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SPECIAL TWO PARAMETER MOTION

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Abstract- In this study all one parameter motions obtained from two parameters motion on the plane, are investigated. It is shown that the pole points which on fixed and moving plane at any position of (λ,μ) are on a line. It is also shown that the velocity vector lengths of these axis are the same. Moreover, the locus of any Hodograph of any point, and accelaration poles of the motion are investigated.

Keywords- Two parameter motion, One parameter motion, Planar motion

1. INTRODUCTION

A general planar motion is given by

$$\begin{aligned} y_1 &= x\cos\theta - y\sin\theta + a \\ y_2 &= x\sin\theta + y\cos\theta + b. \end{aligned} \tag{1.1}$$

If θ, a and b are given by the functions of time parameter t, then this motions is called as one parameter motion. One parameter planar motion given by (1.1) can be written in the form

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \tag{1.2}$$

or

$$Y = AX + C, Y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T,\ X = \begin{bmatrix} x & y \end{bmatrix}^T, C = \begin{bmatrix} a & b \end{bmatrix}^T \tag{1.3}$$

where, $A \in SO(2)$ and Y and X are the position vectors of the same point B, respectively, for the fixed and moving systems, and C is the translation vector. By taking the derivates with respect to t in (1.3), we get

$$\dot{Y} = \dot{A}X + A\dot{X} + \dot{C}\ , \tag{1.4}$$

where the velocities $\overrightarrow{V_a} = \dot{Y}$, $\overrightarrow{V_f} = \dot{A}X + \dot{C}$, $\vec{V_r} = A\dot{X}$ are called absolute, sliding, and relative velocities of the point B, respectively. The solution of the equation $\overrightarrow{V_f} = 0$ gives us the pole points $P = (x_p, y_p)$ on the moving plane. The locus of these points is called the moving pole curve, and correspondingly the locus of pole points on the fixed plane is called the fixed pole curve. The solution of the equation $\ddot{A}X + \ddot{C} = 0$ gives acceleration pole of the motion. If θ, a and b are function of λ and μ in one parameter planar motion (1.1), then this motion is called two parameters motion and it is denoted by M_{II}. Now let us analyse two parameters planar motion given by (1.1). The equations of two parameters planar motion is as follows [1]:

$$\begin{aligned} y_1 &= x\cos\lambda - y\sin\lambda + \ell\mu \\ y_2 &= x\sin\lambda + y\cos\lambda + b(\lambda,\mu)\ . \end{aligned} \tag{1.5}$$

Here if λ and μ are functions of *t*, then one parameter motion is obtained. Therefore this motion is called one parameter motion obtained from M_{II} and shown by M_I [1]. Note that Eq.(1.5) gives Eq.(1.3) for $\lambda = \lambda(t),\ \mu = \mu(t)$. Let us investigate *P* rotation poles of all one parameter M_I motion obtained from two parameters M_{II} motion at a position of (λ,μ).

2. THE POLES OF THE MOTIONS

Theorem 2.1- The pole points of M_I motions obtained from M_{II} on a moving plane lie on a line at each (λ,μ) position.

Proof- Since the point B is fixed on the moving plane, $\overrightarrow{V_r} = 0$ and the same point is also fixed on the fixed plane, then $\vec{V}_f = 0$. From Eq.(1.5), if $\dot{A}X + \dot{C} = 0$ is solved, then we obtain

$$X = P = -(\dot{A})^{-1}\dot{C}, \tag{2.1}$$

$$P = \frac{1}{\dot{\lambda}}\begin{bmatrix} \ell\dot{\mu} - \cos\lambda(b_\lambda\dot{\lambda} + b_\mu\dot{\mu}) \\ \ell\dot{\mu}\cos\lambda + \sin\lambda(b_\lambda\dot{\lambda} + b_\mu\dot{\mu}) \end{bmatrix}\ . \tag{2.2}$$

Therefore the point $P = (x_p, y_p)$ of the moving plane gives the family of the lines

$$(b_\mu\sin\lambda + \ell\cos\lambda)x_p + (-\ell\sin\lambda + b_\mu\cos\lambda)y_p + \ell b_\lambda = 0\ .$$

Corollary 2.1- On moving plane at the $\lambda = \mu = 0$ position, the pole points of M_I motions obtained from M_{II} lie on the line

$$\ell x_p + b_\mu y_p + \ell b_\lambda = 0\ . \tag{2.3}$$

This result is obtained by Bottema [1].

Theorem 2.2- The pole points of M_I motions obtained from M_{II} on a fixed plane lie on a line for each (λ,μ).

Proof- If the point $P = (x_p, y_p)$ is substituted for X in the equation which is $Y = AX + C$, then the pole of the fixed plane is obtained as follows:

$$\overline{P} = \begin{bmatrix} \cos\lambda & -\sin\lambda \\ \sin\lambda & \cos\lambda \end{bmatrix}\begin{bmatrix} -b_\lambda\cos\lambda + \dfrac{\dot{\mu}}{\dot{\lambda}}(\ell\sin\lambda + b_\mu\cos\lambda) \\ b_\lambda\sin\lambda + \dfrac{\dot{\mu}}{\dot{\lambda}}(b_\mu\sin\lambda + \ell\cos\lambda) \end{bmatrix} + \begin{bmatrix} \ell\mu \\ b(\lambda,\mu) \end{bmatrix}$$

and simplifiying

$$\overline{P}=\left(\overline{\mathrm{x}_{\mathrm{p}}},\overline{y_p}\right)=\left(-b_\lambda+\ell\mu-b_\mu\frac{\dot{\mu}}{\dot{\lambda}},b(\lambda,\mu)+\ell\frac{\dot{\mu}}{\dot{\lambda}}\right). \tag{2.4}$$

Therefore the point $\overline{P}=\left(\overline{\mathrm{x}_{\mathrm{p}}},\overline{y_p}\right)$ of the fixed plane gives the lines

$$\ell\overline{x_p}+b_\mu\overline{y_p}+\ell b_\lambda-\ell^2\mu-b_\mu b(\lambda,\mu)=0\ . \tag{2.5}$$

Corollary 2.2- On the fixed plane at the position of $\lambda=\mu=0$, the pole points of M_I motions obtained from M_{II} lies on the line

$$\ell\overline{x_p}+b_\mu\overline{y_p}+\ell b_\lambda=0. \tag{2.6}$$

Corollary 2.3- The pole lines on the fixed and moving planes are coincide at the position of $\lambda=\mu=0$.

Theorem 2.3- For all M_I motions obtained from M_{II}, if the θ is the angle between the pole ray going from the pole P to the point B and the sliding velocity $\overrightarrow{V_f}$ then at a position of (λ,μ), we have $\theta=\frac{\pi}{2}+\lambda$. In addition, these vectors are perpendicular if $\lambda=2k\pi$ (k=0,1,2,...).

Proof- From Eq.(2.1)

$$\dot{C}=-\dot{A}\begin{bmatrix}x_p\\y_p\end{bmatrix}$$

is derived. If this equality is substituted into $\overrightarrow{V_f}=\dot{A}X+\dot{C}$, then following equation is obtained

$$\overrightarrow{V_f}=\dot{\lambda}\big(-\sin\lambda(x-x_p)-\cos\lambda(y-y_p),\ \cos\lambda(x-x_p)-\sin\lambda(y-y_p)\big)\ . \tag{2.7}$$

Since

$$\overrightarrow{PB}=(x-x_p,y-y_p), \tag{2.8}$$

then

$$\left\langle\overrightarrow{PB},\overrightarrow{V_f}\right\rangle=-\dot{\lambda}\sin\lambda\left\|\overrightarrow{PB}\right\|^2.$$

In addition, we have

$$\left\langle\overrightarrow{PB},\overrightarrow{V_f}\right\rangle=\left\|\overrightarrow{PB}\right\|\left\|\overrightarrow{V_f}\right\|.\cos\theta\,.$$

From last two equations we get $\cos\theta=-\sin\lambda$ and thus

$$\theta=\frac{\pi}{2}+\lambda\,.$$

Theorem 2.4- In all M_I motions obtained from M_{II} motion, the length of the sliding velocity vector $\overrightarrow{V_f}$ is

$$\left\|\overrightarrow{V_f}\right\|=\left|\dot{\lambda}\right|\left\|\overrightarrow{PB}\right\|\ .$$

Proof- By taking the norms of vectors $\overrightarrow{PB}$ and $\overrightarrow{V_f}$ in Eq.(2.7) and (2.8), we have

$$\left\|\overrightarrow{PB}\right\| = \sqrt{(x-x_p)^2 + (y-y_p)^2}$$

and

$$\vec{V}_f = \dot{\lambda}\left(-\sin\lambda(x-x_p) - \cos\lambda(y-y_p), \cos\lambda(x-x_p) - \sin\lambda(y-y_p)\right)$$

$$\left\|\overrightarrow{V_f}\right\| = \left|\dot{\lambda}\right|\sqrt{(x-x_p)^2 + (y-y_p)^2}.$$

and hence

$$\left\|\overrightarrow{V_f}\right\| = \left|\dot{\lambda}\right|\left\|\overrightarrow{PB}\right\|.$$

Theorem 2.5- The lengths of the directrix vectors of the pole lines corresponding to each (λ, μ) position of all M_I motions obtained from M_{II} are equal on the fixed and moving planes.

Proof- If the pole P in Eq.(2.2) is written in terms of the parameter $\frac{\dot{\mu}}{\dot{\lambda}}$, then the pole P given by

$$P = \left(-b_\lambda\cos\lambda, -b_\lambda\sin\lambda\right) + \frac{\dot{\mu}}{\dot{\lambda}}\left(\ell\sin\lambda - b_\mu\cos\lambda, b_\mu\sin\lambda + \ell\cos\lambda\right) \tag{2.9}$$

$$= p_1 + \frac{\dot{\mu}}{\dot{\lambda}}\overrightarrow{v_1}$$

gives a family of lines which passes through the point p_1 with the directrix $\overrightarrow{v_1}$. Similarly, if the pole $\overline{P}$ of a fixed plane in Eq.(2.4) is written in terms of the parameter $\frac{\dot{\mu}}{\dot{\lambda}}$, we obtain

$$\overline{P} = \left(-b_\lambda + \ell\mu, b(\lambda,\mu)\right) + \frac{\dot{\mu}}{\dot{\lambda}}\left(-b_\mu, \ell\right) \tag{2.10}$$

$$= \overline{p_2} + \frac{\dot{\mu}}{\dot{\lambda}}\overline{\overrightarrow{v_2}}.$$

As directrixes in Equations (2.9) and (2.10) are being

$$\overrightarrow{v_1} = (\ell\sin\lambda - b_\mu\cos\lambda, b_\mu\sin\lambda + \ell\cos\lambda)$$

and

$$\overline{\overrightarrow{v_2}} = (-b_\mu, \ell)$$

taking the norms, we get

$$\left\|\overrightarrow{v_1}\right\| = \left\|\overrightarrow{v_2}\right\| = \sqrt{\ell^2 + b_\mu{}^2} \ .$$

Moreover, for the translation vector $C = \left(\ell\mu, b(\lambda,\mu)\right)$, since

$$C_\mu = \frac{\partial C}{\partial \mu} = \left(\ell, b_\mu\right),$$

then

$$\left\|C_\mu\right\| = \sqrt{\ell^2 + b_\mu{}^2} \ .$$

Therefore, we have

$$\left\|\overrightarrow{v_1}\right\| = \left\|\overrightarrow{v_2}\right\| = \left\|C_\mu\right\|.$$

Definition 2.1. When the sliding velocity vectors of a fixed point are carried to the starting point, without changing the directions, then the locus of the end points of these vectors, which is a curve, is called hodograph.

Now as a special case, let $\dot{\lambda}^2 + \dot{\mu}^2 = 1$ and analyse any (x,y) point of the locus of the hodographs in all M_I motion obtained from M_{II}, according to the position of λ and μ : are they ellipse, circle or hyperbol?

Let the determinant of coefficents of

$$\begin{aligned} \dot{y}_1 &= (-x\sin\lambda - y\cos\lambda)\,\dot{\lambda} + \ell\,\dot{\mu} \\ \dot{y}_2 &= (x\cos\lambda - y\sin\lambda + b_\lambda)\,\dot{\lambda} + b_\mu\,\dot{\mu} \end{aligned} \tag{2.11}$$

be

$$\det\Delta = b_\mu(-x\sin\lambda - y\cos\lambda) + \ell(-x\cos\lambda + y\sin\lambda - b_\lambda).$$

Then

$$\dot{\lambda} = \frac{b_\mu\,\dot{y}_1 - \ell\,\dot{y}_2}{\det\Delta}$$

and

$$\dot{\mu} = \frac{(-x\sin\lambda - y\cos\lambda)\,\dot{y}_2 - (x\cos\lambda - y\sin\lambda + b_\lambda)\,\dot{y}_1}{\det\Delta}$$

are obtained. Substituting these in $\dot{\lambda}^2 + \dot{\mu}^2 = 1$, then we get

$$\begin{aligned} &\left(b_\mu{}^2 + x^2\cos^2\lambda + y^2\sin^2\lambda + b_\lambda{}^2 - 2xy\sin\lambda\cos\lambda + 2xb_\lambda\cos\lambda - 2yb_\lambda\sin\lambda\right)\dot{y}_1{}^2 + \\ &\left(\ell^2 + x^2\sin^2\lambda + y^2\cos^2\lambda + 2xy\sin\lambda\cos\lambda\right)\dot{y}_2{}^2 + \\ &\left(-2\ell b_\mu + 2xy\sin\lambda\cos\lambda - 2xy\sin^2\lambda + 2xb_\lambda\sin\lambda + 2xy\cos^2\lambda - 2y^2\sin\lambda\cos\lambda + 2yb_\lambda\cos\lambda\right)\dot{y}_1\,\dot{y}_2 \\ &= (\det\Delta)^2 \end{aligned} \tag{2.12}$$

Corollary 2.4- In all M_I motions obtained from M_{II} motions, the locus of the hodographs of any point (x,y) is independent of the selection of μ.

Theorem 2.6- In all M_I motions derived from M_{II} motions, the hodograph of any (x,y) point is an ellipse at the position $\lambda = b_\lambda = b_\mu = 0$ [1].

Proof- Setting $\lambda = b_\lambda = b_\mu = 0$ in Eq.(2.12), we get

$$x^2 \dot{y}_1^2 + (\ell^2 + y^2)\dot{y}_2^2 + 2xy\dot{y}_1\dot{y}_2 = x^2\ell^2. \tag{2.13}$$

which shows an ellipse.

Theorem 2.7- The hodograph, being on the x-axis at the position $\lambda = b_\lambda = b_\mu = 0$ of the symmetric two points, is a circle [1].

Proof- Setting $B_0 = (\ell,0)$ in (2.13), we get

$$\dot{y}_1^2 + \dot{y}_2^2 = \ell^2$$

which is a circle with the radius ℓ. For the orbit of the point $B_0 = (\ell,0)$,

$$y_1 = \ell\cos\lambda + \ell\mu \ ,$$
$$y_2 = \ell\sin\lambda + b(\lambda,\mu)$$

are obtained from Eq.(1.5). This orbit changes according to the each (λ,μ). From the last equation

$$\dot{y}_1 = \ell\,\dot{\mu}$$
$$\dot{y}_2 = \ell\,\dot{\lambda}$$

are obtained for the point $B_0 = (\ell,0)$, from which

$$\dot{y}_1^2 + \dot{y}_2^2 = \ell^2.$$

The scalar velocity of the point B_0 is also given by

$$\left\|(\dot{y}_1,\dot{y}_2)\right\| = \ell$$

which is a constant. Elementary analytic geometry shows us that the area of the ellipse obtained from (2.13) is $\pi.\ell.|x|$. This area is only dependent on x. The area of an ellipse for a point on the pole line is zero. Moreover, hodograph of this point is circle for $B_0 = (-\ell,0)$ and $\lambda = b_\lambda = b_\mu = 0$.

3. THE ACCELERATION POLES OF THE MOTIONS

Now we will investigate the locus of the points which have zero sliding acceleration. We need to solve first, the equation

$$\ddot{A}X + \ddot{C} = 0 \ . \tag{3.1}$$

For the acceleration of M_I motion obtained from M_{II}, from the equation

$$P_{ip} = -\left(\ddot{A}\right)^{-1}\ddot{C}$$

we get

$$\begin{bmatrix} x_{ip} \\ y_{ip} \end{bmatrix} = \frac{1}{(\dot{\lambda})^4 + (\ddot{\lambda})^2}\begin{bmatrix} \ddot{\lambda}\sin\lambda + (\dot{\lambda})^2\cos\lambda & -\ddot{\lambda}\cos\lambda + (\dot{\lambda})^2\sin\lambda \\ \ddot{\lambda}\cos\lambda - (\dot{\lambda})^2\sin\lambda & \ddot{\lambda}\sin\lambda + (\dot{\lambda})^2\cos\lambda \end{bmatrix}\ddot{C}.$$

we get

$$x_{ip} = \frac{1}{(\dot{\lambda})^4 + (\ddot{\lambda})^2}\left[(\ddot{\lambda}\sin\lambda + (\dot{\lambda})^2\cos\lambda)(\ell\ddot{\mu}) + (-\ddot{\lambda}\cos\lambda + (\dot{\lambda})^2\sin\lambda)(b_{\lambda\lambda}(\dot{\lambda})^2 + b_{\mu\mu}(\dot{\mu})^2 + 2b_{\lambda\mu}\dot{\lambda}\dot{\mu} + b_{\lambda}\ddot{\lambda} + b_{\mu}\ddot{\mu})\right]$$

and

$$y_{ip} = \frac{1}{(\dot{\lambda})^4 + (\ddot{\lambda})^2}\left[(\ddot{\lambda}\cos\lambda - (\dot{\lambda})^2\sin\lambda)(\ell\ddot{\mu}) + (\ddot{\lambda}\sin\lambda + (\dot{\lambda})^2\cos\lambda)(b_{\lambda\lambda}(\dot{\lambda})^2 + b_{\mu\mu}(\dot{\mu})^2 + 2b_{\lambda\mu}\dot{\lambda}\dot{\mu} + b_{\lambda}\ddot{\lambda} + b_{\mu}\ddot{\mu})\right]$$

which are the coordinates of the acceleration poles at a position (λ, μ). For $\lambda = \mu = 0$, we have

$$x_{ip} = \frac{\ell\ddot{\mu}(\dot{\lambda})^2}{(\dot{\lambda})^4 + (\ddot{\lambda})^2} - \frac{\ddot{\lambda}}{(\dot{\lambda})^4 + (\ddot{\lambda})^2}(b_{\lambda\lambda}(\dot{\lambda})^2 + b_{\mu\mu}(\dot{\mu})^2 + 2b_{\lambda\mu}\dot{\lambda}\dot{\mu} + b_{\lambda}\ddot{\lambda} + b_{\mu}\ddot{\mu}), \tag{3.2}$$

$$y_{ip} = \frac{\ell\ddot{\mu}\ddot{\lambda}}{(\dot{\lambda})^4 + (\ddot{\lambda})^2} + \frac{(\dot{\lambda})^2}{(\dot{\lambda})^4 + (\ddot{\lambda})^2}(b_{\lambda\lambda}(\dot{\lambda})^2 + b_{\mu\mu}(\dot{\mu})^2 + 2b_{\lambda\mu}\dot{\lambda}\dot{\mu} + b_{\lambda}\ddot{\lambda} + b_{\mu}\ddot{\mu}) \ . \tag{3.3}$$

Theorem 2.8- The acceleration pole of the M_I motions obtained from M_{II} coincides with the pole lines of fixed and moving plane at position $\lambda = \mu = \dot{\lambda} = \dot{\mu} = 0$.

Proof- From (3.2) and (3.3), we get

$$x_{ip} = -\frac{\ddot{\lambda}}{(\ddot{\lambda})^2}(b_{\lambda}\ddot{\lambda} + b_{\mu}\ddot{\mu}) \ , \ y_{ip} = \frac{\ell\ddot{\mu}\ddot{\lambda}}{(\ddot{\lambda})^2}$$

are obtained. If $\frac{\ddot{\mu}}{\ddot{\lambda}}$ is derived from the second equation and substituted in the first equation, then we get

$$\ell x_{ip} + b_{\mu}y_{ip} + \ell b_{\lambda} = 0 .$$

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