag{29}
$$

Owing to the zero initial value, we obtain that $V(0) = 0$, whereas $V(\infty) \geq 0$, thus

$$
\begin{aligned}
J &\leq \sum_{k=0}^{\infty} \mathbf{E}\{\tilde{e}^T(k)\tilde{e}(k) - \gamma^2 w^T(k)w(k) + \nabla V(k)\} \\
&= \sum_{k=0}^{\infty} \mathbf{E}\left\{\sum_{m=1}^{s_2} \sigma_{im}\eta^T(k)\Pi_{im}^4\eta(k) - \tilde{x}^T(k)P_i\tilde{x}(k)\right\} \\
&< \sum_{k=0}^{\infty} \mathbf{E}\left\{\sum_{m=1}^{s_2} \sigma_{im}\eta^T(k)\hat{F}_{im}\eta(k) - \tilde{x}^T(k)P_i\tilde{x}(k)\right\} \\
&= \sum_{k=0}^{\infty} \mathbf{E}\left\{\tilde{x}^T(k)\left(\sum_{m=1}^{s_2} \sigma_{im}F_{im} - P_i\right)\tilde{x}(k)\right\} \\
&< 0
\end{aligned} \tag{30}
$$

in which the two "$<$" are obtained on the basis of (19) and (14), respectively. Then, from (11) and (30), we can readily conclude that the error system $(\mathbb{S}_e)$ has an H_∞ performance γ. Thus, the proof is accomplished. □

Remark 4. *The purpose of introducing the extra matrix F_{im} in Theorem 1 is to simplify matrix inequalities. However, in order to solve the parameters of the filter, the nonlinearity in (13) needs to be further processed so as to transform the matrix inequalities into linear matrix inequalities (LMIs).*

Next, we will devise the filter with the techniques of slack matrix and Projection lemma and obtain Theorem 2.

Theorem 2. *The filtering error dynamic system* ($\mathbb{S}_e$) *based on Assumption 1 is stochastically stable with an* H_∞ *performance* γ , *if there are matrices* $\tilde{A}_{fm}$, $\tilde{B}_{fm}$, $\tilde{C}_{fm}$, $\tilde{D}_{fm}$, *and* G_m, *a scalar* $\tilde{\gamma} > 0$, *diagonal matrices* $N_1 > 0$, $N_2 > 0$, *and* $W_{im} > 0$, *and the following matrices* $P_i = \begin{bmatrix} P_i^1 & P_i^2 \\ * & P_i^3 \end{bmatrix} > 0$, $F_{im} = \begin{bmatrix} F_{im}^1 & F_{im}^2 \\ * & F_{im}^3 \end{bmatrix} > 0$, $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0$, *such that the following two conditions are fulfilled for* $\forall i \in \mathscr{S}_1, m \in \mathscr{S}_{2U}^i$

$$\mathcal{F}_i^K + (1 - \sigma_i^K) F_{im} < P_i \tag{31}$$

and for $\forall i \in \mathscr{S}_1, j \in \mathscr{S}_{1U}^i, m \in \mathscr{S}_2$

$$\Xi_i^T \widehat{\Pi}_{im} \Xi_i < 0,\ \breve{\Pi}_{im} < 0 \tag{32}$$

where

$$\widehat{\Pi}_{im} = \begin{bmatrix} \bar{P}_i - \widehat{G}_m & \widehat{\Pi}_{im}^{12} \\ * & \breve{\Pi}_{im} \end{bmatrix}, \widehat{G}_m = \begin{bmatrix} 0 & G_m \\ * & G_m + G_m^T \end{bmatrix}, \breve{\Pi}_{im} = \begin{bmatrix} \breve{\Pi}_{im}^1 & \breve{U}_m & (\breve{Z}_i)^T \Lambda W_{im} \\ * & -W_{im} & 0 \\ * & * & -W_{im} \end{bmatrix},$$

$$\widehat{\Pi}_{im}^{12} = \begin{bmatrix} 0 & \tilde{B}_{fm} C_{1i} & \tilde{A}_{fm} & 0 & \tilde{B}_{fm} C_{di} & 0 & \tilde{B}_{fm} D_{1i} & \tilde{B}_{fm} & 0 \\ 0 & \tilde{B}_{fm} C_{1i} & \tilde{A}_{fm} & 0 & \tilde{B}_{fm} C_{di} & 0 & \tilde{B}_{fm} D_{1i} & \tilde{B}_{fm} & 0 \end{bmatrix},$$

$$\breve{\Pi}_{im}^1 = \begin{bmatrix} -I & \breve{\Pi}_{im}^{12} & 0 & D_{2i} - \tilde{D}_{fm} D_{1i} \\ * & \breve{\Pi}_{im}^{22} & 0 & 0 \\ * & * & -Q - \Pi^{33} & 0 \\ * & * & * & -\tilde{\gamma} I \end{bmatrix}, \breve{\Pi}_{im}^{12} = \begin{bmatrix} C_{2i} - \tilde{D}_{fm} C_{1i} & -\tilde{C}_{fm} & 0 \end{bmatrix},$$

$$\breve{\Pi}_{im}^{22} = \begin{bmatrix} \tau Q_{11} - Y_1 N_1 - F_{im}^1 & -F_{im}^2 & \tau Q_{12} + Y_2 N_1 \\ * & -F_{im}^3 & 0 \\ * & * & \tau Q_{22} - N_1 \end{bmatrix}, \breve{U}_m = \begin{bmatrix} -\tilde{D}_{fm}^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$\breve{Z}_i = \begin{bmatrix} 0 & C_{1i} & 0 & 0 & C_{di} & 0 & D_{1i} \end{bmatrix}, \bar{\Xi}_i = \begin{bmatrix} 0 & 0 & A_i & 0 & E_i & A_{di} & E_{di} & B_i & 0 & 0 \end{bmatrix},$$

$$\Xi_i = \begin{bmatrix} \bar{\Xi}_i^T & I_{(6n+2p+q+r)} \end{bmatrix}^T, \tilde{\gamma} = \gamma^2.$$

In addition, if (31) *and* (32) *are solvable, the filter matrices of* (5) *can be gained by*

$$\begin{cases} A_{fm} = (G_m)^{-1} \tilde{A}_{fm}, B_{fm} = (G_m)^{-1} \tilde{B}_{fm} \\ C_{fm} = \tilde{C}_{fm}, D_{fm} = \tilde{D}_{fm} \end{cases} \tag{33}$$

Proof. In order to verify Theorem 2, (13) is rewritten as

$$\Pi_{im} = \begin{bmatrix} -(\bar{P}_i)^{-1} & \Psi_{im}^1 \\ * & \Psi_{im}^2 \end{bmatrix} < 0 \tag{34}$$

By comparing (13) and (34), it is easy to obtain Ψ_{im}^1 and Ψ_{im}^2, which is omitted here to save space. To handle the nonlinearity $(\bar{P}_i)^{-1}$ in (34), an invertible slack matrix G_{im} is introduced as follows:

$$G_{im} \triangleq \begin{bmatrix} G_{im}^1 & G_m \\ G_{im}^2 & G_m \end{bmatrix} \tag{35}$$

where G_{im}^1 , G_{im}^2 , G_m are n-dimensional square matrices. Then, (34) is pre-multiplied and post-multiplied by $diag\{G_{im}, I\}$ and its transpose; hence, one has

$$\begin{bmatrix} -G_{im}(\bar{P}_i)^{-1} G_{im}^T & G_{im} \Psi_{im}^1 \\ * & \Psi_{im}^2 \end{bmatrix} < 0 \tag{36}$$

On the other hand, according to the fact that $(\bar{P}_i - G_{im})(\bar{P}_i)^{-1}(\bar{P}_i - G_{im})^T \geq 0$, we can readily obtain that

$$\bar{P}_i - G_{im} - G_{im}^T \geq -G_{im}(\bar{P}_i)^{-1}G_{im}^T \tag{37}$$

Combining (13), (36) and (37), we know that the following condition

$$\tilde{\Pi}_{im} \triangleq \begin{bmatrix} \bar{P}_i - G_{im} - G_{im}^T & G_{im}\Psi_{im}^1 \\ * & \Psi_{im}^2 \end{bmatrix} < 0 \tag{38}$$

is sufficient for (13). Moreover, we define

$$\begin{cases} \tilde{A}_{fm} \triangleq G_m A_{fm}, \tilde{B}_{fm} \triangleq G_m B_{fm} \\ \tilde{C}_{fm} \triangleq C_{fm}, \tilde{D}_{fm} \triangleq D_{fm} \end{cases} \tag{39}$$

and substitute them into (38).

Then, we define

$$\Theta_i = \begin{bmatrix} -I_n & \bar{\Xi}_i \end{bmatrix}, \Gamma = \begin{bmatrix} I_{2n} & 0_{(2n)\times(5n+2p+q+r)} \end{bmatrix}, \Gamma_\perp = \begin{bmatrix} 0_{(2n)\times(5n+2p+q+r)} \\ I_{(5n+2p+q+r)} \end{bmatrix},$$
$$\bar{G}_{im} = \begin{bmatrix} G_{im}^1 \\ G_{im}^2 \end{bmatrix}$$

We can readily derive that Ξ_i and Θ_i, $\Gamma_\perp$ and Γ are orthogonal complements, respectively. Then, $\tilde{\Pi}_{im}$ is decomposed into the following form

$$\tilde{\Pi}_{im} = \widehat{\Pi}_{im} + \Gamma^T \bar{G}_{im}\Theta_i + \Theta_i^T \bar{G}_{im}^T \Gamma \tag{40}$$

In accordance with lemma 2 (i.e., Projection lemma), $\tilde{\Pi}_{im} < 0$ is equivalent to

$$\Xi_i^T \widehat{\Pi}_{im} \Xi_i < 0, \ \Gamma_\perp^T \widehat{\Pi}_{im} \Gamma_\perp < 0 \tag{41}$$

which is obviously equivalent to (32). Furthermore, it can be inferred from (38) that G_{im} and G_m are both nonsingular, so we can deduce (33) from (39). Thus, we have accomplished the proof. □

Remark 5. *In Theorem 2, a filter design scheme is provided such that the error system ($\mathbb{S}_e$) is stochastically stable with an H_∞ performance γ. γ means the H_∞ performance level, a smaller γ indicates a better performance. The optimal performance $\gamma^* = \sqrt{\tilde{\gamma}_{\min}}$ can be yielded by solving the problem of convex optimization as follows:*

$$\begin{cases} \min & \tilde{\gamma} \\ s.t. & (31), (32) \end{cases} \tag{42}$$

Remark 6. *The number of LMIs $\mathcal{N}_l$ in Theorem 2 is*

$$\mathcal{N}_l = \sum_{i=1}^{s_1} \max\left\{1, \left|\mathscr{S}_{2U}^i\right|\right\} + s_2 \cdot \sum_{i=1}^{s_1} \max\left\{1, \left|\mathscr{S}_{1U}^i\right|\right\} + s_1 \cdot s_2 \tag{43}$$

where $|\mathscr{S}_{1U}^i|$ and $|\mathscr{S}_{2U}^i|$ represent the number of elements in the set $\mathscr{S}_{1U}^i$ and $\mathscr{S}_{2U}^i$, respectively. From (43), we clearly find that as the number of unknown entries for TPM Φ and CPM Ω increases, so does the number of LMIs required, thus aggravating the computational burden.

4. Numerical Example

This section will introduce a numerical example to verify the validity of the presented method. A three-mode MJNN($\mathbb{S}_0$) is considered with the parameters as follows, which are partly borrowed from [25]:

Mode 1 :

$A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $A_{d1} = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}$, $E_1 = \begin{bmatrix} 0.3 & -0.2 \\ 0.1 & 0.3 \end{bmatrix}$, $E_{d1} = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & 0.15 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$, $C_{11} = \begin{bmatrix} 0.17 & 0.18 \end{bmatrix}$, $C_{d1} = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$, $D_{11} = 0.1$, $C_{21} = \begin{bmatrix} 0.2 & 0.35 \end{bmatrix}$, $D_{21} = 0.1$.

Mode 2:

$A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}$, $A_{d2} = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.2 \end{bmatrix}$, $E_{d2} = \begin{bmatrix} 0.1 & -0.2 \\ 0 & 0.1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}$, $C_{12} = \begin{bmatrix} 0.42 & 0.90 \end{bmatrix}$, $C_{d2} = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}$, $D_{12} = 0.5$, $C_{22} = \begin{bmatrix} 0.1 & 0.15 \end{bmatrix}$, $D_{22} = 0.15$.

Mode 3:

$A_3 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$, $A_{d3} = \begin{bmatrix} 0.05 & 0 \\ 0 & -0.15 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0.2 & -0.1 \\ 0 & 0.1 \end{bmatrix}$, $E_{d3} = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$, $C_{13} = \begin{bmatrix} 0.12 & 0.5 \end{bmatrix}$, $C_{d3} = \begin{bmatrix} -0.05 & -0.1 \end{bmatrix}$, $D_{13} = 0.3$, $C_{23} = \begin{bmatrix} 0.2 & 0.2 \end{bmatrix}$, $D_{23} = 0.2$.

The nonlinear function is chosen as $g(x) = \tanh(x)$ with the bounds of $l_1 = l_2 = 0$ and $h_1 = h_2 = 1$; the delay $d(k) \in \{1,2,3\}$ is time-varying and random with $\tau = 3$, and the error threshold of the ET scheduler is $\delta = 0.3$.

In the sequel, four different TPM Φ^i and CPM Ω^i $(i \in \{1,2,3,4\})$ will be considered.

$$\Phi^1 = \begin{bmatrix} 0.85 & 0.05 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}, \Phi^2 = \begin{bmatrix} 0.85 & ? & ? \\ 0.2 & 0.5 & 0.3 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}, \Phi^3 = \begin{bmatrix} 0.85 & ? & ? \\ ? & ? & 0.3 \\ 0.5 & 0.1 & 0.4 \end{bmatrix}, \Phi^4 = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}.$$

$$\Omega^1 = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.1 & 0.9 & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \Omega^2 = \begin{bmatrix} 0.9 & ? & ? \\ 0.1 & 0.9 & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \Omega^3 = \begin{bmatrix} 0.9 & ? & ? \\ ? & ? & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \Omega^4 = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}.$$

Notice that Φ^1 and Ω^1 are fully known; Φ^4 and Ω^4 are fully unknown; Φ^3 and Ω^3 have more unknown elements than Φ^2 and Ω^2, respectively.

Firstly, in accordance with Theorem 2 and Remark 5, we can achieve the optimal H_∞ performance for different combinations (Φ^i, Ω^j), $(i, j \in \{1,2,3,4\})$, listed in Table 1.

From Table 1, we can clearly observe that, for a given Φ (or Ω), the optimal γ^* increases gradually when varying Ω from Ω^1 to Ω^4 (or Φ from Φ^1 to Φ^4). In addition, for $(\Phi, \Omega) = (\Phi^1, \Omega^1)$, which denotes the fully known case, γ^* is the smallest, which means that the H_∞ performance is the best. On the contrary, for $(\Phi, \Omega) = (\Phi^4, \Omega^4)$, which represents the fully unknown case, γ^* is the largest, i.e., the H_∞ performance is the worst. Therefore, we can conclude that the less probability information of TPM Φ or CPM Ω is available, the worse the H_∞ performance is. What is more interesting is that for each case of $\Omega = \Omega^4$, we find that the designed filter parameters are the same, e.g., when $(\Phi, \Omega) = (\Phi^2, \Omega^4)$, the solved filter parameters are as follows:

$$A_{fm} = \begin{bmatrix} 0.2266 & -0.9508 \\ 0.0551 & 0.7867 \end{bmatrix}, B_{fm} = \begin{bmatrix} -1.0299 \\ -0.2585 \end{bmatrix}, C_{fm} = \begin{bmatrix} -0.0912 & -0.1673 \end{bmatrix}, D_{fm} = 0.1963.$$

for $m = 1,2,3$, which indicates that the filter is mode independent when Ω is fully unknown.

Table 1. Optimal H_∞ performance for different Φ and Ω with unknown elements.

γ^*		CPM Ω			
		Ω^1	Ω^2	Ω^3	Ω^4
TPM Φ	Φ^1	0.5437	0.5524	0.6300	0.6386
	Φ^2	0.5498	0.5588	0.6393	0.6489
	Φ^3	0.5806	0.5884	0.6733	0.6784
	Φ^4	0.6319	0.6379	0.7440	0.7441

Furthermore, when $(\Phi, \Omega) = (\Phi^3, \Omega^3)$, the designed filter parameters can be obtained as follows:

Filter 1 :

$A_{f1} = \begin{bmatrix} -0.1645 & -1.3380 \\ 0.1701 & 0.9285 \end{bmatrix}$, $B_{f1} = \begin{bmatrix} -1.5449 \\ -0.0437 \end{bmatrix}$, $C_{f1} = \begin{bmatrix} -0.0896 & -0.1532 \end{bmatrix}$, $D_{f1} = 0.1879$.

Filter 2:

$A_{f2} = \begin{bmatrix} 0.0037 & -0.8974 \\ 0.0714 & 0.6686 \end{bmatrix}$, $B_{f2} = \begin{bmatrix} -1.2739 \\ -0.2034 \end{bmatrix}$, $C_{f2} = \begin{bmatrix} -0.0901 & -0.1549 \end{bmatrix}$, $D_{f2} = 0.1867$.

Filter 3:

$A_{f3} = \begin{bmatrix} 0.3964 & -0.0578 \\ -0.0550 & 0.2278 \end{bmatrix}$, $B_{f3} = \begin{bmatrix} 0.7787 \\ -1.8741 \end{bmatrix}$, $C_{f3} = \begin{bmatrix} -0.0792 & -0.1296 \end{bmatrix}$, $D_{f3} = 0.2777$.

We further assume that the initial values of filter $(\mathbb{S}_f)$ and system $(\mathbb{S}_0)$ are $x_f(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ and $x(k_0) = \begin{bmatrix} 0.2 & -0.2 \end{bmatrix}^T$, $k_0 = -3, -2, -1, 0$, $\alpha_0 = 1$, and the external disturbance is $w(k) = 0.9^k \sin(k)$. Based on the above parameters, a simulation is made with the presented ET asynchronous filtering scheme. The mode jumps of the original plant and the filter are plotted in Figure 2 to show the asynchronization between them.

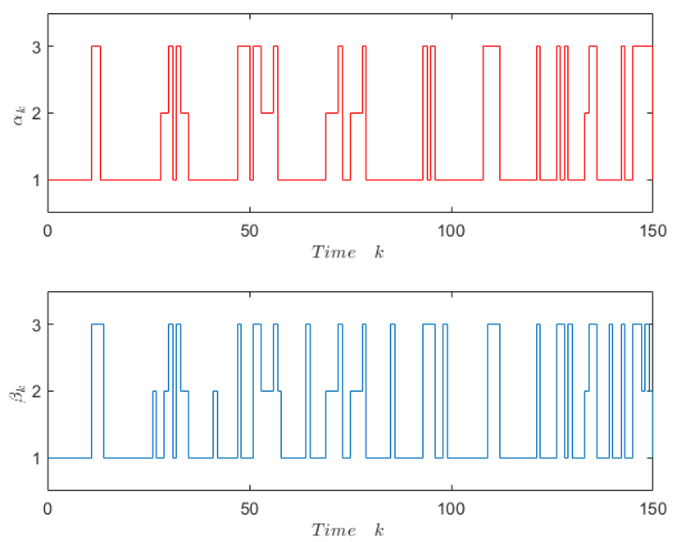


Figure 2. Mode jumps of the original plant and the filter.

The response curves of $z(k)$ and $z_f(k)$, and $\tilde{e}(k) = z(k) - z_f(k)$ are shown in Figures 3 and 4, from which we observe that the filtering error system is stochastically stable. In addition, we obtain $MTR = 0.84$ via calculation with the threshold $\delta = 0.3$, which implies that the ET scheduler can effectively decrease the data-transmission rate of measured outputs. Therefore, it can be observed that the effect of the devised ET filter in Theorem 2 is fine.

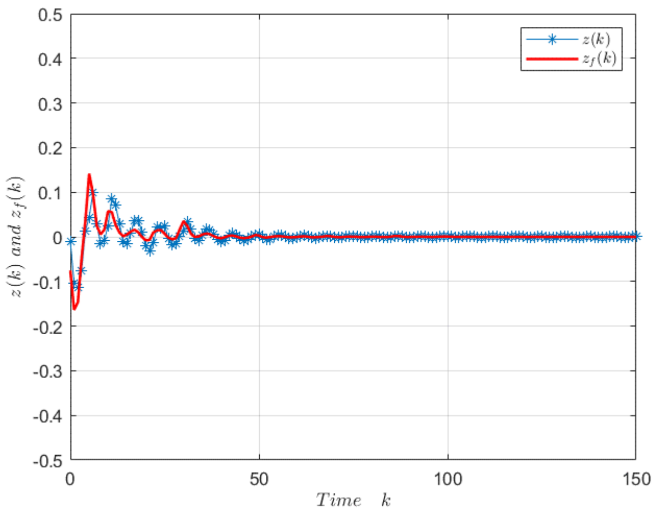


Figure 3. The response curves of $z(k)$ and $z_f(k)$.

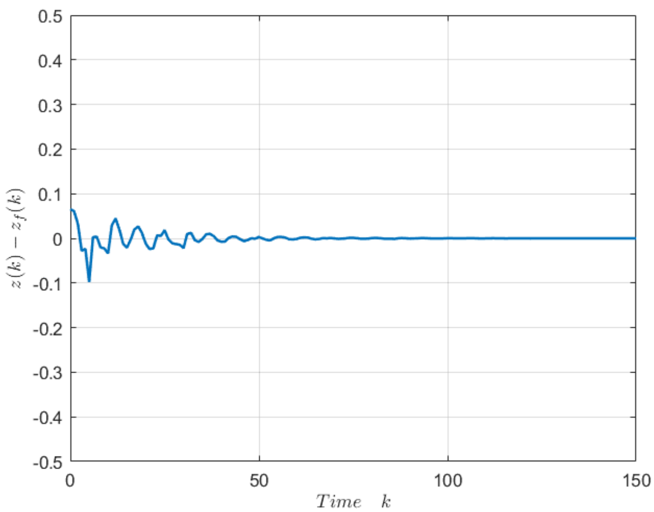


Figure 4. The curve of estimation error $\tilde{e}(k)$.

In our research, the asynchronous issue is characterized as a HMM, the core of which is the CPM, reflecting the asynchronous degree between filter ($\mathbb{S}_f$) and system ($\mathbb{S}_0$). Next, four different CPM Ω^i $(i \in \{a, b, c, d\})$ are chosen to exhibit the influence of asynchronous features on the H_∞ performance of the error system ($\mathbb{S}_e$):

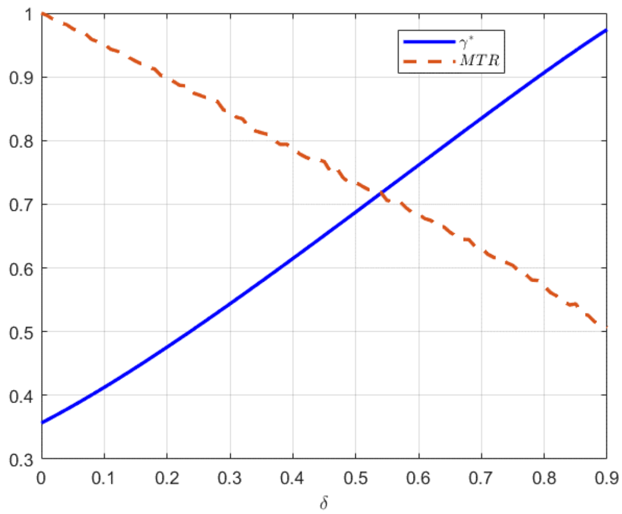
$$\Omega^a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Omega^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \Omega^c = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \Omega^d = \Omega^1.$$

which represent four different cases: synchronization, weak asynchronization, strong asynchronization and full asynchronization. In addition, in order to compare the results of the fully known TPs case and the partly unknown TPs case, we choose TPM Φ as Φ^1 and Φ^3, respectively. By solving the convex optimization in (42) with the LMI toolbox of Matlab,we use the ***mincx*** function to calculate the corresponding optimal γ^*, as shown in Table 2. We can easily see from Table 2 that, for a given Φ, with the increase in asynchronous degree between filter ($\mathbb{S}_f$) and system ($\mathbb{S}_0$), γ^* becomes larger, which implies the decline of the H_∞ performance.

Finally, we will investigate the influence of the ET feature on the H_∞ performance and communication performance with the varying threshold of δ in the ET scheduler. We keep the other parameters fixed, and only vary the threshold parameter δ. The evolution curves of the corresponding H_∞ performance γ^* and communication performance MTR for the cases of (Φ^1, Ω^1) and (Φ^3, Ω^3) are shown in Figure 5. We can easily find that as the parameter δ increases, γ^* becomes larger, which implies that the H_∞ performance decreases, whereas the MTR value shows a trend of getting smaller, which means that the communication performance of measured outputs is becoming better. Considering the trade-off between the H_∞ performance and communication performance, thus we can choose a compromise error threshold of the ET scheduler to achieve a more satisfactory comprehensive performance in practical applications.

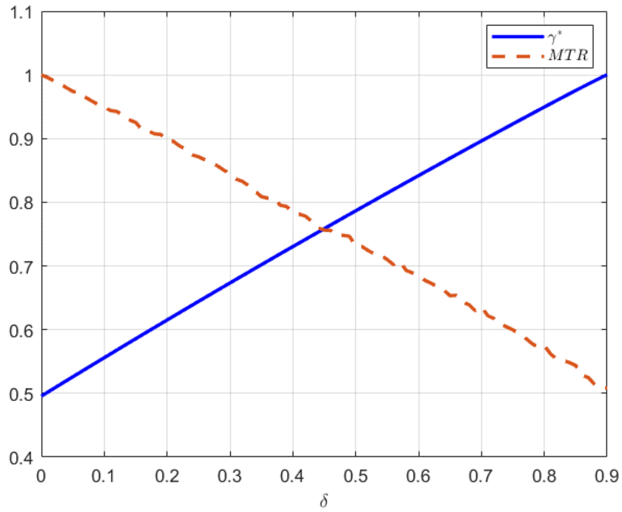
Table 2. Optimal H_∞ performance for Ω with different asynchronous features.

γ^*		CPM Ω			
		Ω^a	Ω^b	Ω^c	Ω^d
TPM Φ	Φ^1	0.4791	0.4912	0.5249	0.5437
	Φ^3	0.5104	0.5219	0.5644	0.5806



(**a**) When $(\Phi, \Omega) = (\Phi^1, \Omega^1)$

Figure 5. *Cont.*



(**b**) When $(\Phi, \Omega) = (\Phi^3, \Omega^3)$

Figure 5. The H_∞ performance and communication performance with varying δ.

5. Conclusions

In this paper, the study of the ET H_∞ asynchronous filtering issue was explored for MJNSs with varying delay and unknown probabilities. An ET scheduling strategy was adopted to decrease the transmission rate of measured outputs, and the filter was mode dependent and asynchronous with the original MJNS, represented by an HMM. Both the TPM of the original system and the CPM of the filter were assumed to be only partly accessible. Under this framework, based on Lyapunov stability and H_∞ theory, a sufficient condition was derived, in which the nonlinearity of the matrix inequalities was further dealt with and a feasible filter was achieved with the techniques of slack matrix and Projection lemma. Lastly, the relationship between the H_∞ performance and the unknown elements of TPM and CPM, the relationship between the H_∞ performance and the asynchronous feature of CPM, and the relationship among the H_∞ performance, communication performance and the ET threshold were discussed and exhibited through a numerical example. The simulation results sufficiently validated the availability of our developed filtering scheme, which will contribute to the further research involving this subject, e.g., control and fault detection. This can also be extended to other dynamic systems, such as singular MJNSs and 2-D MJNSs.

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Abbreviations

The following abbreviations are used in this manuscript:

HMM	hidden Markov model
MJNSs	Markov jump nonlinear systems
MJSs	Markov jump systems
NN	neural network
MJNNs	Markov jump neural networks
NCSs	networked control systems
ET	event-triggered
CPM	conditional probability matrix
TPs	transition probabilities
TPM	transition probability matrix
LKF	Lyapunov–Krasovskii functional
ZOH	zero-order holder
CP	conditional probability
LMIs	linear matrix inequalities

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