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Optimal Control of Technological Processes

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Abstract: The paper formulates conditions under which the roots closest to the imaginary axis (critical roots) of the characteristic equation of a linearized system are real for the maximum possible degree of stability of the closed-loop control system of a technological process with pure delay. For the parameters of the controllers corresponding to the maximum degree of stability, these roots are multiples. Their multiplicity order is one more than the number of coefficients in the transfer function of the controller. It is demonstrated that for a typical technological control object, these conditions are satisfied for all "serial" control laws. This allowed for obtaining analytical expressions for optimal settings and limiting degrees of stability as functions of object parameters for typical dynamic characteristics of technological processes. The paper considers the problem of robust stability for control systems with an object containing pure delay. It has been proven that in the maximum stability problem, the operations of maximizing over controller parameters and minimizing over the set of possible object parameters can be interchanged. Therefore, selecting robust settings amounts to determining the minimum of the maximum stability over the set of possible object parameter values. Controllers with such settings are suitable, without modification, for a whole class of technological processes.

Keywords: maximum stability of linear systems; features of technological objects; pure delay; robustness

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1. Introduction

The stability of dynamic systems and the selection of controller parameters are the most studied topics in control theory. A considerable number of studies were devoted to this theme in the 1950s and 1960s. A.M. Letov [1] paid significant attention to this area, using methods based on the Lyapunov function defined on the state space of the dynamic system. These methods allow for the nonlinearity of equations to be taken into account, but they are only effective for finite-dimensional systems.

Frequency methods are applicable only to linear systems, but they allow for considering systems with delay, which is essential for technological processes as control objects. The dynamics of these objects are mainly determined by heat and mass transfer processes. They are stable, characterized by distributed parameters, and their impulse transfer functions do not change sign and tend to zero or to some constant if the object contains an integrating element. The transfer functions of such objects contain pure delay, and their characteristic equation does not have complex roots. The magnitude and phase of their frequency response decrease monotonically with increasing frequency, with the magnitude tending to zero and the phase to minus infinity (the "monotonicity" effect). For brevity, such objects will be referred to as technological.

Tsypkin introduced the concept of the stability degree of a linear system as the distance from the imaginary axis to the nearest (*critical*) root of the system's characteristic equation. If each of the roots is assigned an index ν, then the stability degree is the absolute value of the maximum over ν of the real part of the root (in a stable system, the real parts of all roots are negative). He used frequency methods in his formulated problem of maximizing the stability of linear systems [2,3]. This topic was also the subject of his dissertation.

The problem of maximizing stability was significantly developed by Shubladze and his colleagues [4,5].

Another feature of controllable technological processes is the change in their dynamic properties when the raw material composition and flow intensity are altered. Therefore, researchers have paid significant attention to the problem of object control with varying parameters and the synthesis of robust controllers that, with fixed settings, can control a whole class of objects or one object within a wide range of parameter changes. For finite-dimensional problems, methods for synthesizing such systems with an estimation of the permissible range of object parameters was developed in the work of Polyak and Shcherbakov (see [6]). However, these methods are not applicable to objects with delays, which includes the majority of technological processes.

The selection of controller parameters based on the condition of maximum stability indirectly ensures the robustness of the system. If the dependence of the maximum stability on the parameters of the object is obtained, and the region in which these parameters can vary is known, then the most "unfavorable" combination of parameters from the region of their possible values is selected. At such parameters, the maximum stability is minimal. This is a guaranteed degree of stability of the system throughout the range of changes in its characteristics.

It is significantly easier to solve these problems if an analytical dependence of the maximum stability on the object's parameters is found. As shown below, such a dependence can be easily obtained when the roots closest to the imaginary axis (critical roots) are real. In [7], the maximum stability in this case is called *aperiodic*, and in the case when the critical roots are complex—*oscillatory*. It is not known in advance which of these cases is true. The controller parameters are chosen based on the condition of maximum aperiodic stability, and then it is checked whether the synthesized system has complex roots that are closer to the imaginary axis than real roots. The paper shows under what conditions the answer to this question is negative, and therefore the relation between the object's parameters and the controller settings, found by the condition of maximum aperiodic stability, can be used to calculate the system.

The problem of calculating the limit aperiodic stability is considered by plotting a hodograph of the extended frequency response of the open-loop system with maximum aperiodic stability.

An important feature of technological processes is that, despite their enormous variety, they are similar to each other in their dynamic characteristics, and in most cases, their dynamics in the Fourier or Laplace transform domain can be approximated by an aperiodic or integrating element with pure delay. This makes it possible to use a range of typical "serial" control laws consisting of proportional (P), integral (I), proportional-integral (PI), and proportional-integral-differential (PID) controllers in control systems.

The paper presents formulas for selecting parameters of typical controllers based on the condition of maximum system stability for objects with delay, single-loop, and two-loop systems. A methodology for selecting robust settings is proposed, and a real-time optimization system is considered to support the optimality conditions of the technological system.

2. Calculation of Optimal Controller Settings

Let us consider a linear single-loop automatic control system (Figure 1).

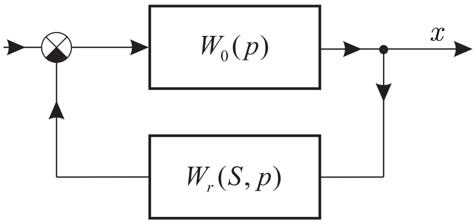

Figure 1. Control system structure.

Here, $W_0(p)$ и $W_r(S,p)$ are the transfer functions of the object and the controller, respectively, where S is the vector of controller parameters. The transfer function of the closed-loop system is

$$W_s(p) = \frac{W_0(p)}{1 + W_0(p)W_r(S,p)}; \tag{1}$$

assume that roots of the characteristic equation do not coincide with the zeros of the object's transfer function.

The controller usually has a standard PID structure

$$W_r(S,p) = \left(S_2 p + S_1 + \frac{S_0}{p}\right). \tag{2}$$

If $S_2 = 0$, then the controller is proportional-integral (PI), if S_2 and S_1 are zero, it is integral (I), and if S_2 and S_0 are zero, it is proportional (P).

An integral component ensures an astatic character of the transition processes. In this case, the characteristic equation has no zero roots. If there is no integral component ($S_0 = 0$), then the characteristic equation has a root at the origin, and the system, when subjected to step-like input, does not return to the equilibrium state and exhibits a "residual static error". In the latter case, when calculating the maximum stability degree, all roots of the characteristic equation but zero are taken into account.

Based on the made assumptions, the characteristic equation of System (1) can be written equivalently as

$$\frac{p}{W_0(p)} + S_2 p^2 + S_1 p + S_0 = 0, \quad p_\nu = \rho_\nu \pm i\omega_\nu, \quad \nu = 1, 2, \ldots, \tag{3}$$

Choosing the value of vector S aims to ensure that the roots of the characteristic equation are located to the left of the imaginary axis while the distance from the nearest roots $\eta^* = |\rho^*|$ is maximized. This requirement is formalized as a minimax.

Problem A:

$$\rho^*(S) = \max_\nu \rho_\nu(S) \to \min_S. \tag{4}$$

The dependency $\rho_\nu(S)$ is determined by solving Equation (3), where the real parts of all roots are known to be negative. If the maximization problem with respect to ν has no solution, then instead of a maximum, the exact upper bound ρ_ν shall be found.

As only the coefficients at low powers of p in the characteristic equation depend on the controller settings, any changes in these settings do not affect the sum of the real parts of its roots

$$\rho_S = \sum_\nu \rho_\nu, \tag{5}$$

which, for the nth power equations, according to Vieta's formulas, is equal to the coefficient at p^{n-1} with the opposite sign. Thus, decreasing the real parts of the roots closest to the imaginary axis leads to an increase in ρ_ν for the remaining roots. At the same time, the vector S primarily affects the roots located closer to the imaginary axis in the p [1] plane. Therefore, the maximum stability

$$\rho^* = \min_S \rho^*(S)$$

corresponds to the case where, for several real roots or several pairs of complex roots, the values of the real parts are the same.

The number of such "critical" roots, if task (4) has a solution, is one more than the number of adjustable controller coefficients. This type of solution structure is typical for minimax problems and, in the case of the problem of uniform approximation, is known as the "Chebyshev alternance" principle.

For the case where all critical roots are real, problem (4) is relatively easy to solve, and in some cases, it can be solved analytically. The limit of aperiodic stability can be found from the condition of multiplicity of the $(m+1)$th critical root

$$\frac{d^m}{(d\rho)^m}\left(\frac{\rho}{W_0(\rho)}\right)_{\rho^*}=0, \tag{6}$$

where m is the number of required controller settings (in Equation (3), $m=3$).

For $S_0=0$, the left-hand side of Equation (3) can be reduced by p. At the same time, a root with $\rho_0=0$, and changing settings S_2, S_1 does not change ρ_0, which leads to a steady-state error. This leads to abrupt changes of $\rho^*(S)$ at $S_0=0$.

If Equation (6) can be solved when $S_0\neq 0$, then the corresponding optimal settings are found as

$$S_2^*=-0.5\frac{d^2}{d\rho^2}\left(\frac{\rho}{W_0(\rho)}\right)_{\rho^*}, \tag{7}$$

$$S_1^*=-\left[2S_2^*\rho^*+\frac{d}{d\rho}\left(\frac{\rho}{W_0(\rho)}\right)_{\rho^*}\right], \tag{8}$$

$$S_0^*=-\rho^*\left(S_1^*+S_2^*\rho^*+\frac{1}{W_0(\rho^*)}\right). \tag{9}$$

For $S_0=0$, settings S_1 and S_2, the PD-controller is chosen in such a way that the roots of the equation

$$\frac{1}{W_0(p)}+S_2p+S_1=0 \tag{10}$$

satisfy requirement (4). The limit of aperiodic stability (taking into account all non-zero roots) and the corresponding settings are found from the conditions:

$$\frac{d^2}{d\rho^2}\left(\frac{1}{W_0(\rho)}\right)_{\rho^*}=0, \tag{11}$$

$$S_2^*=-\frac{d}{d\rho}\left(\frac{1}{W_0(\rho)}\right)_{\rho^*}, \tag{12}$$

$$S_1^*=-\rho^*\left(S_2^*+\frac{1}{W_0(\rho^*)}\right). \tag{13}$$

The settings found in this way, corresponding to the limit of aperiodic stability of the system, are a solution to problem (4) only if all complex roots of the characteristic equation of the system for the settings selected in this way are to the left of the line $-\rho_a^*\pm i\omega$. This leads to

Problem B: Under which conditions does the limit of aperiodic stability ρ_a^* coincide with the maximum stability of the system (i.e., critical roots are real)?

2.1. Conditions of Optimality of the Limit of Aperiodic Stability

Suppose (8) and (9) give ρ_a^*, S_0^*, S_1^*, and S_2^*. They correspond to the frequency response of the open-loop system $W_c(iw)=W_0(i\omega)W_r(S^*,i\omega)$ and its extended frequency response

$$W_c(i\omega,\rho_a^*)=W_0(i\omega-\rho_a^*)W_r(S^*,i\omega-\rho_a^*). \tag{14}$$

Figure 2 shows the location of the roots of the characteristic equation of the system in cases where ρ_a^* corresponds to the maximum stability (a) and when it does not correspond (critical roots are complex) (b).

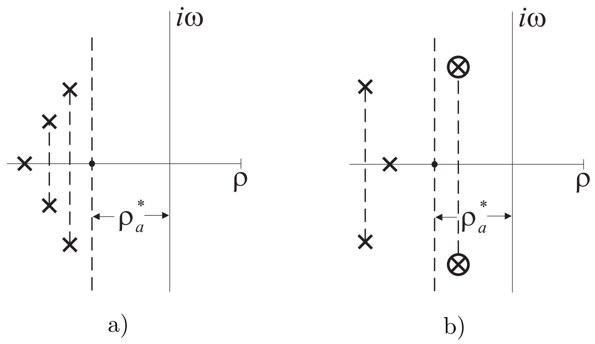

Figure 2. The limit of aperiodic stability may (**a**) or may not (**b**) coincide with the maximum stability.

To determine which case is true, let us construct a Nyquist plot of an extended frequency response of an open-loop system $W_c(i\omega, \rho_a^*)$. It is clear that due to condition (3) at $\omega = 0$, this plot will pass through the point $(-1, i0)$ (Figure 3). *If all the characteristic equation roots of the system lie to the left of the line $-\rho_a^* \pm i\omega$, then when ω changes from zero to infinity, the Nyquist plot $W_c(i\omega, \rho_a^*)$ will not encircle the point $(-1, i0)$* (Figure 3a).

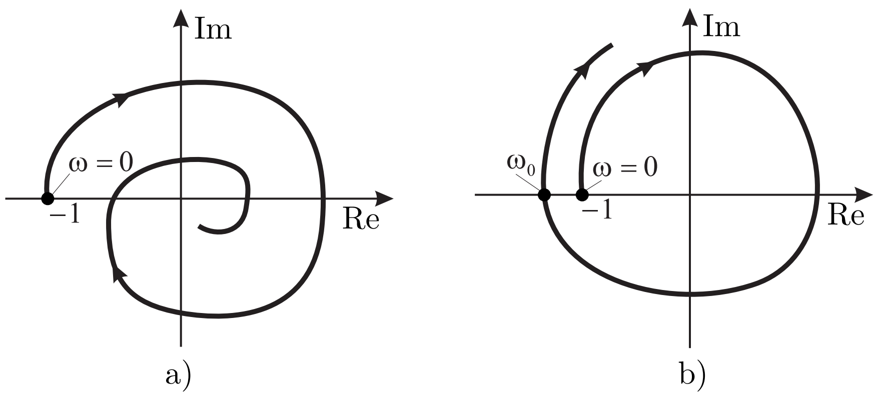

Figure 3. The Nyquist plot of the extended response of an open-loop system for cases where the maximum degree of stability is aperiodic (**a**) and oscillatory (**b**).

Let us denote the frequencies with the phase of frequency $W_c(i\omega, \rho_a^*)$ equal to π as critical frequencies $\omega_{0\nu}$. In this case, $\omega_{00} = 0$; ω_{01} corresponds to a rotation of the plot by $2\pi, \omega_{02}$ by 4π, and so on.

At the frequency ω_{00} $|W_c(i0, \rho_a^*)| = 1$.

The properties of conformal mapping determine the following.

Statement:

The limit of aperiodic stability is the maximum possible stability of the system only if $\forall \nu > 0$

$$|W_c(i\omega_{0\nu}, \rho_a^*)| \leq 1. \tag{15}$$

Particularly, for optimality ρ_a^*, it is sufficient for the modulus of $|W_c(i\omega, \rho_a^*)|$ to decrease monotonically while ω increases. Conversely, the limit of aperiodic stability is not optimal if $|W_c(i\omega, \rho_a^*)|$ increases monotonically while the frequency increases. It is typical for technological processes to exhibit a monotonic decrease of the modulus of the extended frequency response of an open-loop system with frequency.

2.2. Typical Technological Process Control System

A typical industrial subject of control is defined as [8,9]. The dynamics of most processes can be approximately described by a transfer function

$$W_0(p) = \frac{ke^{-p\tau}}{Tp + 1}, \tag{16}$$

as it is close to the dynamics of the subject in question.

To calculate its three coefficients from the curve obtained after applying a step of Δ at the input, a tangent is drawn to this curve at the inflection point. The distance from the point of intersection of this tangent with the x-axis to the origin is τ. The ratio of the steady-state deviation to Δ is k. The tangent of the slope of the tangent is $1/T$.

Let us note that from the transfer function (16) after substituting $k = \frac{T}{\Theta}$ and subsequent limit transition at $T \to \infty$ we obtain an integrating object with delay

$$W_0(p) = \frac{e^{-p\tau}}{\Theta p}. \quad (17)$$

By directing T to zero in (16), we obtain a pure delay object

$$W_0(p) = ke^{-p\tau}. \quad (18)$$

For each of the noted typical objects, analytical expressions for the achievable aperiodic stability ρ_a^* and corresponding controller settings, expressed through the object parameters, were obtained using (7)–(9) and (11)–(13) (see also [2,5]).

Let us determine if the limit of aperiodic stability reaches the maximum possible stability for systems with typical objects (16)–(18) and I, PI, and PID-controllers included in the negative feedback.

2.2.1. The Pure Delay Object

I-controller:

$$\rho_a^* = \frac{1}{\tau}. \quad (19)$$

The only value of the setting parameter

$$S_0 = \frac{1}{k\tau}e^{-1}$$

corresponds to ρ_a^*. The modulus of the extended frequency response of the open-loop system

$$|W_c(i\omega, \rho_a^*)| = \frac{1}{\sqrt{\omega^2\tau^2 + 1}}$$

decreases monotonically from 1 at $\omega = 0$ to zero at $\omega \to \infty$. In this case, the limit of aperiodic stability is the maximum possible.

PI-controller:

$$\rho_a^* = \frac{2}{\tau}. \quad (20)$$

The corresponding setting parameters

$$S_0 = \frac{4}{k\tau}e^{-2}, \quad S_1 = \frac{1}{k}e^{-2}.$$

The modulus of the extended frequency response of the open-loop system is 1. Thus, the critical roots are the real roots and any number of complex roots at critical non-zero frequencies. The limit of stability is ρ_a^*.

PID-controller:

$$\rho_a^* = \frac{3}{\tau}. \quad (21)$$

The corresponding setting parameters

$$S_0 = \frac{27}{2k\tau}e^{-3}, \quad S_1 = \frac{5}{k}e^{-3}, \quad S_2 = \frac{\tau}{2k}e^{-3}.$$

The modulus of the extended frequency response of the open-loop system

$$|W_c(i\omega,\rho_a^*)| = \tau\sqrt{\frac{\left(\frac{6-\tau^2\omega^2}{2\tau}\right)^2+4\omega^2}{\omega^2\tau^2+9}}.$$

It is 1 at $\omega = 0$ and increases monotonically with the increase of ω. Thus, for such a system, the limit of aperiodic stability is not the maximum possible.

2.2.2. An Integrating Object with Delay

PI-controller:

$$\rho_a^* = \frac{2-\sqrt{2}}{\tau}. \tag{22}$$

Optimal controller parameters

$$S_0 = \frac{\Theta}{\tau^2}2\left(\sqrt{2}-1\right)^3 e^{\sqrt{2}-2}, \quad S_1 = \frac{\Theta}{\tau}2\left(\sqrt{2}-1\right)e^{\sqrt{2}-2}.$$

Modulus $W_c(i\omega,\rho_a^*)$

$$|W_c(i\omega,\rho_a^*)| \approx \sqrt{\frac{0.1177+0.6864\omega^2\tau^2}{0.1177+0.6864\omega^2\tau^2+\omega^4\tau^4}}.$$

It monotonically decreases with increasing frequency, therefore, the limit of aperiodic stability is the maximum possible. It is important to note that this modulus does not depend on the object's parameter Θ.

PID-controller:

$$\rho_a^* = \frac{3-\sqrt{3}}{\tau}. \tag{23}$$

Corresponding controller parameters

$$S_0 = \frac{\Theta}{\tau^2}\left(3-\sqrt{3}\right)^3\left(1-\frac{3-\sqrt{3}}{2}\right)e^{\sqrt{3}-3}, \quad S_1 = \frac{\Theta}{\tau}\left(3-\sqrt{3}\right)^2\sqrt{3}e^{\sqrt{3}-3},$$

$$S_2 = \Theta\left(2\left(3-\sqrt{3}\right)-1-\frac{\left(3-\sqrt{3}\right)^2}{2}\right)e^{\sqrt{3}-3}.$$

The modulus of the extended frequency response of the open-loop system

$$|W_c(i\omega,\rho_a^*)| \approx \sqrt{\frac{2.5844+0.8615\omega^2\tau^2}{2.5844+3.2151\omega^2\tau^2+\omega^4\tau^4}}.$$

It is easy to see that the limit of aperiodic stability is the maximum possible, and $|W_c(i\omega,\rho_a^*)|$ is not dependent on Θ.

2.2.3. Aperiodic Object with Delay

Parameters of the aperiodic object with delay are determined as a result of the experiment.

I-controller:

$$\rho_a^* = 1+\frac{1}{2T}-\sqrt{1+\frac{1}{4T^2}}. \tag{24}$$

Corresponding controller parameters

$$S_0 = (1-2T\rho_a^*)e^{-\rho_a^*}.$$

Modulus $W_c(i\omega, \rho_a^*)$ becomes

$$|W_c(i\omega, \rho_a^*)| = \frac{c}{\sqrt{T^2\omega^4 + \omega^2(b^2 - 2aT) + a^2}},$$

where

$$a = \rho_a^*(T\rho_a^* - 1), \quad b = 1 - 2T\rho_a^*, \quad c = |S_0 e^{\rho_a^*}|.$$

Thus, $|W_c(i0, \rho_a^*)| = 1$. Both the numerator and denominator at $\omega = 0$ yield the same expression

$$\left|2T - 2T\sqrt{1 + \frac{1}{4T^2}}\right|.$$

With the increase in frequency $|W_c(i\omega, \rho_a^*)|$ tends towards zero because

$$b^2 > 2aT.$$

Indeed,

$$b^2 = 4T\rho_a^*(T\rho_a^* - 1) + 1,$$
$$2aT = 2T\rho_a^*(T\rho_a^* - 1).$$

As the modulus of the extended frequency response of the open-loop system monotonically decreases with the increase of frequency, the limit of stability is considered aperiodic.

PI-controller:

$$\eta_a^* = \frac{1}{2T} + \frac{2}{\tau} - \sqrt{\frac{1}{4T^2} + \frac{2}{\tau^2}}. \tag{25}$$

Optimal setting parameters:

$$S_0 = \frac{\eta}{k} e^{-\eta\tau}(1 - T\eta), \tag{26}$$

$$S_1 = \frac{0.14 e^{-\frac{\tau}{T}}}{k}\left(4\frac{T}{\tau} + 1\right). \tag{27}$$

The module of the extended frequency response of the open-loop system monotonically decreases with the increase of ω, therefore, the limit of aperiodic stability is the maximum possible.

PID-controller:

$$\eta^* = \frac{1}{2T} + \frac{3}{\tau} - \sqrt{\frac{1}{4T^2} + \frac{3}{\tau^2}}. \tag{28}$$

The corresponding setting parameters

$$S_0 = \frac{\eta^3\tau^2}{k} e^{-\eta\tau}\left(\frac{T}{\tau} - \frac{T}{2}\eta + \frac{1}{2}\right),$$

$$S_1 = \frac{e^{-\eta\tau}}{k}(3T\tau\eta^2 + \tau^2\eta^2 - \tau\eta - T\tau^2\eta^3 - 1),$$

$$S_2 = \frac{\tau e^{-\eta\tau}}{k}\left(\frac{\tau\eta}{2} + 2T\eta - \frac{T}{\tau} - \frac{T\tau\eta^2}{2} - 1\right).$$

The modulus of frequency response of the open-loop system decreases monotonically from 1 to 0 while ω increases, thus, η^* is the maximum possible.

3. The Robust Stability and Settings of Technological Process Control Systems

In recent years, researchers have paid significant attention to the problem of controlling objects with varying dynamic characteristics and synthesizing robust controllers that can control a whole class of objects or a single object over a wide range of its parameters, loads, and others without reconfiguration. Methods for synthesizing such systems with estimation of permissible range of possible object parameters were developed in studies [6–14] and others. Most of these studies are devoted to linearized systems; their characteristic equation's left-hand side is a polynomial of the form

$$P_n(a,p) = a_0 + a_1 p + \cdots + a_n p^n, \tag{29}$$

where the polynomial coefficients can take values belonging to some set V_a. A polynomial (29) is said to be robustly stable if it is stable for any $a \in V_a$. Its coefficients $\forall\, a \in V_a$ should be positive—this is necessary but not sufficient.

In [6,10], the problem of robust stability of the polynomial is solved specifically when the set V_a is a parallelepiped delimited by interval constraints

$$\underline{a_i} \le a_i \le \overline{a_i}, \quad i = 0, \ldots, n. \tag{30}$$

In [10], four polynomials are constructed, and their values a are selected in such a way that their stability guarantees the stability of $P_n(a,p)$. In [11], using the Mikhailov stability criterion based on constraints (30), a system is obtained with characteristics determined by the vector $a_0 \in V_a$ and deviations $|a_i - a_0| \le \gamma\alpha_i$. It is shown there that it is necessary to construct only one Mikhailov plot of this system to determine whether the original polynomial is robustly stable for a given γ, and at what maximum γ this stability is maintained.

For technological processes with pure delay in their transfer functions, the obtained results are not applicable. These technological processes include those in the chemical, metallurgical, food industries, energy, and others.

The denominator of the transfer function of the closed-loop control system of such objects is not a polynomial, and all results of the automatic control theory (ACT), which are based on the properties of polynomials, are not applicable to such systems. Stability criteria such as Routh–Hurwitz, Mikhailov, logarithmic frequency characteristic methods, state-space methods, and others are not applicable as well. These control system features of technological processes have been repeatedly emphasized by Rotach [15].

The monotonicity of a modulus and phase of technological linearized objects is generally valid, even for open-loop control systems, which, as shown below, simplifies the solution of the problem of robust stability and the selection of robust controller settings.

The characteristic equation of the closed-loop control system is

$$1 + W(S,p) = 0.$$

In this equation, $W(S,p)$ is the transfer function of the open-loop system, which is equal to the product of the transfer function of the control object $W_0(p)$ and the transfer function of the controller $W_R(S,p)$, $S \in V_S$, where the feedback is negative.

A sufficient condition for the stability to be equal to x_0 is to fulfill the constraint imposed on the modulus of the extended frequency response of the open-loop system:

$$|W(S(x_0), i\omega - x_0)| < 1, \tag{31}$$

where, as shown in the first section, $S(x_0)$ is the solution of the system of equations containing the derivatives with respect to x_0 of the function $W(S,x_0)$ (conditions for the multiplicity of the real root closest to the imaginary axis).

Below are the conditions for the existence of robust controller settings in technological process control systems, equations defining the boundary of the robust D-decomposition, and a methodology for selecting controller parameters for a limited and closed set V_a of possible values of the transfer function parameters a of the object. The conditions are specified for systems with typical technological objects and controllers.

Let us consider single-loop systems, where the transfer function of the object depends on the coefficients $a \in V_a$. We will assume that the open-loop systems are stable or neutral, and that the modulus M and phase φ satisfy the monotonicity conditions

$$M(a, S, \omega_k) \geq M(a, S, \omega_{k+1}), \tag{32}$$

where ω_k is the solution of the equation

$$\varphi(a, S, \omega_k) = -\pi(1 + 2k), \quad k = 0, 1, \ldots, \quad \frac{\partial \varphi}{\partial \omega} < 0 \quad \forall a \in V_a. \tag{33}$$

For a technological object and any controller, for which frequency response modulus and phase do not increase with frequency, these conditions are met by default. However, if the controller contains a differentiating element, the monotonicity conditions can be verified by inequality (31).

According to the Nyquist criterion, a dynamic system with feedback is stable if the open-loop system is stable, and the Nyquist plot of the open-loop system $W(a, S, i\omega) = W_R(S, i\omega) W_0(a, i\omega)$ does not encircle the point $(-1, i0)$ while varying ω from zero to infinity.

For systems with frequency response modulus that monotonically decreases with increasing ω, this means that the conditions

$$\varphi(a, S, \omega_0) = -\pi, \quad M(a, S, \omega_0) < 1 \tag{34}$$

are satisfied at the first intersection w ith real axis at frequency $\omega_0 > 0$. For brevity, let us denote the frequency response modulus of the open-loop at its intersection with the negative real axis as $M_\pi(a, S, \omega)$.

The expression $r(a, S) = 1 - M_\pi(a, S, \omega)$ is referred to as *stability margin*. Condition (3) can be expressed in the form of inequality $M_\pi(a, S, \omega_0) < 1$.

Systems satisfying conditions (32) and (33) have a Nyquist plot of the open-loop system, as shown in Figure 4.

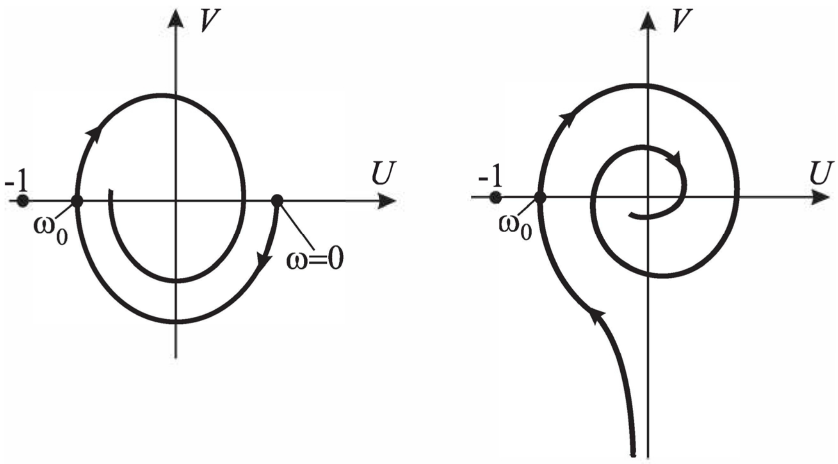

Figure 4. Nyquist plot of the open-loop control system.

Let us provide several definitions related to robust systems.

Definition 1. *1. A closed-loop system is said to be robustly stable if there exists a permissible vector S such that* $r(a, S) > 0 \quad \forall a \in V_a$.

For systems that satisfy conditions (32) and (33), this implies the existence of a vector of controller parameters $S^* \in V_S$ *such that the minimum over S of the maximum* $M_\pi(a, C, \omega)$ *over a is less than 1:*

$$\min_{S \in V_S} \left(\max_{a \in V_a} M_\pi(a, S, \omega) \right) < 1. \tag{35}$$

2. A robust D-partition in the parameter space of the controller is defined as the set of all parameter values for which the closed-loop system is robustly stable. Therefore, the system is robustly stable if the set defined by the D-partition is not empty.

The boundary of the region of robust stability in the parameter space of the controller S is defined by the condition

$$\min_{S\in V_S}\left(\max_{a\in V_a} M_\pi(a,S,\omega)\right)=1. \tag{36}$$

The set V_s may include non-negative values of S.

3. Let the system be robustly stable at a certain value of S. Let us define the robust degree of stability η as a non-negative number such that the system that has an extended frequency response of the open-loop system $W(a,S,i\omega-\eta)$ is stable for all $a\in V_a$ except for the set of values $a=a^*$, for which it is on the stability boundary.

For this system,

$$\max_{a\in V_a,\omega} M(a,S,\eta,\omega)=1\Big/\varphi(a,S,\eta,\omega)=-\pi. \tag{37}$$

Here, $M(a,S,\eta,\omega)$ and $\varphi(a,S,\omega)$ are the modulus and phase of the extended frequency response of the open-loop system.

The region bounded by the extended frequency response is a mapping of all the points in the root locus plane of the closed-loop system, lying to the right of a line parallel to the imaginary axis with an x-axis of $-\eta$. Due to the properties of conformal mapping, this region expands with the growth of η. Therefore, the module of the extended frequency response $M(a,S,\eta,\omega_0(\eta))$ increases with the growth of η for each fixed value of the phase of the extended frequency response

$$\frac{\partial M_\pi(a,S,\eta,\omega)}{\partial\eta}>0. \tag{38}$$

From inequality (38), it follows that for the system to be robustly stable, it is necessary and sufficient for there to exist such $\eta>0$ for which conditions in (37) are satisfied.

The problem of selecting the controller parameters based on the conditions of maximum robust stability can be formulated as follows, taking into account the introduced definitions:

$$\eta^*=\max_S \min_{a\in V_a,\omega}\eta(S,a)\Big/M_\pi(a,S,\eta,\omega)=1. \tag{39}$$

The problem is that the function $\eta(S,a)$ cannot be expressed in analytical form.

On the set determined by the condition $M_\pi(a,S,\eta,\omega)=1$, the derivatives

$$\partial\eta/\partial a=-\frac{\partial M_\pi/\partial a}{\partial M_\pi/\partial\eta};\quad \partial\eta/\partial S=-\frac{\partial M_\pi/\partial S}{\partial M_\pi/\partial\eta}$$

due to condition (38) are opposite in sign to the derivatives of M_π over these variables, which leads to an equivalent form of the problem of maximizing robust stability:

$$\min_S \max_{a\in V_a,\omega} M_\pi(a,S,\eta^*,\omega)=1. \tag{40}$$

Here, η^* is known not to exceed (see [16])

$$\eta^0=\min_{a\in V_a,\omega}\max_S M_\pi(a,S,\eta^0,\omega)=1. \tag{41}$$

From the inequality $\eta^*\le\eta^0$, it follows that *for the system to be robustly stable, it is necessary to have a positive value η^0 of problem* (41).

For processes with the response of the open-loop system monotonically dependent on the frequency, the stability and its margin monotonically depend on each other. This leads to the equation η^* и η^0. This lets us use expressions for maximum stability obtained in the previous section when calculating robust settings.

Problems (39) and (41) are equivalent, and $\eta^* = \eta^0$ when the solution of the internal problem in (39) a^* does not depend on S. That is, the minimum stability of the system over a is achieved at the same value of the object parameter vector for any controller settings. It happens when the function $M_\pi(a, S, \eta, \omega)$ has a multiplicative or additive form.

In the first case,

$$M_\pi(a, S, \eta, \omega) = F_a(a, \eta, \omega) F_S(S, \eta, \omega). \tag{42}$$

Let us denote the maximum of F_a over $a \in V_a$ and the minimum F_S over S as $F_a^*(\omega, \eta)$ and $F_S^*(\omega, \eta)$. The condition of the maximum over ω of the product of these functions being equal to 1 determines the maximum possible robust stability. The order of the maximum and minimum operations does not affect the form of the functions F_a^* и F_S^*, which means that problems (39) and (41) are equivalent, and the robust stability can be found by initially selecting the controller settings based on the minimum condition M_π (or maximum η) for any admissible object parameters, and then finding the maximum M_π (or minimum $\eta^*(a)$) over $a \in V_a$. Similar reasoning is applicable to the additive form of the function M_π.

The modulus of the frequency response of the open-loop control system is a product of the moduli of the frequency characteristics of the object, which depend only on the parameters a, and the controller, which depends on the vector S. Therefore, this expression is multiplicative, and the achievable stability margin does not depend on the order of operations of finding the maximum of this modulus over a and the minimum over S. Because the stability and its margin are monotonically dependent on each other, the same statement holds for the stability itself. This implies an algorithm for calculating robust controller settings.

It should be noted that choosing the controller parameters based on the maximum robust stability condition is more natural than choosing based on the maximum stability margin condition as η^* is directly related to the duration of the transition in the system [17].

3.1. Algorithm for Selecting Robust Controller Settings for Technological Processes

1. The dependences of the controller's optimal settings $S^*(a)$ and corresponding maximum stability $\eta^*(a)$ on the parameters of the transfer function of the control object are found based on the conditions of the proximity of real roots of the closed-loop control system to the imaginary axis (Appendix A1).

2. The minimum value of the function $\min_{a \in V_a} \eta^*(a)$ and its corresponding (critical) parameter values a^* are found. If the obtained minimum value $\eta^*(a^*)$ is positive, the system is robustly stable, and the corresponding settings $S^*(a^*)$ are the desired ones.

It should be noted that the set V_a can be any closed and bounded set, and $\eta^*(a)$ is a continuous and bounded from the function below, which guarantees the existence of a minimum.

3.2. Robust Control System for an Aperiodic Object with Pure Delay

As an example, let us consider a system consisting of an aperiodic object with delay and a PI-controller.

The transfer function of the open-loop system is

$$W(a, p, S) = \frac{Ke^{-p\tau}}{Tp + 1} W_R(S, p), \quad a = (K, T, \tau). \tag{43}$$

Here, S is the vector of controller parameters.

The limit of stability is determined by expression (25):

$$\eta^* = \frac{1}{2T} + \frac{2}{\tau} - \sqrt{\frac{1}{4T^2} + \frac{2}{\tau^2}}. \tag{44}$$

The constraints on T and τ highlight the set V_0 of their possible values. Figure 5 shows how critical values T^* and τ^* are determined in this case. For these values, η^* is minimal (usually corresponding to the maximum of the ratio τ/T).

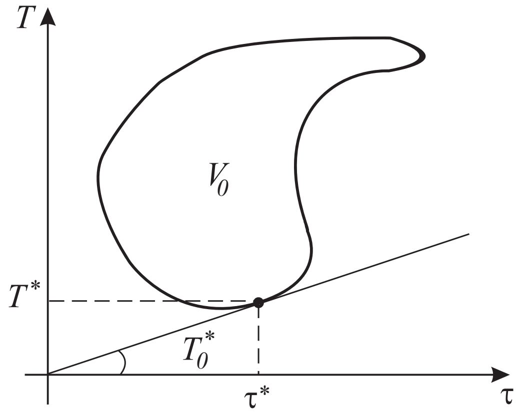

Figure 5. Set V_a of possible values of T and τ and the choice of their critical values.

After substituting the critical object parameters into expressions (26) and (27), the robust controller settings are determined.

3.3. Example

Consider a closed system with a PI controller and an object with a transfer function

$$W(p) = \frac{e^{-p}}{1.5p+1}.$$

The calculation of the limiting degree of stability and the corresponding parameters of the regulator using the formulas given in the Table gives

$$\eta^* = 0.88, S_0 = 0.37, S_1 = 0.56.$$

The transient process corresponding to a single perturbation and the maximum degree of stability is shown in Figure 6, curve (a).

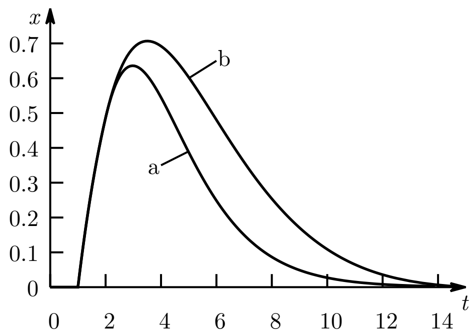

Figure 6. Process examples.

Let the parameters of the object be changed within: $0.5 \leq \tau \leq 1.5, \quad 1 \leq T \leq 2$. Robust settings correspond to the values

$$T = 1, \tau = 1,5, \quad \eta^r = 0.77, S_0 = 0.25, S_1 = 0.25.$$

The transient process in a closed system with and robust settings is shown in the same figure, curve (b).

4. Main Results

Conditions were obtained under which the limit of aperiodic stability was achieved. The problem of robust stability for single-loop linear feedback systems was considered. In these systems, the modulus and phase of the frequency response of the open-loop system monotonically decrease with frequency, and the object contains delay, so that the denominator of the transfer function of the closed-loop system is not a polynomial. An algorithm for robust selection of controller settings was proposed for control systems and technological objects containing delay.

It was proven that it is possible to choose robust settings for typical industrial controllers by minimizing the set of possible object parameters (not necessarily characterized by interval constraints) for the maximum achievable stability, which depends on these parameters.

It was shown that for positive bounded values of τ^*, T^* of the transfer function of typical technological objects, the stability η^* is positive. This means that there are parameters of typical controllers that ensure robust stability. The obtained conditions allow for solving the inverse problem: to find the range of variation of object parameters for which the robust stability of the control system will be equal or higher than the specified one. This makes it possible to design controllers with fixed settings for an entire class of objects.

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Appendix A

Table A1. The settings that correspond to the maximum of aperiodic stability η for a typical technological process.

Controller	η S	$W(p) = \frac{ke^{-p\tau}}{Tp+1}$	$W(p) = \frac{e^{-p\tau}}{\Theta p}$	$W(p) = ke^{-\tau}$
1	2	3	4	5
P	η	$\frac{1}{\tau} + \frac{1}{T}$	$\frac{1}{\tau}$	—
	S_1	$\frac{0.37T}{k\tau} e^{-\frac{\tau}{T}}$	$0.37\frac{\Theta}{\tau}$	—
PI	η	$\frac{1}{2T} + \frac{2}{\tau} - \sqrt{\frac{1}{4T^2} + \frac{2}{\tau^2}}$	$\frac{0.59}{\tau}$	$\frac{2}{\tau}$
	S_1	$\frac{e^{-\eta\tau}}{k}(2T\eta + \tau\eta - T\tau\eta^2 - 1) \approx$ $\approx \frac{0.23}{k}\left(2\frac{T}{\tau} + 1.6e^{-0.9\frac{T}{\tau}} - 1\right)$	$\frac{0.54\Theta}{\tau}$	$\frac{0.14}{k}$
	S_0	$\frac{\tau\eta^2 e^{-\eta\tau}}{k}\left(\frac{T}{\tau} - T\eta + 1\right) \approx$ $\approx \frac{0.19}{k\tau}\left(0.41\frac{T}{\tau} + 1.83e^{-1.2\frac{T}{\tau}} + 1\right)$	$0.08\frac{\Theta}{\tau^2}$	$\frac{0.54}{\tau k}$
I	η	$\frac{1}{2T} + \frac{1}{\tau} - \sqrt{\frac{1}{4T^2} + \frac{1}{\tau^2}}$	—	$\frac{1}{\tau}$
	S_0	$\frac{\eta}{k} e^{-\eta\tau}(1 - T\eta)$	—	$\frac{0.37}{\tau k}$
PD	η	$\frac{1}{T} + \frac{2}{\tau}$	$\frac{2}{\tau}$	—
	S_1	$\frac{0.14e^{-\frac{\tau}{T}}}{k}\left(4\frac{T}{\tau} + 1\right)$	$0.54\frac{\Theta}{\tau}$	—
	S_2	$\frac{0.14T}{k} e^{-\frac{\tau}{T}}$	0.14Θ	—

Table A1. *Cont.*

PID	η	$\frac{1}{2T}+\frac{3}{\tau}-\sqrt{\frac{1}{4T^2}+\frac{3}{\tau^2}}$	$\frac{1.27}{\tau}$	$\frac{3}{\tau}$
	S_1	$\frac{e^{-\eta\tau}}{k}(3T\tau\eta^2+\tau^2\eta^2-\tau\eta-T\tau^2\eta^3-1)\approx$ $\approx\frac{0.19}{k}\left(4.13\frac{T}{\tau}+2.3e^{-1.4\frac{T}{\tau}}-1\right)$	$0.79\frac{\Theta}{\tau}$	$\frac{0.25}{k}$
	S_0	$\frac{\eta^3\tau^2}{k}e^{-\eta\tau}\left(\frac{T}{\tau}-\frac{T}{2}\eta+\frac{1}{2}\right)\approx$ $\approx\frac{0.29}{k\tau}\left(0.73\frac{T}{\tau}+1.33e^{-1.5\frac{T}{\tau}}+1\right)$	$0.19\frac{\Theta}{\tau^2}$	$\frac{0.67}{\tau k}$
	S_2	$\frac{\tau e^{-\eta\tau}}{k}\left(\frac{\tau\eta}{2}+2T\eta-\frac{T}{\tau}-\frac{T\tau\eta^2}{2}-1\right)\approx$ $\approx\frac{0.1\tau}{k}\left(2\frac{T}{\tau}+1.25e^{-1.1\frac{T}{\tau}}-1\right)$	0.2Θ	$\frac{0.025\tau}{k}$

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