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Robust Observer-Based Proportional Derivative Fuzzy Control Approach for Discrete-Time Nonlinear Descriptor Systems with Transient Response Requirements

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Abstract: This paper proposes an observer-based proportional Derivative (O-BPD) fuzzy controller for uncertain discrete-time nonlinear descriptor systems (NDSs). Representing NDSs with the Takagi–Sugeno fuzzy model (T-SFM), the proportional derivative (PD) feedback method can be utilized in the fuzzy controller design via the Parallel Distributed Compensation (PDC) concept, such that the noncausal problem and impulse behavior are avoided. A fuzzy observer is proposed to obtain unmeasured states to fulfill the PD fuzzy controller. Moreover, uncertainties and transient response performances are taken into account for the NDSs. Then, a stability analysis process and corresponding stability conditions are derived from the Lyapunov theory with the robust control method and the pole constraint. Different from existing research, the Singular Value Decomposition (SVD) and the projection lemma are utilized to transfer the stability conditions into the Linear Matrix Inequation (LMI) form. Because of this reason, the conservatism of the analysis process can be reduced by eliminating the restriction on the positive definite matrix in the Lyapunov function. By giving the proper center and radius parameters of the pole constraint in the O-BPD fuzzy controller design process, the expected transient responses can be obtained for different designers and different practical applications. Finally, the effectiveness and applicability of the proposed O-BPD fuzzy controller are demonstrated by two examples of the simulation.

Keywords: Takagi–Sugeno fuzzy model; discrete-time nonlinear descriptor systems; proportional derivative feedback; observer-based control; uncertainties; regional pole placement constraint

Citation: Lin, T.-A.; Lee, Y.-C.; Chang, W.-J.; Lin, Y.-H. Robust Observer-Based Proportional Derivative Fuzzy Control Approach for Discrete-Time Nonlinear Descriptor Systems with Transient Response Requirements. *Processes* **2024**, *12*, 540. https://doi.org/10.3390/pr12030540

Academic Editors: Stefania Tronci, Massimiliano Errico, Riccardo Bacci di Capaci, Ingmar Nopens, Elena Torfs and Michael Short

Received: 23 January 2024
Revised: 27 February 2024
Accepted: 6 March 2024
Published: 9 March 2024

1. Introduction

Because of the generality in describing real physical phenomena, research on nonlinear descriptor systems (NDSs) has accelerated rapidly in recent years [1]. Including but not limited to aerospace engineering systems [2], DC motor systems [3], and circuit systems [4], NDSs have already been successfully applied to describe the dynamic behaviors of various practical systems. Different from linear systems, nonlinear systems are not proportional to the input–output relation and lack the superposition property, which make them harder to analyze. However, many innate phenomena of practical industrial and natural systems occur as nonlinear behaviors. More and more scholars devote their efforts to researching analysis methods for nonlinear systems. In 1985, the Takagi–Sugeno fuzzy model (T-SFM) was proposed to express the dynamic behaviors of nonlinear systems [5]. In accordance with the membership function of each rule, a nonlinear system can be described with fuzzy subsystems in locally linear input–output relations by IF-THEN rules. This feature can efficiently reduce the control and analysis problems of nonlinear systems to a linear problem, such that the intricacy of the fuzzy controller design method can be decreased. Because of this advantage, many performance requirements have already been considered for the control problem of nonlinear systems based on the T-SFM, such as time delay and external disturbance [6,7]. Using the representation of T-SFM, the fuzzy observer and controller can also be developed for an NDS with many valuable linear design theories.

For a descriptor system, the structure is established with two expressions, including difference equations and algebra equations [8,9], which are also called slow subsystems and fast subsystems. Owing to this advantage, the descriptor system is able to express a wider variety of physical behaviors than the normal system only constructed with differential equations. Moreover, the control method for typical linear systems can also be applied based on the representation of the descriptor system, such as time delay [10]. However, this kind of structure also leads to impulse behavior and the noncausal problem in descriptor systems. The impulse behavior is caused by the infinite eigenvalues of descriptor systems, which will cause a huge variety in a second and even lead to the instability of systems. Additionally, the noncausal problem of descriptor systems may violate the time sequence, which means the state is not only related to the initial states and past inputs but also influenced by the future inputs at any time [11,12]. This feature will lead to failure when the normal control methods are applied to descriptor systems. Moreover, the fast-changing characteristic caused by the description of fast subsystems may also increase the possibility of the system collapse. Although the controller design method becomes more challenging, the control issue of NDS is still worth being widely discussed because of the general description for various practical systems.

With the expression of the T-SFM, the T-S fuzzy descriptor system (T-SFDS) can also be constructed for NDSs to avoid the non-trivial controller design process. According to references [13,14], the Parallel Distributed Compensation (PDC) concept is introduced for the fuzzy controller design of T-SFM. Note that each linear fuzzy controller shares the same premise part of the subsystem in T-SFM. To solve the control problem of descriptor systems with a singular matrix, the Proportional Difference (PD) state feedback technique has been proposed [15,16]. In study [17], derivative feedback was proven to efficiently avoid impulse behavior and keep causality and regularity in descriptor systems. Via the derivative element in the PD state feedback technique, the invertibility of a singular matrix can be satisfied such that the description of descriptor systems can be recast to the normal system. Applying the PDC-based PD state feedback controller, the impulse behaviors and noncausal problems are solved for the NDS. To further fulfill the feedback signal in the PD controller, the unmeasurable states of the control systems are obtained by the observer [18]. With the advantage of the observer and the PD feedback, the O-BPD fuzzy controller can be designed for the NDS.

Improving the robustness of NDSs, uncertainties such as aging equipment, rust or wear and tear, are also considered. The effect of uncertainties may decrease the sensitivity when the system operates, and even lead to unexpected system failures. Moreover, modeling error between the NDS and practical systems also often exists, which will degrade the control performance with the designed controller. Because of this reason, a robust control method is introduced to solve uncertain problems such that the designed controller can be more consistent with practical applications [19,20]. Combining the linear robust control method into the PD fuzzy controller design process, the control performance of the uncertain T-SFDS (UT-SFDS) can be ensured. To implement the stability analysis process, Lyapunov theory is considered in terms of the energy concept [21]. However, analyzing the stability condition and designing the fuzzy controller design for UT-SFDS are more challenging than the typical T-SFM.

In terms of practical control systems, it is also necessary to consider the performance of transient responses. That is, the overshoot caused by the designed controller may exceed the tolerance level of the systems. This will decrease the rationality and applicability of the control method. In the present day, the pole placement approach is still widely applied to improve the transient responses because of its intuitiveness and effectiveness [22]. Compared with the method that directly assigns the pole location, the regional pole placement constraint can offer a more flexible scheme to achieve the required transient responses [23]. Moreover, performance parameters such as rising time, settling time, and overshoot can be improved and traded off by just adjusting the regional constraint. For discrete-time control systems, the pole placement constraint has been proposed to limit the poles of systems in a disk within a unit circle [24]. In [25], pole placement was successfully applied to an NDS, and a controller was designed with the T-S fuzzy model. In [26], a PD observer design method was proposed for linear descriptor systems with pole placement. For nonlinear discrete-time systems with uncertainties, the pole placement method was also combined with the covariance control theory in [27]. However, there are still scarcely any existing papers discussing the control issue of NDSs via the O-BPD fuzzy controller with the robust control and pole placement methods. Despite the improvement in robustness and transient responses, the stability condition is much more difficult to transfer into the LMI form due to the consideration of a singular matrix, uncertainties, and the pole constraint. To solve the analysis problem, mathematical techniques including projection lemma [28], the Singular Value Decomposition (SVD) technique [29], and the Schur complement [30] are merged.

Integrating the concepts mentioned above, an O-BPD fuzzy controller design method is proposed in this paper for NDSs with the requirement of robustness and transient response. Based on the representation of UT-SFDS, NDSs with uncertain problems are expressed with several linear fuzzy subsystems. To avoid impulse behavior and noncausal problems for NDSs, the PD technique is applied to design each linear fuzzy controller by using the PDC method. Moreover, a fuzzy observer is also formulated for the estimation of unmeasurable states, which is supplied for the feedback signal of the PD fuzzy controller. To achieve the expected performance of transient responses, the pole placement constraint in a disk is also considered. Then, the Lyapunov theory is applied to derive the stability conditions and analyze the stability of UT-SFDS. The robust control method and pole placement constraint are simultaneously combined into the analysis process. Via the SVD technique, Schur complement, and projection lemma, not only can the stability conditions be transferred into the LMI form, but a less conservative analysis process can also be provided. That is, the limitation of the diagonal form for the common positive definite matrix of the Lyapunov function is no longer required. Since the conditions are successfully transferred in LMI form with the mathematical techniques, the convex optimization algorithm [31] is able to be efficiently applied to solve the control problem. The contributions compared with existing research of O-BPD fuzzy controllers [32] for discrete-time NDSs are presented as follows.

(1) The uncertain problem is considered for NDSs and the robust control method is utilized in the O-BPD fuzzy controller design process to ensure robustness.
(2) Via the combination of pole constraints, transient responses can be improved for the different requirements across various practical NDSs. Moreover, the tradeoff of transient performance parameters such as rising time, settling time, and maximum overshoot can be conveniently achieved by adjusting the center and radius parameters of the stability disk.
(3) Due to the consideration of a singular matrix, uncertainties, and the pole constraint, the O-BPD fuzzy controller design problem becomes more challenging. Different from the analysis method in [32], the SVD technique is applied with a projection lemma to solve the proposed problem. Thus, the restriction on the positive definite matrix, which is required to be set as a diagonal form in the analysis process in [32], can be relaxed via the proposed design method.
(4) Although the requirement of robustness and transient responses is improved, the robust control method and pole constraint lead to a more conservative O-BPD design process. By eliminating the restriction of the positive definite matrix, a more relaxed analysis process is investigated to ensure the requirements.

Finally, simulations are provided with two examples including a numerical and bio-economic NDS to prove the effectiveness of the proposed O-BPD fuzzy controller.

The arrangement of this paper is outlined as follows. In Section 2, the UT-SFDS is built for the uncertain NDS and the O-BPD fuzzy controller is developed. In Section 3, a stability theorem is proposed for the closed-loop UT-SFDS with Lyapunov theory. In Section 4, the simulation results of a numerical descriptor system and bio-economy descriptor system are presented with the designed robust O-BPD fuzzy controller. In Section 5, some conclusions are provided for this research.

Notation: R^{m_x} and $R^{m_x \times m_y}$ *represent the vector with dimension* m_x *and the matrix with dimension* $m_x \times m_y$*, respectively.* $\mathbf{I}$ *is the identity matrix with the appropriate dimension.* $sym\{\Im\}$ *represents the short notation for* $\Im + \Im^{\mathrm{T}}$. $*$ *represents the symmetric term in the matrix.* $\Im(h)$ *represents the short notation for* $\sum_{i=1}^{m} h_i\{\Im\}$. $rank(\Im)$ *represents the rank of* $\Im$.

2. System Formulation and Problem Statement

To develop the robust O-BPD fuzzy controller, the UT-SFDS was constructed for the representation of NDSs. To improve the applicability of the robust O-BPD fuzzy controller, a fuzzy observer was also established to estimate the unmeasurable states in practical control systems. Then, an O-BPD fuzzy controller was developed with the observer state by using the PDC method. Moreover, mathematical transformation techniques including the SVD technique and projection lemma were considered to solve the derivation problem in the stability analysis. Firstly, the UT-SFDS was established with IF-THEN rules as follows.

Plant Rule *i*:

IF $\Theta_1(k)$ is M_{i1} and … and $\Theta_n(k)$ is M_{in}, THEN

$$\mathbf{E}x(k+1) = (\mathbf{A}_i + \boldsymbol{\Delta}\mathbf{A}_i(k))x(k) + (\mathbf{B}_i + \boldsymbol{\Delta}\mathbf{B}_i(k))u(k) \tag{1}$$

$$y(k) = \mathbf{C}x(k) \tag{2}$$

where premise variables are expressed as $\Theta_z(k)$, the fuzzy sets are expressed as M_{iz}, $z = 1, 2 \ \ldots, n$ is the number of premise variables, $i = 1, 2 \ \ldots, r$ is the number of fuzzy rules, and the state vector, the control input vector, and the output vector are expressed as $x(k) \in R^{m_x}$, $u(k) \in R^{m_u}$, and $y(k) \in R^{m_y}$. $\mathbf{A}_i \in R^{m_x \times m_x}$, $\mathbf{B}_i \in R^{m_x \times m_u}$, and $\mathbf{C} \in R^{m_y \times m_x}$ are constant matrices, $\mathbf{E}$ is a constant matrix with $rank(\mathbf{E}) < m_x$, and $\Delta\mathbf{A}_i(k) \in R^{m_x \times m_x}$ and $\Delta\mathbf{B}_i(k) \in R^{m_x \times m_u}$ are the time-varying uncertainties. Note that these uncertainties are expressed as $\Delta\mathbf{A}_i$ and $\Delta\mathbf{B}_i$ in the rest of this paper to save space. Then, the structure of uncertainties are described as follows:

$$\begin{bmatrix} \boldsymbol{\Delta}\mathbf{A}_i & \boldsymbol{\Delta}\mathbf{B}_i \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{Ai}\boldsymbol{\Delta}(t)\mathbf{W}_{Ai} & \mathbf{H}_{Bi}\boldsymbol{\Delta}(t)\mathbf{W}_{Bi} \end{bmatrix} \tag{3}$$

where $\mathbf{H}_{Ai}$, $\mathbf{W}_{Ai}$, $\mathbf{H}_{Bi}$, and $\mathbf{W}_{Bi}$ are real constant matrices with appropriate dimensions, and $\boldsymbol{\Delta}(t)$ is the unknown time-varying function satisfying $\boldsymbol{\Delta}^{\mathrm{T}}(t)\boldsymbol{\Delta}(t) \leq \mathbf{I}$. By the defuzzification process, the UT-SFDS (1)-(2) is obtained as follows:

$$\mathbf{E}x(k+1) = \sum_{i=1}^{r} h_i(\Theta(k))\{(\mathbf{A}_i + \boldsymbol{\Delta}\mathbf{A}_i)x(k) + (\mathbf{B}_i + \boldsymbol{\Delta}\mathbf{B}_i)u(k)\} \tag{4}$$

$$y(k) = \sum_{i=1}^{r} h_i(\Theta(k))\{\mathbf{C}x(k)\} \tag{5}$$

where $h_i(\Theta(k)) = \frac{\prod_{z=1}^{n} \mathrm{M}_{iz}(\Theta_z(k))}{\sum_{i=1}^{r}\prod_{z=1}^{n} \mathrm{M}_{iz}(\Theta_z(k))}$, $\mathrm{M}_{iz}(\Theta_z(k))$ is the grade of membership of $\Theta_z(k)$ in M_{iz}, $h_i(\Theta(k)) \geq 0$ and $\sum_{i=1}^{r} h_i(\Theta(k)) = 1$.

Based on UT-SFDS (4)-(5), the O-BPD fuzzy controller was designed to ensure the stability of NDSs with the requirement of robustness and transient responses. According to UT-SFDS (4)-(5), the fuzzy observer was also developed to estimate unmeasured states. For the problem of uncertainties in UT-SFDS (4)-(5), the robust control method was also combined into the design process. Moreover, the pole placement constraint was applied to constrain the poles of the dominant term of UT-SFDS (4)-(5) into a specific disk region.

To ensure the observability and controllability of UT-SFDS (4)-(5), a definition is provided as follows.

Definition 1. *If the relationships* $rank\begin{bmatrix} v\mathbf{E}-\mathbf{A}_i \\ \mathbf{C} \end{bmatrix} = rank\begin{bmatrix} \mathbf{E} \\ \mathbf{C} \end{bmatrix} = m_x$ *and* $rank\begin{bmatrix} v\mathbf{E}-\mathbf{A}_i & \mathbf{B}_i \end{bmatrix} = rank\begin{bmatrix} \mathbf{E} & \mathbf{B}_i \end{bmatrix} = m_x$*, where* $|v| > 1$ *and* $\forall v \in C$*, are all satisfied, the observability and controllability of UT-SFDS (4)-(5) are guaranteed.*

Similar to uncertainties, $h_i(\Theta(k))$ is defined as h_i in the following context to save space. In order to obtain the unmeasured states and complete the PD fuzzy controller, the fuzzy observer was designed as follows.

Observer Rule *i*:

IF $\Theta_1(k)$ is M_{i1} and ... and $\Theta_n(k)$ is M_{in}, THEN

$$\mathbf{E}\widehat{x}(k+1) = \mathbf{A}_i\widehat{x}(k) + \mathbf{B}_i u(k) + \mathbf{L}_{pi}\Big(y(k) - \widehat{y}(k)\Big) + \mathbf{L}_{di}\Big(y(k+1) - \widehat{y}(k+1)\Big) \tag{6}$$

$$\widehat{y}(k) = \mathbf{C}\widehat{x}(k) \tag{7}$$

where $\widehat{x}(k) \in \Re^{m_x}$ is the estimated state vector and $\widehat{y}(k) \in \Re^{m_y}$ is the output vector, and the matrices $\mathbf{L}_{pi}$ and $\mathbf{L}_{di}$ are the observer gains. With defuzzification, fuzzy observers (6)-(7) can be represented as follows.

$$\mathbf{E}\widehat{x}(k+1) = \sum_{i=1}^{r} h_i\Big\{\mathbf{A}_i\widehat{x}(k) + \mathbf{B}_i u(k) + \mathbf{L}_{pi}\Big(y(k) - \widehat{y}(k)\Big) + \mathbf{L}_{di}\Big(y(k+1) - \widehat{y}(k+1)\Big)\Big\} \tag{8}$$

$$\widehat{y}(k) = \sum_{i=1}^{r} h_i\Big\{\mathbf{C}\widehat{x}(k)\Big\} \tag{9}$$

According to the PDC method and the designed fuzzy observers (8)-(9), the O-BPD fuzzy controller was developed as follows.

Controller Rule *i*:

IF $\Theta_1(k)$ is M_{i1} and ... and $\Theta_n(k)$ is M_{in}, THEN

$$u(k) = -\mathbf{F}_{pi}\widehat{x}(k) - \mathbf{F}_{di}\widehat{x}(k+1) \tag{10}$$

where $\mathbf{F}_{pi}$ and $\mathbf{F}_{di}$ are the gains for the proportional and derivative controller. Referring to study [33], one can know that the estimated states $\widehat{x}(k+1)$ and $\widehat{x}(k)$ are, respectively, defined for the current time and the previous time.

Similarly, PD fuzzy controller (10) was also constructed as follows by defuzzification.

$$u(k) = -\sum_{i=1}^{r} h_i\Big\{\mathbf{F}_{pi}\widehat{x}(k) + \mathbf{F}_{di}\widehat{x}(k+1)\Big\} \tag{11}$$

By substituting fuzzy controller (11) into UT-SFDS (4)-(5), the following fuzzy system was obtained.

$$\mathbf{E}x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j\Big\{(\mathbf{A}_i + \Delta\mathbf{A}_i)x(k) - (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{F}_{pj}\widehat{x}(k) - (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{F}_{dj}\widehat{x}(k+1)\Big\} \tag{12}$$

In order to confirm the estimated error can achieve convergence, the error vector between real states and estimated states are $e(k) = x(k) - \widehat{x}(k)$ and $e(k+1) = x(k+1) - \widehat{x}(k+1)$. Applying the estimated error, the fuzzy system (12) can be expressed as follows.

$$\mathbf{E}x(k+1) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Big\{ (\Delta\mathbf{A}_i - \Delta\mathbf{B}_i\mathbf{F}_{pj})x(k) - \Big(\Delta\mathbf{B}_i\mathbf{F}_{dj}\Big)x(k+1) \\ +(\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{F}_{pj}e(k) + (\mathbf{B}_i + \Delta\mathbf{B}_i)\mathbf{F}_{dj}e(k+1) \Big\} \tag{13}$$

Then, the following error dynamic system was obtained by subtracting fuzzy observer (8) from fuzzy system (13).

$$\begin{aligned}\mathbf{E}e(k+1) &= \mathbf{E}x(k+1) - \mathbf{E}\widehat{x}(k+1) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Big\{ (\Delta\mathbf{A}_i - \Delta\mathbf{B}_i\mathbf{F}_{pj})x(k) - \Big(\Delta\mathbf{B}_i\mathbf{F}_{dj}\Big)x(k+1) \\ &\quad + (\mathbf{A}_i + \Delta\mathbf{B}_i\mathbf{F}_{pj} - \mathbf{L}_{pi})e(k) + \Big(\Delta\mathbf{B}_i\mathbf{F}_{dj} - \mathbf{L}_{di}\mathbf{C}\Big)e(k+1) \Big\}\end{aligned} \tag{14}$$

Consequently, the following augmented system was built according to fuzzy system (4) and error dynamic system (14).

$$\begin{bmatrix}\mathbf{E} & 0 \\ 0 & \mathbf{E}\end{bmatrix}\begin{bmatrix}x(k+1) \\ e(k+1)\end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Bigg\{ \begin{bmatrix}\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_{pj} + \Delta\mathbf{A}_i - \Delta\mathbf{B}_i\mathbf{F}_{pj} & \mathbf{B}_i\mathbf{F}_{pj} + \Delta\mathbf{B}_i\mathbf{F}_{pj} \\ \Delta\mathbf{A}_i - \Delta\mathbf{B}_i\mathbf{F}_{pj} & \mathbf{A}_i - \mathbf{L}_{pi}\mathbf{C} + \Delta\mathbf{B}_i\mathbf{F}_{pj}\end{bmatrix}\begin{bmatrix}x(k) \\ e(k)\end{bmatrix} \\ + \begin{bmatrix}-\mathbf{B}_i\mathbf{F}_{dj} - \Delta\mathbf{B}_i\mathbf{F}_{dj} & \mathbf{B}_i\mathbf{F}_{dj} + \Delta\mathbf{B}_i\mathbf{F}_{dj} \\ -\Delta\mathbf{B}_i\mathbf{F}_{dj} & -\mathbf{L}_{di}\mathbf{C} + \Delta\mathbf{B}_i\mathbf{F}_{dj}\end{bmatrix}\begin{bmatrix}x(k+1) \\ e(k+1)\end{bmatrix} \Bigg\} \tag{15}$$

With definition (3), fuzzy system (15) can be further represented as follows.

$$\tilde{\mathbf{E}}_{Rij}(h)\tilde{x}(k+1) = \tilde{\mathbf{A}}_{Rij}(h)\tilde{x}(k) \tag{16}$$

where $\tilde{x}(k) = \begin{bmatrix}x^{\mathrm{T}}(k) & e^{\mathrm{T}}(k)\end{bmatrix}^{\mathrm{T}}$, $\tilde{x}(k+1) = \begin{bmatrix}x^{\mathrm{T}}(k+1) & e^{\mathrm{T}}(k+1)\end{bmatrix}^{\mathrm{T}}$, $\tilde{\mathbf{E}}_{Rij}(h) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Big\{\tilde{\mathbf{E}}_{Rij}\Big\}$, $\tilde{\mathbf{E}}_{Rij}(h) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j \Big\{\tilde{\mathbf{E}}_{Rij}\Big\}$, $\tilde{\mathbf{E}}_{Rij} = \mathbf{E}_{Rij} + \tilde{\mathbf{H}}_{Bi}\mathbf{\Delta}(t)\tilde{\mathbf{W}}_{Bdi}$, $\mathbf{E}_{Rij} = \begin{bmatrix}\mathbf{E} + \mathbf{B}_i\mathbf{F}_{dj} & -\mathbf{B}_i\mathbf{F}_{dj} \\ 0 & \mathbf{E} + \mathbf{L}_{di}\mathbf{C}\end{bmatrix}$, $\tilde{\mathbf{H}}_{Bi} = \begin{bmatrix}\mathbf{H}_{Bi} \\ \mathbf{H}_{Bi}\end{bmatrix}$, $\tilde{\mathbf{W}}_{Bdi} = \begin{bmatrix}\mathbf{W}_{Bi}\mathbf{F}_{dj} & -\mathbf{W}_{Bi}\mathbf{F}_{dj}\end{bmatrix}$, $\tilde{\mathbf{A}}_{Rij} = \mathbf{A}_{Rij} + \tilde{\mathbf{H}}_{Ai}\mathbf{\Delta}(t)\tilde{\mathbf{W}}_{Ai} + \tilde{\mathbf{H}}_{Bi}\mathbf{\Delta}(t)\tilde{\mathbf{W}}_{Bpi}$, $\mathbf{A}_{Rij} = \begin{bmatrix}\mathbf{A}_i - \mathbf{B}_i\mathbf{F}_{pj} & \mathbf{B}_i\mathbf{F}_{pj} \\ 0 & \mathbf{A}_i - \mathbf{L}_{pi}\mathbf{C}\end{bmatrix}$, $\tilde{\mathbf{H}}_{Ai} = \begin{bmatrix}\mathbf{H}_{Ai} \\ \mathbf{H}_{Ai}\end{bmatrix}$, $\tilde{\mathbf{W}}_{Ai} = \begin{bmatrix}\mathbf{W}_{Ai} & 0\end{bmatrix}$ and $\tilde{\mathbf{W}}_{Bpi} = \begin{bmatrix}-\mathbf{W}_{Bi}\mathbf{F}_{pj} & \mathbf{W}_{Bi}\mathbf{F}_{pj}\end{bmatrix}$.

It is worth noting that the original singular matrix $\mathbf{E}$ in the UT-SFD (4)-(5) become an invertible matrix $\tilde{\mathbf{E}}_{Rij}(h)$ in (16) because of the PD fuzzy controller and observer. Because of this reason, the inverse matrix $\tilde{\mathbf{E}}_{Rij}^{-1}(h)$ can be multiplied on the left-hand side of (16), and the following UT-SFDS can be obtained.

$$\tilde{x}(k+1) = \tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h)\tilde{x}(k) \tag{17}$$

According to Definition 1, it is seen that UT-SFDS (17) can meet the requirements of controllability and observability. Moreover, the impulse behavior and noncausal problem of UT-SFDS (4)-(5) can also be solved by the description of (17). For the uncertainties existing in the stability analysis process, the following lemma is introduced for the time-varying term.

Lemma 1 ([19]). *By giving the appropriate dimension of matrices* $\mathbf{H}$ *and* $\mathbf{W}$ *satisfying* $\mathbf{\Delta}^{\mathrm{T}}(t)\mathbf{\Delta}(t) \le \mathbf{I}$ *for the time-varying term* $\mathbf{\Delta}(t)$*, the following relationship with a scalar* $\varepsilon > 0$ *can be found.*

$$\mathbf{H}^{\mathrm{T}}\mathbf{\Delta}^{\mathrm{T}}(t)\mathbf{W} + \mathbf{W}^{\mathrm{T}}\mathbf{\Delta}(t)\mathbf{H} \le \varepsilon\mathbf{H}^{\mathrm{T}}\mathbf{H} + \varepsilon^{-1}\mathbf{W}^{\mathrm{T}}\mathbf{W} \tag{18}$$

For the stability analysis of UT-SFDS (17), the projection lemma is also provided for the derivation as follows.

Lemma 2 ([28]). *For any matrix* $\mathbf{\Psi}$*, if and only if the given matrices* $\boldsymbol{\rho} \in R^{m_\rho \times m_\chi}$*,* $\boldsymbol{\omega} \in R^{m_\omega \times m_\chi}$*, and the symmetric matrix* $\boldsymbol{\chi} \in R^{m_\chi \times m_\chi}$ *satisfy* $rank(\boldsymbol{\rho}) < m_\chi$ *and* $rank(\boldsymbol{\omega}) < m_\chi$*, such that*

$$\chi + \rho^{\mathrm{T}}\Psi\omega + \omega^{\mathrm{T}}\Psi^{\mathrm{T}}\rho < 0 \tag{19}$$

then the following two inequalities are also held.

$$\rho_{\perp}^{\mathrm{T}}\chi\rho_{\perp} < 0 \text{ and } \omega_{\perp}^{\mathrm{T}}\chi\omega_{\perp} < 0 \tag{20}$$

where $\rho_{\perp}$ and $\omega_{\perp}$ are the null-space matrices of ρ and ω, respectively.

Additionally, the following SVD technique is provided to make it possible for the stability conditions to be transferred into LMI form.

Lemma 3 ([29])**.** *For a time-invariant matrix $\mathbf{C}$, the structure can be decomposed as follows by the SVD technique.*

$$\mathbf{C} = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & \mathbf{0}\end{bmatrix}\mathbf{V}^{\mathrm{T}} \tag{21}$$

where $\mathbf{U} \in R^{m_y \times m_y}$ and $\mathbf{V} \in R^{m_x \times m_x}$ are the orthogonal matrices, and $\boldsymbol{\Sigma} \in R^{m_y \times m_y}$ is the diagonal matrix with positive diagonal elements.

Ensuring better transient responses for NDSs based on fuzzy system (17), the so-called D-stable pole placement constraint is introduced in the following lemma to constrain all the poles in the disk.

Lemma 4 ([24])**.** *The poles of the dominant term in the closed-loop T-S fuzzy system (17) are constrained in a disk $\mathfrak{D}(q,\gamma)$ if there exists a common positive matrix $\mathbf{P}$, such that the following condition is established.*

$$\begin{bmatrix} -\mathbf{P}^{-1} & \boldsymbol{\Gamma}_{ii}(h) - q\mathbf{I} \\ * & -\gamma^2\mathbf{P} \end{bmatrix} < 0 \tag{22}$$

where $\boldsymbol{\Gamma}_{ii}(h) = \tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h)$; q and γ are the center and radius of the disk.

Merging the use of Lemma 1 into Lemma 4 and the Schur complement, the stability criteria are proposed based on closed-loop UT-SFDS (17) by Lyapunov theory to achieve better robustness and transient responses.

3. O-BPD Fuzzy Controller Design Method

In this section, the stability analysis process and design method of O-BPD fuzzy controller (11) are developed for NDSs with UT-SFDS (17) and Lyapunov theory. Moreover, the stability conditions with pole constraint (22) are derived and transferred into LMI form via the robust control method (18), projection lemma (19)-(20), SVD (21), and Schur complement to guarantee the stability of UT-SFDS (17). Therefore, the robustness and transient response performances can be improved for NDSs. Firstly, the following theorem is introduced to ensure stability and constrain all the dominant terms' poles of UT-SFDS (17) into a disk $\mathfrak{D}(q,\gamma)$.

Theorem 1. *If there exist the gains $\mathbf{F}_{pj}$, $\mathbf{F}_{dj}$, $\mathbf{L}_{pi}$ and $\mathbf{L}_{di}$, common positive definite matrix $\mathbf{P}$, $\mathbf{R}$, and α scalar ε, with the given parameters α, $\mathbf{V}$, q, $\tilde{\mathbf{H}}_{Ai}$, $\tilde{\mathbf{H}}_{Bi}$, $\tilde{\mathbf{W}}_{Ai}$, $\tilde{\mathbf{W}}_{Bdi}$, and $\tilde{\mathbf{W}}_{Bpi}$, such that the following sufficient conditions are satisfied, then UT-SFDS (17) is asymptotically stable and the poles are located in the region $\mathfrak{D}(q,\gamma)$*

$$\begin{bmatrix} \mathbf{P} - sym\left\{ \mathbf{E}_{Rij}{}^{\mathrm{T}}\mathbf{\Psi} \right\} + \boldsymbol{\varphi}_{11} & \mathbf{\Psi}^{\mathrm{T}}\mathbf{A}_{Rij} + \boldsymbol{\varphi}_{12} \\ * & -\mathbf{P} + \boldsymbol{\varphi}_{22} \end{bmatrix} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i \neq j \tag{23}$$

$$\begin{bmatrix} \mathbf{P} - sym\{ \mathbf{E}_{Rii}{}^{\mathrm{T}}\mathbf{\Psi} \} + \hat{\boldsymbol{\varphi}}_{11} & \mathbf{\Psi}^{\mathrm{T}}(\mathbf{A}_{Rii} - q\mathbf{E}_{Rii}) + \hat{\boldsymbol{\varphi}}_{12} \\ * & -\gamma^2\mathbf{P} + \hat{\boldsymbol{\varphi}}_{22} \end{bmatrix} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i = j \tag{24}$$

where $\boldsymbol{\varphi}_{11} = \varepsilon\mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi}\tilde{\mathbf{H}}_{Bi}^{\mathrm{T}}\mathbf{\Psi} + \varepsilon^{-1}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}\tilde{\mathbf{W}}_{Bdi} + \varepsilon\mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai}\tilde{\mathbf{H}}_{Ai}^{\mathrm{T}}\mathbf{\Psi}$, $\boldsymbol{\varphi}_{12} = -\varepsilon^{-1}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}\tilde{\mathbf{W}}_{Bdi}$, $\boldsymbol{\varphi}_{22} = \varepsilon^{-1}\tilde{\mathbf{W}}_{Ai}^{\mathrm{T}}\tilde{\mathbf{W}}_{Ai}$, $\boldsymbol{\varphi}_{11} = \hat{\boldsymbol{\varphi}}_{11}$, $\hat{\boldsymbol{\varphi}}_{12} = \varepsilon^{-1}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}\left(\tilde{\mathbf{W}}_{Bpi} - q\tilde{\mathbf{W}}_{Bdi}\right)$, $\hat{\boldsymbol{\varphi}}_{22} = \varepsilon^{-1}\left(\tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} - q\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}\right)\left(\tilde{\mathbf{W}}_{Bpi} - q\tilde{\mathbf{W}}_{Bdi}\right) + \varepsilon^{-1}\tilde{\mathbf{W}}_{\mathrm{A}i}^{\mathrm{T}}\tilde{\mathbf{W}}_{\mathrm{A}i}$, $\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ * & \mathbf{P}_{22} \end{bmatrix}$, and $\mathbf{\Psi} = \begin{bmatrix} \mathbf{VRV}^{\mathrm{T}} & 0 \\ * & \alpha\mathbf{VRV}^{\mathrm{T}} \end{bmatrix}$.

Proof of Theorem 1. Firstly, the quadratic Lyapunov function is defined as follows for the energy analysis of UT-SFDS (17).

$$V(\tilde{x}(k)) = \tilde{x}^{\mathrm{T}}(k)\mathbf{P}\tilde{x}(k) \tag{25}$$

The difference of Lyapunov Function (25) are obtained as follows.

$$\begin{aligned} \Delta V(\tilde{x}(k)) &= V(\tilde{x}(k+1)) - V(\tilde{x}(k)) = \tilde{x}^{\mathrm{T}}(k+1)\mathbf{P}\tilde{x}(k+1) - \tilde{x}^{\mathrm{T}}(k)\mathbf{P}\tilde{x}(k) \\ &= \left(\tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h)\tilde{x}(k)\right)^{\mathrm{T}}\mathbf{P}\left(\tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h)\tilde{x}(k)\right) - \tilde{x}^{\mathrm{T}}(k)\mathbf{P}\tilde{x}(k) \\ &= \tilde{x}^{\mathrm{T}}(k)\left(\tilde{\mathbf{A}}_{Rij}^{\mathrm{T}}(h)\tilde{\mathbf{E}}_{Rij}^{-\mathrm{T}}(h)\mathbf{P}\tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h) - \mathbf{P}\right)\tilde{x}(k) \end{aligned} \tag{26}$$

If conditions (23) and (24) are satisfied by Theorem 1, then the following two conditions can be obtained by separating the items related to uncertainties (3).

$$\begin{aligned} &\begin{bmatrix} \mathbf{P} - sym\left\{ \mathbf{E}_{Rij}{}^{\mathrm{T}}\mathbf{\Psi} \right\} & \mathbf{\Psi}^{\mathrm{T}}\mathbf{A}_{Rij} \\ * & -\mathbf{P} \end{bmatrix} + \varepsilon\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi} \\ 0 \end{bmatrix}\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}\begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ &+\varepsilon\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai} \\ 0 \end{bmatrix}\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i \neq j \end{aligned} \tag{27}$$

$$\begin{aligned} &\begin{bmatrix} \mathbf{P} - sym\left\{ \mathbf{E}_{Rij}{}^{\mathrm{T}}\mathbf{\Psi} \right\} & \mathbf{\Psi}^{\mathrm{T}}\left(\mathbf{A}_{Rij} - q\mathbf{E}_{Rij}\right) \\ * & -\gamma^2\mathbf{P} \end{bmatrix} + \varepsilon\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi} \\ 0 \end{bmatrix}\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}\begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} - q\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} - q\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \\ &+\varepsilon\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai} \\ 0 \end{bmatrix}\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai} \\ 0 \end{bmatrix}^{\mathrm{T}} + \varepsilon^{-1}\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i = j \end{aligned} \tag{28}$$

Applying Lemma 1, condition (27) can also be satisfied by the definition of membership functions $\sum_{i=1}^{r} h_i = 1, 0 \leq h_i \leq 1$.

$$\begin{aligned} &\begin{bmatrix} \mathbf{P} - sym\left\{ \mathbf{E}_{Rij}^{\mathrm{T}}(h)\mathbf{\Psi} \right\} & \mathbf{\Psi}^{\mathrm{T}}\mathbf{A}_{Rij}(h) \\ * & -\mathbf{P} \end{bmatrix} + \begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai}(h) \\ 0 \end{bmatrix}\mathbf{\Delta}(t)\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}}(h) \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}(h) \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}}(h) \end{bmatrix}\mathbf{\Delta}^{\mathrm{T}}(t)\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi}(h) \\ 0 \end{bmatrix}^{\mathrm{T}} \\ &+\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}}(h) \end{bmatrix}\mathbf{\Delta}^{\mathrm{T}}(t)\begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai}(h) \\ 0 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} \mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi}(h) \\ 0 \end{bmatrix}\mathbf{\Delta}(t)\begin{bmatrix} -\tilde{\mathbf{W}}_{\mathrm{B}di}^{\mathrm{T}}(h) \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}}(h) \end{bmatrix}^{\mathrm{T}} \\ &= \boldsymbol{\chi} + \boldsymbol{\rho}^{\mathrm{T}}\mathbf{\Psi}\boldsymbol{\omega} + \boldsymbol{\omega}^{\mathrm{T}}\mathbf{\Psi}^{\mathrm{T}}\boldsymbol{\rho} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i \neq j \end{aligned} \tag{29}$$

where $\boldsymbol{\rho} = \begin{bmatrix} -\tilde{\mathbf{E}}_{Rij}(h) & \tilde{\mathbf{A}}_{Rij}(h) \end{bmatrix}$, $\boldsymbol{\omega} = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}$ and $\boldsymbol{\chi} = \begin{bmatrix} \mathbf{P} & 0 \\ * & -\mathbf{P} \end{bmatrix}$.

To develop stability analysis with the projection lemma, the orthogonal vectors $\boldsymbol{\rho}_{\perp}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{A}}_{Rij}^{\mathrm{T}}(h)\tilde{\mathbf{E}}_{Rij}^{-\mathrm{T}}(h) & \mathbf{I} \end{bmatrix}$ and $\boldsymbol{\omega}_{\perp}^{\mathrm{T}} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}$ are chosen according to the matrices $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ in condition (29). Based on the orthogonal vectors, the following two equivalent conditions can also be satisfied with Lemma 2.

$$\tilde{\mathbf{A}}_{Rij}^{\mathrm{T}}(h)\tilde{\mathbf{E}}_{Rij}^{-\mathrm{T}}(h)\mathbf{P}\tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h) - \mathbf{P} = \boldsymbol{\rho}_{\perp}^{\mathrm{T}}\boldsymbol{\chi}\boldsymbol{\rho}_{\perp} < 0 \tag{30}$$

and

$$-\mathbf{P} = \boldsymbol{\omega}_{\perp}^{\mathrm{T}}\boldsymbol{\chi}\boldsymbol{\omega}_{\perp} < 0 \tag{31}$$

With the same concept from (27) to (31), the condition (28) can be derived into the following form by applying Lemmas 1 and 2.

$$\begin{aligned}
&\begin{bmatrix} \mathbf{P} - sym\left\{\mathbf{E}_{Rij}^{\mathrm{T}}(h)\boldsymbol{\Psi}\right\} & \boldsymbol{\Psi}^{\mathrm{T}}\left(\mathbf{A}_{Rij}(h) - q\mathbf{E}_{Rij}(h)\right) \\ * & -\gamma^2\mathbf{P} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai}(h) \\ 0 \end{bmatrix}\boldsymbol{\Delta}(t)\begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}}(h) \end{bmatrix}^{\mathrm{T}} \\
&+ \begin{bmatrix} -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}(h) \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}}(h) - q\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}}(h) \end{bmatrix}\boldsymbol{\Delta}^{\mathrm{T}}(t)\begin{bmatrix} \boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi}(h) \\ 0 \end{bmatrix}^{\mathrm{T}} + \begin{bmatrix} 0 \\ \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}}(h) \end{bmatrix}\boldsymbol{\Delta}^{\mathrm{T}}(t)\begin{bmatrix} \boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai}(h) \\ 0 \end{bmatrix}^{\mathrm{T}} \\
&+ \begin{bmatrix} \boldsymbol{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi}(h) \\ 0 \end{bmatrix}\boldsymbol{\Delta}(t)\begin{bmatrix} -\tilde{\mathbf{W}}_{\mathrm{B}di}^{\mathrm{T}}(h) \\ \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}}(h) - q\tilde{\mathbf{W}}_{\mathrm{B}di}^{\mathrm{T}}(h) \end{bmatrix}^{\mathrm{T}} \\
&= \hat{\boldsymbol{\chi}} + \hat{\boldsymbol{\rho}}^{\mathrm{T}}\boldsymbol{\Psi}\hat{\boldsymbol{\omega}} + \hat{\boldsymbol{\omega}}^{\mathrm{T}}\boldsymbol{\Psi}^{\mathrm{T}}\hat{\boldsymbol{\rho}} < 0, \text{ for } i,j = 1,2,\ldots,r \text{ and } i = j
\end{aligned} \tag{32}$$

where $\hat{\boldsymbol{\rho}} = \begin{bmatrix} -\tilde{\mathbf{E}}_{Rij}(h) & \tilde{\mathbf{A}}_{Rij}(h) - q\tilde{\mathbf{E}}_{Rij}(h) \end{bmatrix}$, $\hat{\boldsymbol{\omega}} = \begin{bmatrix} \mathbf{I} & 0 \end{bmatrix}$, and $\hat{\boldsymbol{\chi}} = \begin{bmatrix} \mathbf{P} & 0 \\ * & -\gamma^2\mathbf{P} \end{bmatrix}$. The orthogonal vectors $\hat{\boldsymbol{\rho}}_{\perp}^{\mathrm{T}} = \begin{bmatrix} \tilde{\mathbf{A}}_{Rij}^{\mathrm{T}}(h)\tilde{\mathbf{E}}_{Rij}^{-\mathrm{T}}(h) - q\mathbf{I} & \mathbf{I} \end{bmatrix}$ and $\hat{\boldsymbol{\omega}}_{\perp}^{\mathrm{T}} = \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix}$ are chosen in condition (32). The following two equivalent conditions are established.

$$\left(\tilde{\mathbf{A}}_{Rij}^{\mathrm{T}}(h)\tilde{\mathbf{E}}_{Rij}^{-\mathrm{T}}(h) - q\mathbf{I}\right)\mathbf{P}\left(\tilde{\mathbf{E}}_{Rij}^{-1}(h)\tilde{\mathbf{A}}_{Rij}(h) - q\mathbf{I}\right) - \gamma^2\mathbf{P} = \hat{\boldsymbol{\rho}}_{\perp}^{\mathrm{T}}\hat{\boldsymbol{\chi}}\hat{\boldsymbol{\rho}}_{\perp} < 0 \tag{33}$$

and

$$-\mathbf{P} = \hat{\boldsymbol{\omega}}_{\perp}^{\mathrm{T}}\hat{\boldsymbol{\chi}}\hat{\boldsymbol{\omega}}_{\perp} < 0 \tag{34}$$

Therefore, one can know that conditions (30) and (31) can be achieved by satisfying condition (23) in Theorem 1 because of Lemmas 1 and 2. Similarly, conditions (33) and (34) can also be satisfied by condition (24). Via the Schur complement, it is obvious that the pole placement constraint (22) is satisfied by condition (33). Therefore, all the poles of dominant term in (17) can be forced in the designed disk region $\mathfrak{D}(q,\gamma)$. Moreover, if the center and radius parameters of the pole constraint are set as $q = 0$ and $\gamma = 1$, the stability condition with the form (30) for the case $i = j$ is also satisfied by condition (33). This also means that the fact $\Delta V(\tilde{x}(k)) < 0$ is also achieved via (26). It is worth noting that the case of all the pairs of center and radius, which are located in the unit circle, can also achieve the stability and pole constraint (22). It can be concluded that if conditions (23) and (24) are satisfied by the O-BPD fuzzy controller design method in Theorem 1, then UT-SFDS (17) is asymptotically stable and all the poles of dominant terms are placed in the disk $\mathfrak{D}(q,\gamma)$ by satisfying the pole constraint (22).

However, stability conditions (23) and (24) still are not presented in LMI form and are unable to be solved with the convex optimization algorithm in programs. Because of this reason, the SVD technique and Schur complement are applied in the following theorem to convert conditions (23) and (24) into LMI form. □

Theorem 2. *If there exist the matrices* $\mathbf{Q}_{dj}$, $\mathbf{Q}_{pj}$, $\mathbf{G}_{di}$, *and* $\mathbf{G}_{pi}$, *definite matrices* $\tilde{\mathbf{P}}_{11}$ *and* $\tilde{\mathbf{P}}_{22}$, matrices $\tilde{\mathbf{P}}_{12}$ *and* $\mathbf{R} = \begin{bmatrix} \mathbf{Z}_{11} & 0 \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}$, *and a scalar* ε, *with the given parameters* α, $\mathbf{V}$, q, γ, $\mathbf{H}_{Ai}$, $\mathbf{H}_{Bi}$, $\mathbf{W}_{Ai}$, *and* $\mathbf{W}_{Bi}$, *such that the following sufficient conditions are satisfied, then the closed-loop UT-SFDS (17) is asymptotically stable and the pole placement constraint (22) can be satisfied.*

$$\begin{bmatrix} \mathbf{\Omega}_{11} & \tilde{\mathbf{P}}_3 + \alpha \mathbf{B}_i \mathbf{Q}_{dj} & \mathbf{\Omega}_{13} & \alpha \mathbf{B}_i \mathbf{Q}_{pj} & \varepsilon \mathbf{H}_{Bi} & \varepsilon \mathbf{H}_{Ai} & -\mathbf{Q}_{dj}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & \mathbf{\Omega}_{22} & 0 & \mathbf{\Omega}_{24} & \varepsilon \mathbf{H}_{Bi} & \varepsilon \mathbf{H}_{Ai} & \alpha \mathbf{Q}_{dj}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & * & -\tilde{\mathbf{P}}_{11} & -\tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pj}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & \mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\ * & * & * & -\tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha \mathbf{Q}_{pj}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & * & * & * & -\varepsilon \mathbf{I} & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon \mathbf{I} & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon \mathbf{I} & 0 \\ * & * & * & * & * & * & * & -\varepsilon \mathbf{I} \end{bmatrix} < 0 \tag{35}$$
$$\text{for } i, j = 1, 2, \ldots, r \text{ and } i \neq j$$

$$\begin{bmatrix} \mathbf{\Omega}_{11} & \tilde{\mathbf{P}}_{12} + \alpha \mathbf{B}_i \mathbf{Q}_{di} & \hat{\mathbf{\Omega}}_{13} & \alpha \mathbf{B}_i \mathbf{Q}_{pi} + \alpha q \mathbf{B}_i \mathbf{Q}_{di} & \varepsilon \mathbf{H}_{Bi} & \varepsilon \mathbf{H}_{Ai} & -\mathbf{Q}_{di}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & \mathbf{\Omega}_{22} & 0 & \hat{\mathbf{\Omega}}_{24} & \varepsilon \mathbf{H}_{Bi} & \varepsilon \mathbf{H}_{Ai} & \alpha \mathbf{Q}_{di}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & * & -\gamma^2 \tilde{\mathbf{P}}_{11} & -\gamma^2 \tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pi}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} - \hat{\mathbf{\Omega}}_7 & \mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\ * & * & * & -\gamma^2 \tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha \mathbf{Q}_{pi}^{\mathrm{T}} \mathbf{W}_{Bi}^{\mathrm{T}} + \hat{\mathbf{\Omega}}_7 & 0 \\ * & * & * & * & -\varepsilon \mathbf{I} & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon \mathbf{I} & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon \mathbf{I} & 0 \\ * & * & * & * & * & * & * & -\varepsilon \mathbf{I} \end{bmatrix} < 0 \tag{36}$$
$$\text{for } i, j = 1, 2, \ldots, r \text{ and } i = j$$

where $\mathbf{\Omega}_{11} = \tilde{\mathbf{P}}_{11} - sym\left\{ \mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} + \mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}} \right\}$, $\mathbf{\Omega}_{13} = \mathbf{A}_i\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \mathbf{B}_i\mathbf{Q}_{pj}$, $\mathbf{\Omega}_{22} = \tilde{\mathbf{P}}_{22} - sym\left\{ \alpha\mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} + \alpha\mathbf{C}^{\mathrm{T}}\mathbf{G}_{di}^{\mathrm{T}} \right\}$, $\mathbf{\Omega}_{24} = \alpha\mathbf{A}_i\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \alpha\mathbf{G}_{pi}\mathbf{C}$, $\hat{\mathbf{\Omega}}_{13} = \mathbf{A}_i\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \mathbf{B}_i\mathbf{Q}_{pi} - q\mathbf{E}\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - q\mathbf{B}_i\mathbf{Q}_{di}$, $\hat{\mathbf{\Omega}}_{24} = \alpha\mathbf{A}_i\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \alpha\mathbf{G}_{di}\mathbf{C} - \alpha q\mathbf{E}\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \alpha q\mathbf{G}_{di}\mathbf{C}$, $\hat{\mathbf{\Omega}}_7 = q\mathbf{Q}_{di}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}$, $\mathbf{G}_{di} = \mathbf{L}_{di}\overset{\frown}{\mathbf{R}}$, and $\mathbf{G}_{pi} = \mathbf{L}_{pi}\overset{\frown}{\mathbf{R}}$.

Proof of Theorem 2. Applying the Schur complement to conditions (23) and (24), the following two inequalities can be obtained.

$$\begin{bmatrix} \mathbf{P} - sym\left\{ \mathbf{E}_{Rij}^{\mathrm{T}} \mathbf{\Psi} \right\} & \mathbf{\Psi}^{\mathrm{T}} \mathbf{A}_{Rij} & \varepsilon \mathbf{\Psi}^{\mathrm{T}} \tilde{\mathbf{H}}_{Bi} & \varepsilon \mathbf{\Psi}^{\mathrm{T}} \tilde{\mathbf{H}}_{Ai} & -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & 0 \\ * & -\mathbf{P} & 0 & 0 & \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} & \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \\ * & * & -\varepsilon \mathbf{I} & 0 & 0 & 0 \\ * & * & * & -\varepsilon \mathbf{I} & 0 & 0 \\ * & * & * & * & -\varepsilon \mathbf{I} & 0 \\ * & * & * & * & * & -\varepsilon \mathbf{I} \end{bmatrix} < 0, \tag{37}$$
$$\text{for } i, j = 1, 2, \ldots, r \text{ and } i \neq j$$

$$\begin{bmatrix} \mathbf{P} - sym\left\{\mathbf{E}_{Rij}^{\mathrm{T}}\mathbf{\Psi}\right\} & \mathbf{\Psi}^{\mathrm{T}}\left(\mathbf{A}_{Rij} - q\mathbf{E}_{Rij}\right) & \varepsilon\mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Bi} & \varepsilon\mathbf{\Psi}^{\mathrm{T}}\tilde{\mathbf{H}}_{Ai} & -\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & 0 \\ * & -\gamma^2\mathbf{P} & 0 & 0 & \tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} - q - \tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & \tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \\ * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \quad \text{for } i,j = 1,2,\ldots,r \text{ and } i = j \tag{38}$$

Then, the following condition can also be obtained by multiplying on the left-hand side of (37)-(38) by $diag\{\mathbf{\Psi}^{-\mathrm{T}}, \mathbf{\Psi}^{-\mathrm{T}}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}\}$ and on the right-hand side by $diag\{\mathbf{\Psi}^{-1}, \mathbf{\Psi}^{-1}, \mathbf{I}, \mathbf{I}, \mathbf{I}, \mathbf{I}\}$ respectively.

$$\begin{bmatrix} \tilde{\mathbf{P}} - sym\left\{\mathbf{\Psi}^{-\mathrm{T}}\mathbf{E}_{Rij}^{\mathrm{T}}\right\} & \mathbf{A}_{Rij}\mathbf{\Psi}^{-1} & \varepsilon\tilde{\mathbf{H}}_{Bi} & \varepsilon\tilde{\mathbf{H}}_{Ai} & -\mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & 0 \\ * & -\tilde{\mathbf{P}} & 0 & 0 & \mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} & \mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \\ * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \quad \text{for } i,j = 1,2,\ldots,r \text{ and } i \neq j \tag{39}$$

$$\begin{bmatrix} \tilde{\mathbf{P}} - sym\left\{\mathbf{\Psi}^{-\mathrm{T}}\mathbf{E}_{Rij}^{\mathrm{T}}\right\} & \left(\mathbf{A}_{Rij} - q\mathbf{E}_{Rij}\right)\mathbf{\Psi}^{-1} & \varepsilon\tilde{\mathbf{H}}_{Bi} & \varepsilon\tilde{\mathbf{H}}_{Ai} & -\mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & 0 \\ * & -\gamma^2\tilde{\mathbf{P}} & 0 & 0 & \mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Bpi}^{\mathrm{T}} - q\mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Bdi}^{\mathrm{T}} & \mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{W}}_{Ai}^{\mathrm{T}} \\ * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\ * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \quad \text{for } i,j = 1,2,\ldots,r \text{ and } i = j \tag{40}$$

where $\tilde{\mathbf{P}} = \mathbf{\Psi}^{-\mathrm{T}}\mathbf{P}\mathbf{\Psi}^{-1}$ and $\tilde{\mathbf{P}} = \begin{bmatrix} \tilde{\mathbf{P}}_{11} & \tilde{\mathbf{P}}_{12} \\ * & \tilde{\mathbf{P}}_{22} \end{bmatrix}$. Then, defining the $\mathbf{Q}_{dj}$ and $\mathbf{Q}_{pj}$ to substitute $\mathbf{F}_{dj}\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}}$ and $\mathbf{F}_{pj}\mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}}$, inequalities (39) and (40) can be rewritten as follows.

$$\begin{bmatrix} \tilde{\mathbf{\Omega}}_{11} & \tilde{\mathbf{P}}_{12} + \alpha\mathbf{B}_i\mathbf{Q}_{dj} & \tilde{\mathbf{\Omega}}_{13} & \alpha\mathbf{B}_i\mathbf{Q}_{pj} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & -\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & \tilde{\mathbf{\Omega}}_{22} & 0 & \tilde{\mathbf{\Omega}}_{24} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & \alpha\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & * & -\tilde{\mathbf{P}}_{11} & -\tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & \mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\ * & * & * & -\tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\ * & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon\mathbf{I} & 0 \\ * & * & * & * & * & * & * & -\varepsilon\mathbf{I} \end{bmatrix} < 0, \quad \text{for } i,j = 1,2,\ldots,r \text{ and } i \neq j \tag{41}$$

$$\begin{bmatrix}
\tilde{\boldsymbol{\Omega}}_{11} & \tilde{\mathbf{P}}_{12}+\alpha\mathbf{B}_i\mathbf{Q}_{dj} & \hat{\boldsymbol{\Omega}}_{13} & \alpha\mathbf{B}_i\mathbf{Q}_{pj}+q\alpha\mathbf{B}_i\mathbf{Q}_{dj} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & -\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & \tilde{\boldsymbol{\Omega}}_{22} & 0 & \hat{\boldsymbol{\Omega}}_{24} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & \alpha\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & * & -\gamma^2\tilde{\mathbf{P}}_{11} & -\gamma^2\tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}-\hat{\boldsymbol{\Omega}}_7 & \mathbf{VR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\
* & * & * & -\gamma^2\tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}+\alpha\hat{\boldsymbol{\Omega}}_7 & 0 \\
* & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon\mathbf{I} & 0 \\
* & * & * & * & * & * & * & -\varepsilon\mathbf{I}
\end{bmatrix} < 0, \quad \text{for } i,j=1,2,\ldots,r \text{ and } i=j \tag{42}$$

where $\tilde{\boldsymbol{\Omega}}_{11} = \tilde{\mathbf{P}}_{11} - sym\left\{\mathbf{VR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} + \mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{B}_i^{\mathrm{T}}\right\}$, $\tilde{\boldsymbol{\Omega}}_{13} = \mathbf{A}_i\mathbf{VRV}^{\mathrm{T}} - \mathbf{B}_i\mathbf{Q}_{pj}$, $\tilde{\boldsymbol{\Omega}}_{22} = \tilde{\mathbf{P}}_{22} - sym\left\{\alpha\mathbf{VR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} + \alpha\mathbf{C}^{\mathrm{T}}\overset{\frown}{\mathbf{R}}^{\mathrm{T}}\mathbf{L}_{di}^{\mathrm{T}}\right\}$, $\tilde{\boldsymbol{\Omega}}_{24} = \alpha\mathbf{A}_i\mathbf{VRV}^{\mathrm{T}} - \alpha\mathbf{L}_{pi}\overset{\frown}{\mathbf{R}}\mathbf{C}$, $\hat{\boldsymbol{\Omega}}_{13} = \mathbf{A}_i\mathbf{VRV}^{\mathrm{T}} - \mathbf{B}_i\mathbf{Q}_{pj} - q\mathbf{E}_i\mathbf{VRV}^{\mathrm{T}} - q\mathbf{B}_i\mathbf{Q}_{dj}$, $\hat{\boldsymbol{\Omega}}_{24} = \alpha\mathbf{A}_i\mathbf{VRV}^{\mathrm{T}} - \alpha\mathbf{L}_{pi}\mathbf{CVRV}^{\mathrm{T}} - q\alpha\mathbf{E}_i\mathbf{VRV}^{\mathrm{T}} - q\alpha\mathbf{L}_{di}\mathbf{CVRV}^{\mathrm{T}}$, and $\hat{\boldsymbol{\Omega}}_7 = q\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}$.

Using the SVD technique (21) in Lemma 3, the matrix $\mathbf{CVRV}^{\mathrm{T}}$ in (41) and (42) can be decomposed as the matrices $\mathbf{U}$, $\boldsymbol{\Sigma}$, and $\mathbf{V}$ as follows:

$$\mathbf{CVRV}^{\mathrm{T}} = \mathbf{U}\begin{bmatrix}\boldsymbol{\Sigma} & 0\end{bmatrix}\mathbf{V}^{\mathrm{T}}\mathbf{V}\begin{bmatrix}\mathbf{Z}_{11} & 0 \\ \mathbf{Z}_{21} & \mathbf{Z}_{22}\end{bmatrix}\mathbf{V}^{\mathrm{T}} = \overset{\frown}{\mathbf{R}}\mathbf{C} \tag{43}$$

where $\overset{\frown}{\mathbf{R}} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{Z}_{11}\boldsymbol{\Sigma}^{-1}\mathbf{U}^{\mathrm{T}}$

$$\begin{bmatrix}
\tilde{\boldsymbol{\Omega}}_{11} & \tilde{\mathbf{P}}_{12}+\alpha\mathbf{B}_i\mathbf{Q}_{dj} & \tilde{\boldsymbol{\Omega}}_{13} & \alpha\mathbf{B}_i\mathbf{Q}_{pj} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & -\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & \tilde{\tilde{\boldsymbol{\Omega}}}_{22} & 0 & \tilde{\tilde{\boldsymbol{\Omega}}}_{24} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & \alpha\mathbf{Q}_{dj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & * & -\tilde{\mathbf{P}}_{11} & -\tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & \mathbf{VR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\
* & * & * & -\tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha\mathbf{Q}_{pj}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon\mathbf{I} & 0 \\
* & * & * & * & * & * & * & -\varepsilon\mathbf{I}
\end{bmatrix} < 0 \quad \text{for } i,j=1,2,\ldots,r \text{ and } i\neq j \tag{44}$$

$$\begin{bmatrix}
\tilde{\boldsymbol{\Omega}}_{11} & \tilde{\mathbf{P}}_{12}+\alpha\mathbf{B}_i\mathbf{Q}_{di} & \hat{\boldsymbol{\Omega}}_{13} & \alpha\mathbf{B}_i\mathbf{Q}_{pi}+\alpha q\mathbf{B}_i\mathbf{Q}_{di} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & -\mathbf{Q}_{di}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & \tilde{\tilde{\boldsymbol{\Omega}}}_{22} & 0 & \hat{\hat{\boldsymbol{\Omega}}}_{24} & \varepsilon\mathbf{H}_{Bi} & \varepsilon\mathbf{H}_{Ai} & \alpha\mathbf{Q}_{di}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}} & 0 \\
* & * & -\gamma^2\tilde{\mathbf{P}}_{11} & -\gamma^2\tilde{\mathbf{P}}_{12} & 0 & 0 & -\mathbf{Q}_{pi}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}-\tilde{\boldsymbol{\Phi}}_7 & \mathbf{VR}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{W}_{Ai}^{\mathrm{T}} \\
* & * & * & -\gamma^2\tilde{\mathbf{P}}_{22} & 0 & 0 & \alpha\mathbf{Q}_{pi}^{\mathrm{T}}\mathbf{W}_{Bi}^{\mathrm{T}}+\alpha\tilde{\boldsymbol{\Phi}}_7 & 0 \\
* & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 & 0 \\
* & * & * & * & * & -\varepsilon\mathbf{I} & 0 & 0 \\
* & * & * & * & * & * & -\varepsilon\mathbf{I} & 0 \\
* & * & * & * & * & * & * & -\varepsilon\mathbf{I}
\end{bmatrix} < 0 \quad \text{for } i,j=1,2,\ldots,r \text{ and } i=j \tag{45}$$

where $\tilde{\tilde{\mathbf{\Omega}}}_{22} = \tilde{\mathbf{P}}_{22} - sym\left\{\alpha \mathbf{V}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\mathbf{E}^{\mathrm{T}} + \alpha \mathbf{C}^{\mathrm{T}}\mathbf{G}_{di}^{\mathrm{T}}\right\}$, $\tilde{\tilde{\mathbf{\Omega}}}_{24} = \alpha \mathbf{A}_i \mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \alpha \mathbf{G}_{pi}\mathbf{C}$, and $\hat{\hat{\mathbf{\Omega}}}_{24} = \alpha \mathbf{A}_i \mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - \alpha \mathbf{G}_{pi}\mathbf{C} - q\alpha \mathbf{E}_i \mathbf{V}\mathbf{R}\mathbf{V}^{\mathrm{T}} - q\alpha \mathbf{G}_{di}\mathbf{C}$.

Obviously, conditions (44) and (45) are satisfied if conditions (35) and (36) are achieved by Theorem 2. Then, conditions (41) and (42) are achieved by the SVD technique. Via the Schur complement, conditions (37) and (38) can also be achieved. Consequently, the closed-loop UT-SFDS (13) can achieve the asymptotically stability and the pole placement constraint (22) according to the stability analysis process of Theorem 1. □

After LMI stability conditions (35) and (36) are derived, the control gains $\mathbf{F}_{dj}$ and $\mathbf{F}_{pj}$ can be obtained with $\mathbf{Q}_{dj}$ and $\mathbf{Q}_{pj}$ by solving the control problem of Theorem 1 with the convex optimization algorithm. The observer gains $\mathbf{L}_{di}$ and $\mathbf{L}_{pi}$ can also be obtained by $\mathbf{G}_{di}$ and $\mathbf{G}_{pi}$. Moreover, the positive definite matrix is obtained with $\mathbf{P} = \mathbf{\Psi}^{-\mathrm{T}}\tilde{\mathbf{P}}\mathbf{\Psi}^{-1} = \begin{bmatrix} \tilde{\mathbf{P}}_{11} & \tilde{\mathbf{P}}_{12} \\ * & \tilde{\mathbf{P}}_{22} \end{bmatrix}$. Different from the design method in [32], the matrix $\mathbf{P}$ is no longer required to be set as the diagonal form.

Via the O-BPD fuzzy controller design method in Theorem 2, asymptotic stability in NDSs can be achieved and the robustness can be improved. The impulse behavior and noncausal problems are also avoided by the PD feedback technique. Moreover, better transient responses can be achieved for NDSs by selecting more suitable center and radius parameters for the pole constraint. In the next section, the O-BPD fuzzy controller is designed by Theorem 2 and applied to numerical and bio-economic NDSs for simulations.

4. Simulation of Numerical NDS and Bio-Economic NDS

In this section, the proposed O-BPD controller is validated by two simulations including a numerical NDS and a practical bio-economic NDS. The first simulation of a numerical NDS is provided to validate the applicability of the O-BPD fuzzy controller and the fuzzy observer designed by Theorem 2. In the second simulation, the effectiveness of the proposed O-BPD fuzzy controller is illustrated based on a bio-economic NDS by adjusting the center and radius parameters of the pole constraint. Moreover, the better O-BPD fuzzy controller is applied to compare with the design method in [32] to further verify the advantage of Theorem 2 in solving the control problem of the NDS.

4.1. Numerical NDS

In the first simulation, a numerical NDS was considered to verify the proposed robust O-BPD fuzzy controller design method. Therefore, the following UT-SFDS was constructed for the representation of a numerical NDS.

$$\mathbf{E}x(k+1) = \sum_{i=1}^{2} h_i(x_2(k))\{(\mathbf{A}_i + \mathbf{\Delta A}_i)x(k) + (\mathbf{B}_i + \mathbf{\Delta B}_i)u(k)\} \tag{46}$$

$$y(k) = \sum_{i=1}^{2} h_i(x_2(k))\{\mathbf{C}x(k)\} \tag{47}$$

where $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{A}_1 = \begin{bmatrix} 1.1 & 0.3 & 0.3 \\ 0.7 & -0.9 & 0 \\ 0.5 & 0 & -1 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 1.2 & 0.5 & 0 \\ -0.4 & 0.8 & 0 \\ -0.5 & 0 & 1.2 \end{bmatrix}$, $\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The structure of the uncertainties $\mathbf{\Delta A}_i$ and $\mathbf{\Delta B}_i$ according to (3) are considered with $\mathbf{H}_{A1} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{H}_{A2} = \begin{bmatrix} -0.03 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{H}_{B1} = \begin{bmatrix} 0 \\ 0 \\ 0.05 \end{bmatrix}$, $\mathbf{H}_{B2} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$,

$\mathbf{W}_{A1}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0.02 \\ 0 \end{bmatrix}$, $\mathbf{W}_{A2}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0.01 \\ 0 \end{bmatrix}$, $\mathbf{W}_{B1}^{\mathrm{T}} = \begin{bmatrix} 0 \\ -0.02 \end{bmatrix}$, $\mathbf{W}_{B1}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}$, and $\mathbf{\Delta}(t) = \sin(t)$. The membership functions for each rule are given as $h_1(x_2(k)) = \left(1 - \frac{x_2(k)}{3}\right)/2$ and $h_2(x_2(k)) = \left(1 + \frac{x_2(k)}{3}\right)/2$. Based on Lemma 3, the matrix $\mathbf{C}$ is decomposed as follows.

$$\mathbf{C} = \mathbf{U}\begin{bmatrix} \mathbf{\Sigma} & 0 \end{bmatrix}\mathbf{V}^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1.4142 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -0.7071 & -0.7071 & 0 \\ 0 & 0 & -1 \\ 0.7071 & -0.7071 & 0 \end{bmatrix}^{\mathrm{T}} \tag{48}$$

From the model matrices $\mathbf{A}_i$, $\mathbf{B}_i$, and $\mathbf{C}$, it is verified that the controllability and observability are guaranteed according to Definition 1. Note that UT-SFDS (46)-(47) is an unstable system. Therefore, the purpose in this simulation was to apply the O-BPD fuzzy controller designed by Theorem 2 to achieve stability and to obtain a fast settling time. According to the consideration, the disk region was set as $\mathfrak{D}(q, \gamma) = (0.1, 0.9)$ for the pole constraint of Lemma 4 in Theorem 2. Then, setting $\alpha = 0.5$, the following gains of the fuzzy controller and fuzzy observer were obtained via solving Theorem 2 with the convex optimization algorithm.

$$\begin{aligned}
&\mathbf{F}_{d1} = \begin{bmatrix} 5.5795 & 5.8707 & -2.5616 \\ 19.1983 & 19.4214 & -226.3391 \end{bmatrix}, \mathbf{F}_{d2} = \begin{bmatrix} 5.4775 & 5.7454 & -2.2351 \\ 19.0814 & 19.2917 & -265.7760 \end{bmatrix}, \\
&\mathbf{F}_{p1} = \begin{bmatrix} 0.0793 & -0.3120 & 1.1000 \\ -0.9957 & -1.0263 & 13.9918 \end{bmatrix}, \mathbf{F}_{p2} = \begin{bmatrix} 0.1944 & -0.1185 & 0.4570 \\ -1.0105 & -0.9919 & 14.1978 \end{bmatrix}, \\
&\mathbf{L}_{d1} = \begin{bmatrix} 10.1769 & 21.4084 \\ 14.0270 & -4.0086 \\ 41.8403 & -584.1124 \end{bmatrix}, \mathbf{L}_{d2} = \begin{bmatrix} 7.3280 & 46.3471 \\ 11.4947 & -2.8957 \\ 41.6191 & -588.8856 \end{bmatrix}, \\
&\mathbf{L}_{p1} = \begin{bmatrix} -0.1837 & 0.4413 \\ -1.4235 & 2.9356 \\ -2.1078 & 30.2008 \end{bmatrix} \text{ and } \mathbf{L}_{p2} = \begin{bmatrix} 0.2707 & -1.8929 \\ 0.0179 & -1.3867 \\ -2.3055 & 31.4310 \end{bmatrix}
\end{aligned} \tag{49}$$

Giving the initial conditions for the system and observer as $x(0) = \begin{bmatrix} 0.7 & 0.2 & 0.3 \end{bmatrix}^{\mathrm{T}}$ and $\widehat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, the responses of system states and observer states are presented in Figure 1 by applying the O-BPD fuzzy controller (11) and the fuzzy observer (8)-(9) with the gains in (49). From the simulation results in Figure 1, it can be seen that the system state can achieve stability within 0.5 s. Additionally, the fuzzy observer also can estimate the system state rapidly. This is because the pole placement constraint is also combined into the stability conditions for the estimated error in Theorems 1 and 2. Consequently, the estimation performance for fuzzy observer is also enhanced. Moreover, the effect of uncertainties is also suppressed for the responses of the numerical NDS. After the applicability and efficiency of the proposed O-BPD fuzzy controller design method in Theorem 2 are verified, comparison results with study [32] are provided in the following simulation.

4.2. Bio-Economical NDS

To further demonstrate the advantage of the proposed O-BPD fuzzy controller design method in this paper, the simulation results of a practical bio-economic NDS compared with [32] are presented in this subsection. It was noticed that the research in [32] developed a O-BPD fuzzy controller design method for NDSs without the consideration of uncertain problems and pole constraints. Then, the simulation of comparison was implemented as follows. Firstly, the following bio-economic NDS was considered as follows to describe the relationship between the biological population and the harvest economy.

$$\dot{x}_1(t) = -ax_1(t) + bx_2(t) \tag{50}$$

$$\dot{x}_2(t) = \beta x_1(t) - \lambda x_2^2(t) - x_2(t)x_3(t) + u_1(t) \tag{51}$$

$$0 = x_3(t)(\gamma x_2(t) - c) - f + u_2(t) \tag{52}$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$, respectively, denote the density of the immature population, the density of the mature population, and the effort of capturing the mature population; $u_1(t)$ and $u_2(t)$ are the control inputs such as tax to balance the biological resource. To compare with the research in [32], the same parameters $a = 0.2$, $b = 0.7$, $\beta = 0.05$, $\lambda = 0.1$, $\gamma = 1$, c $= 30$, and $f = 0$ were selected for the bio-economic NDS (50)-(52) in this simulation.

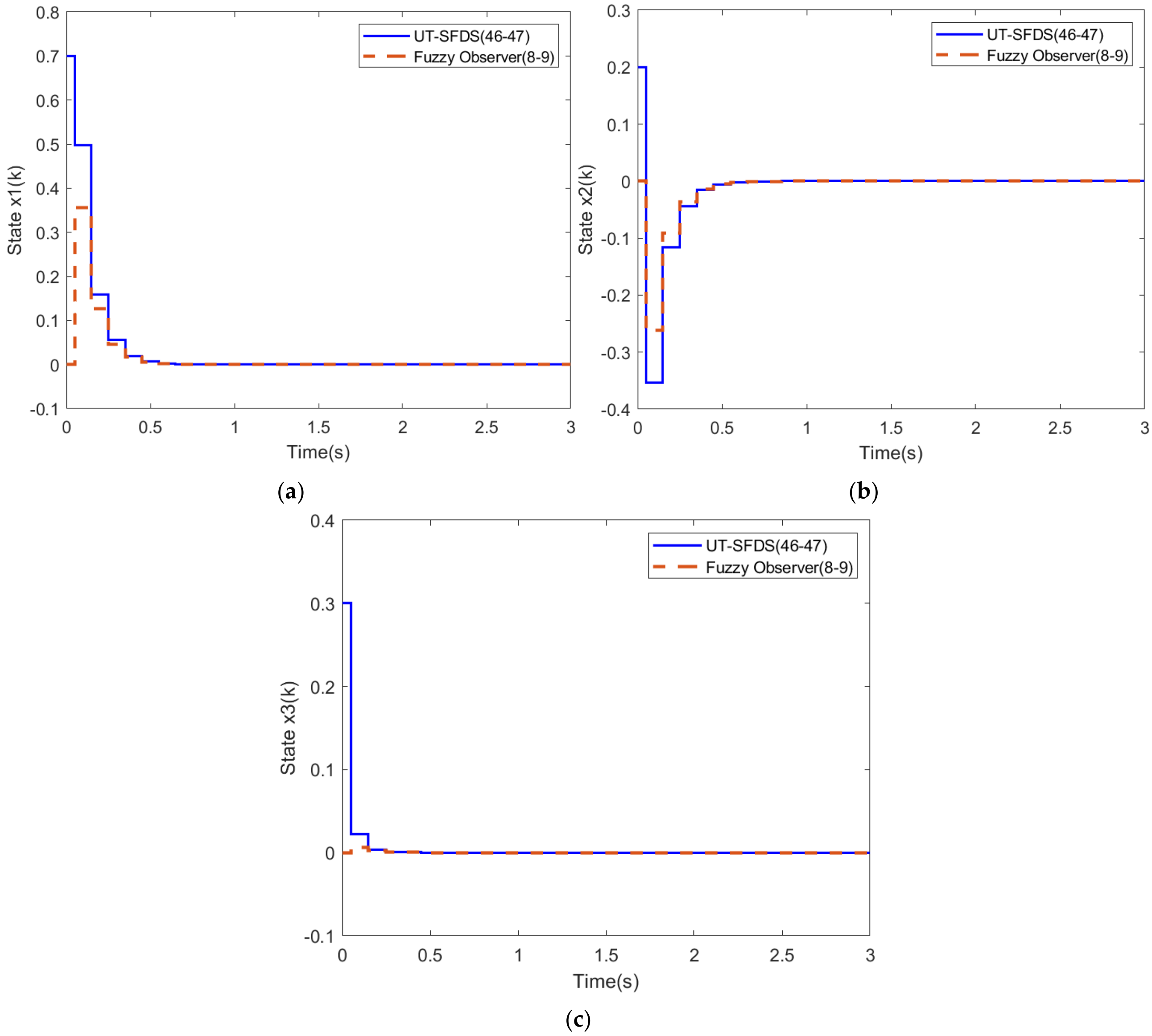

Figure 1. (**a**) Response of system state $x_1(k)$ and observer state $\widehat{x}_1(k)$; (**b**) response of system state $x_2(k)$ and observer state $\widehat{x}_2(k)$; (**c**) response of system state $x_3(k)$ and observer state $\widehat{x}_3(k)$.

Referring to reference [32], the Euler discretization technique and the T-S fuzzy modelling method were applied to obtain the following discrete-time UT-SFDS with the sampling time T $= 0.05$s.

$$\mathbf{E}x(k+1) = \sum_{i=1}^{2} h_i(x(k))\{(\mathbf{A}_i + \mathbf{\Delta A}_i)x(k) + (\mathbf{B}_i + \mathbf{\Delta B}_i)u(k)\} \tag{53}$$

$$y(k) = \sum_{i=1}^{2} h_i(x(k))\{\mathbf{C}x(k)\} \tag{54}$$

where $x_2(k) \in [-\mu,\ \mu]$ is considered for the UT-SFDS, the states and input vectors are denoted as $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & x_3(t) \end{bmatrix}^{\mathrm{T}}$ and $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^{\mathrm{T}}$, and the matrices are given as $\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{A}_1 = \begin{bmatrix} 0.8690 & 0.0090 & 0.0011 \\ 0.0023 & 0.9228 & 0.2279 \\ 0 & 0 & -31.5 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 0.8690 & 0.0087 & -0.0011 \\ 0.0022 & 0.8778 & -0.2222 \\ 0 & 0 & -22.5 \end{bmatrix}$, $\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0.05 & 0 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. According to (3), the uncertainties $\mathbf{\Delta A}_i$ and $\mathbf{\Delta B}_i$ are constructed with $\mathbf{H}_{A1} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{H}_{A2} = \begin{bmatrix} -0.01 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{H}_{B1} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix}$, $\mathbf{H}_{B2} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix}$, $\mathbf{W}_{A1}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ -0.3 \end{bmatrix}$, $\mathbf{W}_{A2}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}$, $\mathbf{W}_{B1}^{\mathrm{T}} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$, $\mathbf{W}_{B2}^{\mathrm{T}} = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}$, and $\mathbf{\Delta}(t) = \sin(t)$. The membership functions for each rule are denoted as $h_1(x_2(k)) = \left(1 - \frac{x_2(k)}{\mu}\right)/2$ and $h_2(x_2(k)) = \left(1 + \frac{x_2(k)}{\mu}\right)/2$, where $\mu = 5$ is chosen for $x_2(k)$ of UT-SFDS (53)-(54). According to the SVD technique of Lemma 3, the matrix $\mathbf{C}$ is decomposed as follows.

$$\mathbf{C} = \mathbf{U}\begin{bmatrix} \mathbf{\Sigma} & 0 \end{bmatrix}\mathbf{V}^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1.4142 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -0.7071 & 0 & 0.7071 \\ -0.7071 & 0 & -0.7071 \\ 0 & -1 & 0 \end{bmatrix}^{\mathrm{T}} \tag{55}$$

Similar to Section 4.1, the model matrices of UT-SFDS (53)-(54) satisfy the condition of controllability and observability in Definition 1. Moreover, the UT-SFDS (53)-(54) is also an unstable system. For the simulations of a bio-economic NDS, the purpose was to achieve stability and obtain more proper transient responses by applying the O-BPD fuzzy controller designed by Theorem 2. It is expected that better and more reasonable responses can be obtained by the proposed Theorem 2 than in [32], which did not consider the robust control performance and the pole constraint in the O-BPD fuzzy controller design process.

To illustrate the function and advantage of the pole placement method in Theorem 2, the simulation results for different pairs of center and radius parameters are first presented. Because of this reason, three cases of the disk region, $\mathfrak{D}(q,\gamma) = (0,1)$, $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$, and $\mathfrak{D}(q,\gamma) = (0.4, 0.6)$, were selected for the pole placement method. Then, solving Theorem 2 by the convex optimization algorithm with the setting $\alpha = 0.5$ and, respectively, with the parameters q and γ of three cases, the gains of the fuzzy controller and fuzzy observer were obtained as follows.

For the disk region $\mathfrak{D}(q,\gamma) = (0,1)$

$$\begin{aligned}
&\mathbf{F}_{d1} = \begin{bmatrix} -0.5796 & -0.6107 & -1262.8598 \\ 3717.2356 & 3717.2355 & 7677698.0422 \end{bmatrix}, \\
&\mathbf{F}_{d2} = \begin{bmatrix} -0.5796 & -0.6107 & -1262.8598 \\ 3717.2356 & 3717.2355 & 7677698.0422 \end{bmatrix}, \ \mathbf{F}_{p1} = \begin{bmatrix} -0.2189 & -0.1037 & -481.1946 \\ 0.0279 & 0.0277 & 81.3524 \end{bmatrix}, \\
&\mathbf{F}_{p2} = \begin{bmatrix} -0.2189 & -0.1037 & -481.1946 \\ 0.0279 & 0.0277 & 81.3524 \end{bmatrix}, \ \mathbf{L}_{d1} = \begin{bmatrix} -615.3827 & -1272047.6197 \\ -615.8080 & -1273013.1170 \\ -9628.6077 & -19886702.2077 \end{bmatrix}, \\
&\mathbf{L}_{d2} = \begin{bmatrix} 349.1854 & 720173.4031 \\ 348.7086 & 719156.8067 \\ -9627.8376 & -19885942.7480 \end{bmatrix}, \ \mathbf{L}_{p1} = \begin{bmatrix} 0.4211 & -42.0172 \\ 0.4791 & 39.0784 \\ -0.0521 & -153.5534 \end{bmatrix}, \text{ and} \\
&\mathbf{L}_{p2} = \begin{bmatrix} 0.4184 & -47.4161 \\ 0.4607 & 46.9822 \\ 0.0346 & 34.6102 \end{bmatrix}
\end{aligned} \tag{56}$$

For the disk region $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$

$$\begin{aligned}
\mathbf{F}_{d1} &= \begin{bmatrix} 0.0319 & -0.0170 & 0.9793 \\ -7.3295 & -7.2396 & -18622.6250 \end{bmatrix}, \mathbf{F}_{d2} = \begin{bmatrix} 0.0343 & -0.0113 & 5.7403 \\ -7.3304 & -7.3306 & -18625.0568 \end{bmatrix}, \\
\mathbf{F}_{p1} &= \begin{bmatrix} 0.0045 & 0.1231 & 0.8691 \\ -0.9047 & 0.9046 & 2323.1495 \end{bmatrix}, \mathbf{F}_{p2} = \begin{bmatrix} 0.0050 & 0.1234 & 0.2827 \\ 0.9045 & 0.9044 & 2321.6365 \end{bmatrix}, \\
\mathbf{L}_{d1} &= \begin{bmatrix} 0.4909 & 18.5034 \\ 0.5358 & 21.0806 \\ 18.9761 & 48221.3062 \end{bmatrix}, \mathbf{L}_{d2} = \begin{bmatrix} 0.5088 & 28.4811 \\ 0.5250 & 31.1130 \\ 18.9728 & 48215.0429 \end{bmatrix}, \\
\mathbf{L}_{p1} &= \begin{bmatrix} 0.3210 & -1.9911 \\ 0.3353 & -2.3404 \\ -2.3351 & -5979.9267 \end{bmatrix}, \text{ and } \mathbf{L}_{p2} = \begin{bmatrix} 0.3187 & -3.0793 \\ 0.3142 & -3.9985 \\ -2.3284 & -5953.7510 \end{bmatrix}
\end{aligned} \tag{57}$$

For the disk region $\mathfrak{D}(q,\gamma) = (0.4, 0.6)$

$$\begin{aligned}
\mathbf{F}_{d1} &= \begin{bmatrix} 0.0200 & -0.0156 & -0.1413 \\ -2.5519 & -2.5520 & -8111.9903 \end{bmatrix}, \mathbf{F}_{d2} = \begin{bmatrix} 0.0225 & -0.0079 & 3.0627 \\ -2.5572 & -2.5574 & -8128.8580 \end{bmatrix}, \\
\mathbf{F}_{p1} &= \begin{bmatrix} 0.0055 & 0.0902 & 0.4217 \\ 0.7495 & 0.7494 & 2407.4104 \end{bmatrix}, \mathbf{F}_{p2} = \begin{bmatrix} 0.0054 & 0.0907 & -0.2606 \\ 0.7466 & 0.7465 & 2396.1923 \end{bmatrix}, \\
\mathbf{L}_{d1} &= \begin{bmatrix} 0.4871 & 17.5114 \\ 0.5370 & 18.9687 \\ 6.5929 & 20971.1937 \end{bmatrix}, \mathbf{L}_{d2} = \begin{bmatrix} 0.5035 & 20.8348 \\ 0.5215 & 22.3058 \\ 6.5820 & 20938.1836 \end{bmatrix}, \\
\mathbf{L}_{p1} &= \begin{bmatrix} 0.1546 & -4.7652 \\ 0.1598 & -5.1148 \\ -1.8623 & -5969.3581 \end{bmatrix} \text{ and } \mathbf{L}_{p2} = \begin{bmatrix} 0.1499 & -5.5019 \\ 0.1421 & -6.2784 \\ -1.8622 & -5959.9672 \end{bmatrix}
\end{aligned} \tag{58}$$

Selecting the initial states $x(0) = \begin{bmatrix} 10 & 4 & 1 \end{bmatrix}^{\mathrm{T}}$ for the bio-economical NDS (50)–(52) and $\widehat{x}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ for the fuzzy observer (8)-(9), the state responses can be presented as follows by applying O-BPD fuzzy controller (11) with gains (56)–(58).

It is obvious that the maximum overshoot of the state responses is much larger when applying the O-BPD fuzzy controller with the pole-constrained region $\mathfrak{D}(q,\gamma) = (0,1)$. Although the overshoot is not caused by the O-BPD fuzzy controller designed with the regions $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$ and $\mathfrak{D}(q,\gamma) = (0.4, 0.6)$, the settling time obtained by setting $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$ is better than $\mathfrak{D}(q,\gamma) = (0.4, 0.6)$. To clearly present the transient performances of responses obtained by the three disks, the following table is provided.

From Table 1, it is not difficult to see that the setting of disk $\mathfrak{D}(q,\gamma) = (0,1)$ can achieve a faster rising time than other two cases. Note that the rising time is defined as the time at which the value first deviates from 0 by 0.05. However, the value of maximum overshoot is also higher than the other two cases. Comparing the cases $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$ and $\mathfrak{D}(q,\gamma) = (0.4, 0.6)$, the faster rising time can be improved by the setting of $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$, which only causes a slight overshoot. Moreover, the overshoot of state is $x_1(k)$ smaller. Via the simulation results in Figure 2 and Table 1, when adjusting the parameters of center and radius for the pole constraints, the disk region $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$ is selected for the O-BPD fuzzy controller design approach of Theorem 2. Applying the O-BPD fuzzy controller (11) with the gains in (57) of case $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$, the state responses compared with the research [32] are presented as follows.

Table 1. A comparison of transient performances of each state for different disks.

States	Transient Performances	$D(q,\gamma)$ = (0, 1)	$D(q,\gamma)$ = (0.2, 0.8)	$D(q,\gamma)$ = (0.4, 0.6)
$x_1(k)$	Rising Time Maximum Overshoot	3.75 s −0.001	3.95 s −0.0001	3.95 s −0.001
$x_2(k)$	Rising Time Maximum Overshoot	1.45 s −0.1319	3.65 s −0.001	3.85 s −0.001
$x_3(k)$	Rising Time Maximum Overshoot	0.05 s −0.0059	0.20 s −0.0030	0.25 s −0.0015

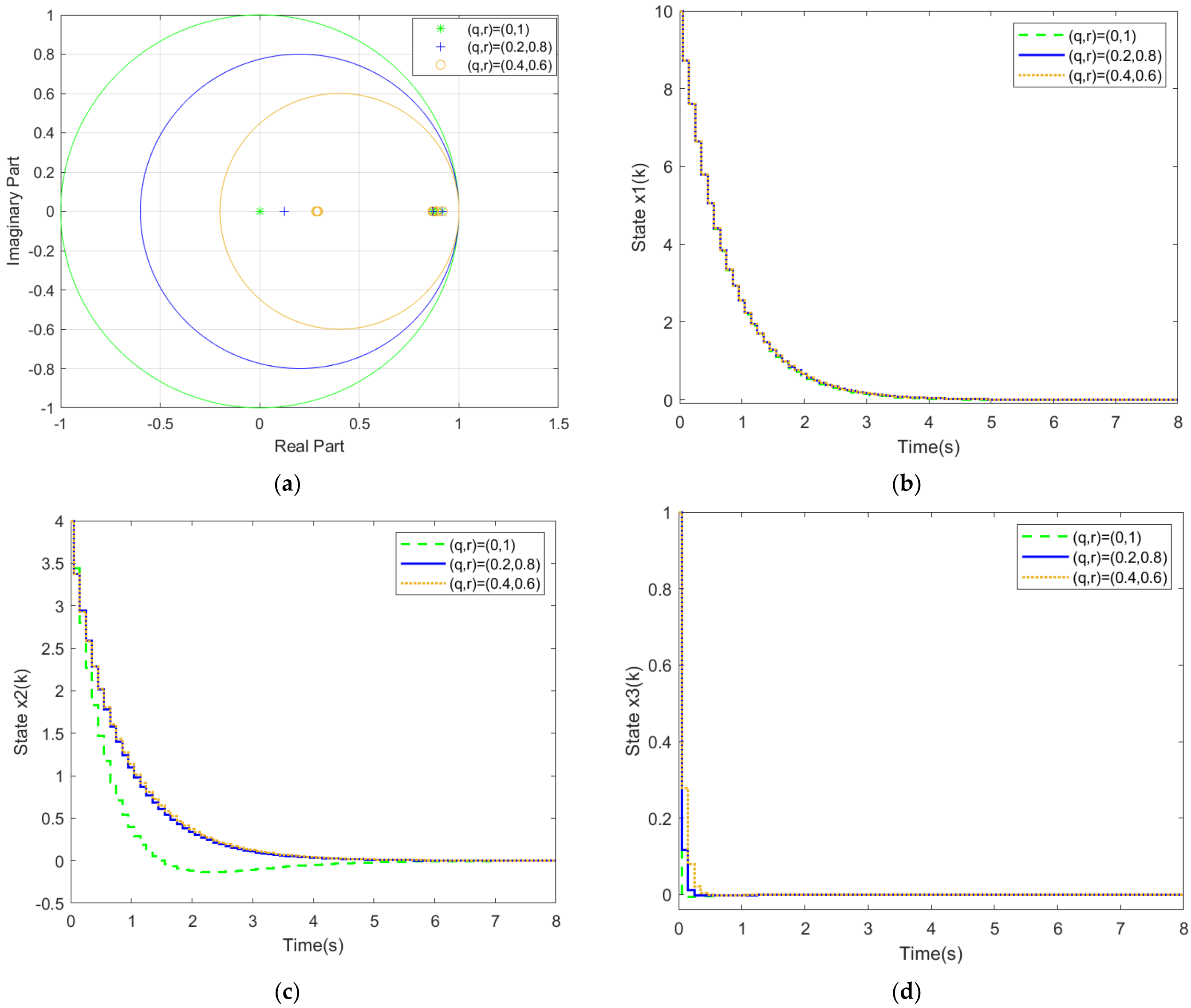

Figure 2. (**a**) Pole location of three different disk regions; (**b**) responses of system state $x_1(k)$ with different disk regions; (**c**) responses of system state $x_2(k)$ with different disk regions; (**d**) responses of system state $x_3(k)$ with different disk regions.

According to the simulation results in Figure 3, the proposed O-BPD fuzzy controller can obtain smoother responses for the bio-economical NDS (50)–(52), even under the effect of uncertainties. For the practical situation, uncertainties such as weather and temperature changes' effects on immature and mature populations are also inevitable in bio-economic NDSs (50)–(52). Therefore, the proposed O-BPD fuzzy controller can ensure the robustness of practical NDSs. From the simulation result in Figure 3a, one can see that the response of state $x_1(k)$ obtained by the proposed O-BPD fuzzy controller is same as the responses obtained by [32]. This is because it is difficult to further improve the transient response of state $x_1(k)$, which can be seen from the results in Figure 2b.

From the responses of the second state in Figure 3b, it is witnessed that the O-BPD fuzzy controller designed by [32] can achieve a faster settling time. However, this also causes a larger maximum overshoot. From the perspective of practical applications, this unreasonably drastic change cannot be tolerated by any biological system. On the contrary, the proposed O-BPD fuzzy controller design method can obtain smoother responses to achieve balance for the bio-economic NDS (50)–(52) by selecting the region $\mathfrak{D}(q,\gamma) = (0.2, 0.8)$ for pole constraint (22).

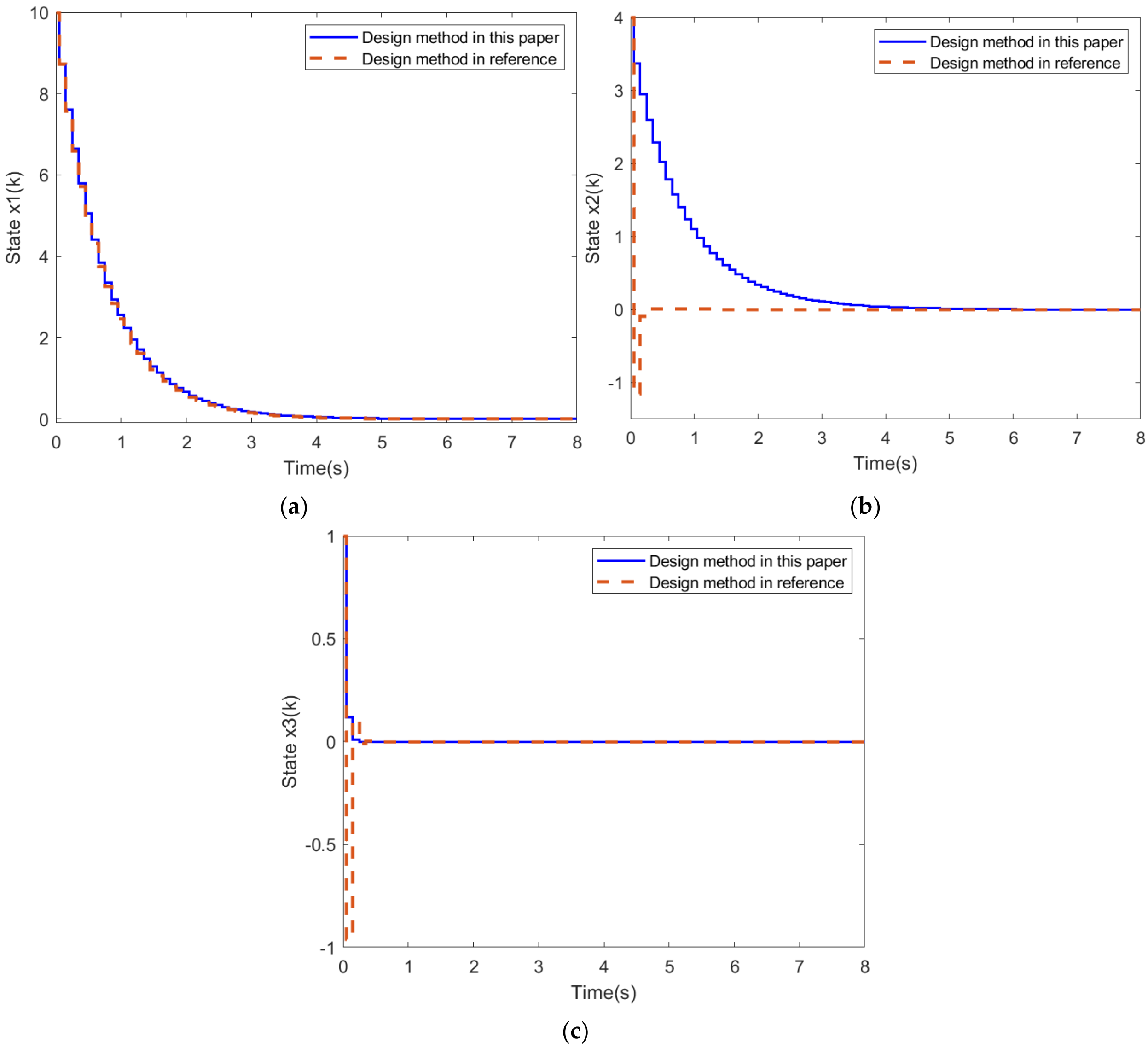

Figure 3. (**a**) Responses of system state $x_1(k)$ compared with [32]; (**b**) responses of system state $x_2(k)$ compared with [32]; (**c**) responses of system state $x_3(k)$ compared with [32].

In addition, the effort of capturing the mature population can achieve balance rapidly and avoid the overshoot via the proposed O-BPD fuzzy controller in Figure 3c. This advantage also means that it will not waste the redundant resources for capturing. Notably, the effort of capturing the mature population $x_3(t)$ is difficult to quantify. Therefore, the application of the fuzzy observer in our research can also allow an estimation of the state and fulfill the O-BPD fuzzy controller in practical situations. Based on the comparison results, it can be said that better and smoother economic balance can be achieved for the bio-economic NDS (50)–(52) by the proposed O-BPD fuzzy controller in this paper.

5. Conclusions

An O-BPD fuzzy controller design method is proposed in this paper to solve the control problem of discrete-time NDSs with the requirements of robustness and transient responses. Based on the UT-SFDS, an NDS with an uncertain problem is represented. To improve applicability and fulfill the PD fuzzy control method, a fuzzy observer is also combined to estimate the state variables, which are possibly difficult to measure in practical situations. Using the PDC concept, a PD fuzzy controller is developed based on the UT-SFDS and fuzzy observer. Ensuring that the expected transient responses can be obtained, pole placement constraints are also combined into the design of the O-BPD fuzzy controller. Via Lyapunov theory, a stability criterion based on the UT-SFDS is also proposed to achieve stability with the pole constraint. Via simultaneously applying the SVD technique, projection lemma, robust control method, and Schur complement, the stability conditions are efficiently recast into LMI form. More importantly, the conservativeness caused by the diagonal positive definite matrix in the existing research is also eliminated by the use of SVD and projection lemma. The simulation results in the two examples verify that the proposed O-BPD fuzzy controller can provide a smoother and more reasonable response for a bio-economic NDS to achieve convergence, such that biologically immature and mature populations can adequately strike a balance with the harvest economy. It is worth noting that a tradeoff between the different transient properties such as maximum overshoot, settling time, and rising time can be conveniently achieved for the requirements of different practical applications by adjusting the center and radius parameters of the pole constraint. In the future, an extension to the proposed O-PBD fuzzy controller can be developed with the aim of exploring issues such as time delay and disturbance, which would further improve the applicability of the control method.

Author Contributions: Conceptualization, Y.-C.L., W.-J.C. and T.-A.L.; Methodology, T.-A.L., W.-J.C. and Y.-H.L.; Software, T.-A.L. and Y.-H.L.; Validation, W.-J.C.; Investigation, W.-J.C.; Resources, Y.-C.L.; Writing—original draft, T.-A.L.; Writing—review and editing, Y.-C.L.; Supervision, Y.-C.L. and W.-J.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the National Science and Technology Council of the Republic of China under Contract NSTC112-2221-E-019-057.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The authors would like to express their sincere gratitude to the anonymous reviewers who gave us many constructive comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kchaou, M.; Jerbi, H. Reliable H∞ and passive fuzzy observer-based sliding mode control for nonlinear descriptor systems subject to actuator failure. *Int. J. Fuzzy Syst.* **2022**, *24*, 105–120. [CrossRef]
2. Ran, C.J.; Sun, S.L.; Dou, Y.F. WMF reduced-order robust estimators for multisensor descriptor systems. *IET Control Theory Appl.* **2018**, *12*, 2232–2244. [CrossRef]
3. Bessaoudi, T.; Hmida, F.B.; Hsieh, C.-S. Robust state and fault estimation for linear descriptor stochastic systems with disturbances: A DC motor application. *IET Control Theory Appl.* **2017**, *11*, 601–610. [CrossRef]
4. Wang, Y.; Karimi, H.R.; Lam, H.K.; Yan, H. Fuzzy output tracking control and filtering for nonlinear discrete-time descriptor systems under unreliable communication links. *IEEE Trans. Cybern.* **2020**, *50*, 2369–2379. [CrossRef]
5. Takagi, T.; Sugeno, M. Fuzzy identification of systems and its applications to modeling and control. *IEEE Trans. Syst. Man Cybern.* **1985**, *SMC-15*, 116–132. [CrossRef]
6. Yan, S.; Gu, Z.; Ding, L.; Park, J.H.; Xie, X. Weighted memory H∞ stabilization of time-varying delayed Takagi-Sugeno fuzzy systems. *IEEE Trans. Fuzzy Syst.* **2023**, *32*, 337–342. [CrossRef]
7. Chang, W.J.; Huang, Y.M.; Lin, Y.H. Observer-based fuzzy control of uncertain nonlinear singular systems under multi-performance requirements. *Mathematics* **2023**, *11*, 2632. [CrossRef]
8. Dai, L. *Singular Control Systems*; Springer: Berlin/Heidelberg, Germany, 1989.
9. Nguyen, A.; Campos, V.; Guerra, T.; Pan, J.; Xie, W. Takagi–Sugeno fuzzy observer design for nonlinear descriptor systems with unmeasured premise variables and unknown inputs. *Int. J. Robust Nonlinear Control* **2021**, *31*, 8353–8372. [CrossRef]
10. Yan, S.; Gu, Z.; Park, J.H.; Xie, X. A delay-kernel-dependent approach to saturated control of linear systems with mixed delays. *Automatica* **2023**, *152*, 110984. [CrossRef]
11. Wagner, A. Causality in complex systems. *Biol. Philos.* **1999**, *14*, 83–101. [CrossRef]
12. Corinthios, M. *Signals, Systems, Transforms, and Digital Signal Processing With MATLAB*; CRC Press: Boca Raton, FL, USA, 2018.
13. Wang, H.; Tanaka, K.; Griffin, M. An approach to fuzzy control of nonlinear systems: Stability and design issues. *IEEE Trans. Fuzzy Syst.* **1996**, *4*, 14–23. [CrossRef]

14. Ghorbel, C.; Tiga, A.; Braiek, N.B. Proportional PDC design-based robust stabilization and tracking control Strategies for uncertain and disturbed T-S model. *Complexity* **2020**, *2020*, 1–9. [CrossRef]
15. Ku, C.C.; Chang, W.J.; Huang, Y.M. Robust observer-based fuzzy control via proportional derivative feedback method for singular Takagi–Sugeno fuzzy systems. *Int. J. Fuzzy Syst.* **2022**, *24*, 3349–3365. [CrossRef]
16. Mu, Y.; Zhang, H.; Sun, J.; Ren, J. Proportional plus derivative state feedback controller design for a class of fuzzy descriptor systems. *Int. J. Syst. Sci.* **2019**, *50*, 2249–2260. [CrossRef]
17. Zaheer, M.H.; Arthur, K.M.; Yoon, S.Y.P. Derivative feedback control of nonlinear systems with uncertain equilibrium states and actuator constraints. *Automatica* **2021**, *127*, 109495. [CrossRef]
18. Li, J.; Wang, Z.; Shen, Y.; Wang, Y. Interval observer design for discrete-time uncertain Takagi–Sugeno fuzzy systems. *IEEE Trans. Fuzzy Syst.* **2019**, *27*, 816–823. [CrossRef]
19. Mu, Y.; Zhang, H.; Sun, S.; Ren, J. Robust non-fragile proportional plus derivative state feedback control for a class of uncertain Takagi-Sugeno fuzzy singular systems. *J. Frankl. Inst.* **2019**, *356*, 6208–6225. [CrossRef]
20. Luan, T.T.; Sun, M.X.; Hu, Z.Y.; Fu, Q.; Wang, H. A novel T-S fuzzy robust control for part transportation of aircraft carrier considering transportation time and stochastic demand. *Aerosp. Sci. Technol.* **2021**, *119*, 107096. [CrossRef]
21. Bof, N.; Carli, R.; Schenato, L. Lyapunov theory for discrete time systems. *arXiv* **2018**, arXiv:1809.05289.
22. Peres, W.; Coelho, F.C.R.; Costa, J.N.N. A pole placement approach for multi-band power system stabilizer tuning. *Int. Trans. Electr. Energy Syst.* **2020**, *30*, e12548. [CrossRef]
23. Garcia, G.; Bernussou, J. Pole assignment for uncertain systems in a specified disk by state feedback. *IEEE Trans. Autom. Control* **1995**, *40*, 184–190. [CrossRef]
24. Chilali, M.; Gahinet, P. H∞ design with pole placement constraints: An LMI approach. *IEEE Trans. Autom. Control* **1996**, *41*, 358–367. [CrossRef]
25. Lu, R.; Wang, M.; Bai, J.; Xue, A.; Zou, H. Pole placement with LMI constraint of fuzzy descriptor system. *J. Frankl. Inst.* **2015**, *352*, 2665–2678. [CrossRef]
26. Gupta, M.K.; Tomar, N.K.; Sharma, D.; Jaiswal, J. PD observer design for descriptor systems with unknown inputs: Application to infinite bus system. In Proceedings of the 2020 5th IEEE International Conference on Recent Advances and Innovations in Engineering (ICRAIE), Jaipur, India, 1–3 December 2020; pp. 1–5.
27. Chang, W.J.; Chang, C.M.; Lin, Y.H. Robust fuzzy control for nonlinear discrete-time systems with internal and external noises subject to multi-variance constraints and pole location constraints. *J. Intell. Fuzzy Syst.* **2020**, *38*, 4959–4975. [CrossRef]
28. Apkarian, P.; Tuan, H.D.; Bernussou, J. Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Trans. Autom. Control* **2001**, *46*, 1941–1946. [CrossRef]
29. Dongarra, J.; Gates, M.; Haidar, A.; Kurzak, J.; Luszczek, P.; Tomov, S.; Yamazaki, I. The singular value decomposition: Anatomy of optimizing an algorithm for extreme scale. *SIAM Rev.* **2018**, *60*, 808–865. [CrossRef]
30. Zhang, F. *The Schur Complement and Its Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2006.
31. Nesterov, Y. *Lectures on Convex Optimization*; Springer International Publishing: Berlin/Heidelberg, Germany, 2018.
32. Chang, W.J.; Tsai, M.H.; Pen, C.L. Observer-based fuzzy controller design for nonlinear discrete-time singular systems via proportional derivative feedback scheme. *Appl. Sci.* **2021**, *11*, 2833. [CrossRef]
33. Gao, Z.; Liu, Y.; Wang, Z. On stabilization of linear switched singular systems via P-D state feedback. *IEEE Access* **2020**, *8*, 97007–97015. [CrossRef]