fractal and fractional

Article

Fractional Diffusion Models for the Atmosphere of Mars

Salvador Jiménez [1], David Usero [2], Luis Vázquez [2] and Maria Pilar Velasco [1,*]

1 Department of Applied Mathematics to the Information and Communications Technologies, Universidad Politécnica de Madrid, 28040 Madrid, Spain; s.jimenez@upm.es
2 Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid, Spain; umdavid@mat.ucm.es (D.U.); lvazquez@fdi.ucm.es (L.V.)
* Correspondence: mp.velasco@upm.es; Tel.: +34-91-336-7837

Received: 5 December 2017; Accepted: 24 December 2017; Published: 28 December 2017

Abstract: The dust aerosols floating in the atmosphere of Mars cause an attenuation of the solar radiation traversing the atmosphere that cannot be modeled through the use of classical diffusion processes. However, the definition of a type of fractional diffusion equation offers a more accurate model for this dynamic and the second order moment of this equation allows one to establish a connection between the fractional equation and the Ångstrom law that models the attenuation of the solar radiation. In this work we consider both one and three dimensional wavelength-fractional diffusion equations, and we obtain the analytical solutions and numerical methods using two different approaches of the fractional derivative.

Keywords: fractional calculus; Mittag-Leffler type functions; fractional ordinary and partial differential equations; dust; solar radiation

1. Introduction

The scattering of solar radiation by the dust particles floating in the atmosphere is a relevant phenomenon in the study of Mars' atmosphere [1,2]. The dust aerosols have a direct effect on basic aspects of the atmospheric dynamic: the surface and the atmospheric heating rates. It is known that the mechanism is more complex than a simple diffusive process associated to a Brownian motion.

The attenuation of the solar radiation traversing the atmosphere is modeled by the Lambert-Beer-Bouguer law, which establishes that the direct solar irradiance $F(\lambda)$ at the Mars' surface at wavelength λ is given by

$$F(\lambda) = DF_0(\lambda)e^{-\tau(\lambda)m} \tag{1}$$

where $F_0(\lambda)$ is the spectral irradiance at the top of the atmosphere, m is the absolute air mass, D is the correction factor for the earth-sun distance, and $\tau(\lambda)$ is the total optical thickness at wavelength λ.

This total optical thickness is the sum of three components: the molecular scattering optical thickness $\tau_r(\lambda)$, the absorption optical thickness for atmospheric gases $\tau_g(\lambda)$, and the aerosol optical thickness $\tau_a(\lambda)$. The last component, $\tau_a(\lambda)$, depends directly on the medium and its value can be obtained by direct solar spectral irradiance measurements. Mathematically, the dependency of the optical thickness of an aerosol on the wavelength is expressed through the Ångström law [3],

$$\tau_a^{-1} = \frac{\lambda^\alpha}{\beta} \tag{2}$$

where α and β are parameters related to the size and the content of the aerosol, respectively. So, β is the extinction coefficient corresponding to a 1 µm wavelength, which depends on the concentration of aerosols in the atmosphere, and α is the wavelength exponent which is closely correlated to the size distribution of the scattering particles.

In the case of a classical diffusive process the governing equation is the classical diffusion equation,

$$\frac{\partial \varphi}{\partial \lambda} = c\frac{\partial^2 \varphi}{\partial x^2} \tag{3}$$

which gives a dependency of the mean square value proportional to λ, to be compared with (2) in the case of $\alpha = 1$. However, radiative transfer analysis for Mars atmosphere [2] gives the values $\alpha = 1.2$ and $\beta = 0.3$, corresponding to an aerosol optical thickness $\tau_a = 0.6$ which cannot be accounted for with a model such as (17).

In this context where the classical model of the process does not provide a good enough description of this kind of dynamics, the use of Fractional Calculus could provide an adequate description and simulation of the complex systems and processes observed [4–6]. The reason is that the fractional operators are non-local and they involve convolution kernels which act as memory factors. Consequently, the Fractional Calculus offers new scenarios of modeling to describe physical phenomena, like the dynamic of the Martian atmosphere.

In this manuscript, we continue the work started in [7] and we define both one and three dimensional wavelength-fractional diffusion equations to obtain a more accurate model of the attenuation of the solar radiation traversing the atmosphere. With this fractional model, a generalization of the classical Angstrom law is obtained through the second order moment of the fractional equation.

This paper is organized as follow. In Section 2, we expose some fundamental concepts of Fractional Calculus that are necessary in this work. Next, in Section 3, a brief overview of the classical and Caputo time-fractional diffusion processes is shown, with the objective of explaining a three dimensional wavelength-fractional diffusion model for the dynamic of the attenuation of the solar radiation traversing the Martian atmosphere in Section 4. Finally, in Section 5, we consider numerical methods for the three dimensional problem using two different approaches to the fractional derivative.

2. Some Tools of Fractional Calculus

In this section, we introduce some of the basic tools of Fractional Calculus. There exist several definitions of fractional operators. We show the fractional derivatives and integrals used in this work and some properties and related special functions [4–6].

Definition 1. *Let $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$ and let f be a suitable real function (for example, it suffices if $f \in L_1(a,b)$). The Riemann-Liouville fractional operators are:*

$$({}^{RL}I^{\alpha}_{t,a+}f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\, dt \qquad (t > a) \tag{4}$$

$$({}^{RL}D^{\alpha}_{t,a+}f)(t) = D^n(I^{n-\alpha}_{t,a+}f)(t) \qquad (t > a) \tag{5}$$

$$({}^{RL}I^{\alpha}_{t,b-}f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1} f(s)\, dt \qquad (x < b) \tag{6}$$

$$({}^{RL}D^{\alpha}_{t,b-}f)(t) = D^n(I^{n-\alpha}_{t,b-}f)(t), \qquad (x < b) \tag{7}$$

where D is the usual differential operator.

Definition 2. *Let $\alpha > 0$, with $n-1 < \alpha < n$ and $n \in \mathbb{N}$, $[a,b] \subset \mathbb{R}$ and let f be a suitable real function (for example, it suffices if $f \in L_1(a,b)$). The Caputo fractional derivative is:*

$$({}^{C}D^{\alpha}_{t,a+}f)(x) = ({}^{RL}I^{n-\alpha}_{t,a+}D^n f)(t) \qquad (x > a) \tag{8}$$

$$({}^{C}D^{\alpha}_{t,b-}f)(x) = ({}^{RL}I^{n-\alpha}_{t,b-}D^n f)(t) \qquad (x < b) \tag{9}$$

The following identity is well-known for a suitable function f (for example, f n-times derivable):

$$({}^{RL}D^{\alpha}_{t,a+}f)(t) = ({}^{C}D^{\alpha}_{t,a+}f)(t) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)}(t-a)^{j-\alpha}. \tag{10}$$

Thus, we have:

$$({}^{C}D^{\alpha}_{t,a+}1) = 0 \qquad ({}^{RL}D^{\alpha}_{t,a+}1) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{11}$$

Definition 3. *Let be $\alpha, \beta > 0$. The Mittag-Leffler functions are defined as:*

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \tag{12}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{13}$$

Definition 4. *By using the previous definitions, we can obtain a function of type Mittag-Leffler as:*

$$e^{\lambda z}_{\alpha} := z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha}) \qquad (\alpha > 0, \lambda \in \mathbb{C}) \tag{14}$$

These functions allow us to generalize the classical exponential functions. So:

Property 1. *Let $\alpha > 0$ and $\lambda \in \mathbb{C}$, then:*

$${}^{C}D^{\alpha}_{t,0+} E_{\alpha}(\lambda t^{\alpha}) = \lambda E_{\alpha}(\lambda t^{\alpha}), \tag{15}$$

$${}^{RL}D^{\alpha}_{t,0+} e^{\lambda t}_{\alpha} = \lambda e^{\lambda t}_{\alpha} \tag{16}$$

3. Classical and Time-Fractional Diffusion Processes

For the classical diffusion equation,

$$\frac{\partial \varphi}{\partial t} = c \frac{\partial^2 \varphi}{\partial x^2} \tag{17}$$

where c is the diffusion coefficient, the two-order moment of the solution is obtained as

$$< X^2 >= \int_{-\infty}^{\infty} x^2 \varphi dx, \qquad \text{tal que} \qquad \int_{-\infty}^{\infty} \varphi dx = 1 \tag{18}$$

Since,

$$\frac{d}{dt} < X^2 >= \int_{-\infty}^{\infty} x^2 \varphi_t dx = \int_{-\infty}^{\infty} x^2 c \varphi_{xx} dx = 2c \tag{19}$$

then, we obtain

$$< X^2 >= 2ct \tag{20}$$

Consequently, we can obtain the moments of any order without knowing the exact expression of the solution, only by knowing that the solution is 0 in the infinity and the solution verifies the normalization condition.

The classical diffusion equation can be generalized to obtain a fractional diffusion equation which could model processes of anomalous diffusion:

$$D^{\alpha}_{t} \varphi = c D^{\beta}_{x} \varphi, \qquad 0 < \alpha < 2, \beta > 0 \tag{21}$$

In fact, this equation represents an interpolation between the classical diffusion and wave equations, that are recovered for $\alpha = 1, \beta = 2$ and $\alpha = \beta = 2$, respectively.

From the Equation (21), we can study the particular case of the time-fractional diffusion equation with Caputo fractional derivative:

$$ {}^{C}D^{\alpha}_{t,0+}\varphi = c\frac{\partial^2 \varphi}{\partial x^2}, \qquad 0 < \alpha < 2 \tag{22}$$

The Cauchy problem associated to this time-fractional diffusion equation is well defined through the following conditions:

$$\lim_{x\to\pm\infty} \varphi(t,x) = 0, \qquad t > 0 \tag{23}$$

$$\varphi(0+,x) = g(x), \qquad x \in \mathbb{R} \tag{24}$$

$$\frac{\partial}{\partial t}\varphi(t,x)\bigg|_{t=0} = 0 \quad \text{(additional condition for } 1 < \alpha < 2) \tag{25}$$

This problem has been deeply studied [4,6,8–11] and their exact solution has been obtained as:

$$\varphi(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,1}(-ck^2t^{\alpha})G(k)e^{-ikx}dk \tag{26}$$

where $G(k)$ is the Fourier transform of the initial condition $g(x)$. Or, equivalently, by using the expression of the Mittag-Leffler function:

$$\varphi(t,x) = \sum_{j=0}^{\infty} \frac{(ct^{\alpha})^j}{\Gamma(\alpha j+1)} D^{2j}g(x) \tag{27}$$

In the particular case of the function $g(x)$ is the Dirac delta function $\delta(x)$, the solution of the Cauchy problem (22)–(25) is known as fundamental solution or Green function and it is expressed as:

$$\varphi(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_{\alpha,1}(-ck^2t^{\alpha})e^{-ikx}dk \tag{28}$$

Whose explicit representation in terms of the Wright function is:

$$\varphi(t,x) = \frac{1}{2\pi t^{\alpha/2}} W\left(-\frac{|x|}{t^{\alpha/2}c^{1/2}}; -\frac{\alpha}{2}; 1-\frac{\alpha}{2}\right) \tag{29}$$

that recovers the classical Gaussian function for $\alpha = 1$.

This fundamental solution verifies

$$\varphi(t,x) \geq 0, \quad \forall t > 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \varphi(t,x)dx = 1 \tag{30}$$

and their moments are obtained as:

$$< X^{2n} >= \int_{-\infty}^{\infty} x^{2n}\varphi(t,x)dx = \frac{\Gamma(2n+1)}{\Gamma(\alpha n+1)}(ct^{\alpha})^n, \qquad n = 0,1,2... \tag{31}$$

In particular, the second order moment ($n = 1$) is:

$$< X^2 >= \int_{-\infty}^{\infty} x^2\varphi(t,x)dx = \frac{2c}{\Gamma(\alpha+1)}t^{\alpha} \tag{32}$$

4. Wavelength-Fractional Diffusion Model

The propagation of the solar radiation in the atmosphere is a complex process. The dynamics is governed by different time/space scales. Thus, it is natural to think about integro-differential equations to describe a better modeling [7,12]. In this context, we assume the following:

$$ {}^{C}D^{\alpha}_{\lambda,0+}\varphi = \frac{\Gamma(\alpha+1)}{2\beta}\frac{\partial^2\varphi}{\partial x^2}, \qquad 0<\alpha<2. \tag{33} $$

$$ \begin{cases} \lim\limits_{x\to\pm\infty}\varphi(\lambda,x)=0, & \lambda>0 \\ \varphi(0+,x)=\delta(x), & x\in\mathbb{R} \\ \dfrac{\partial}{\partial\lambda}\varphi(\lambda,x)\Big|_{\lambda=0}=0 & (\text{condition for } 1<\alpha<2) \end{cases} \tag{34} $$

Their exact solution is known as fundamental solution or Green function, and it is expressed through the Mittag-Leffler and Wright functions:

$$ \varphi(\lambda,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E_\alpha\left(-\frac{\Gamma(\alpha+1)}{2\beta}\lambda^\alpha\right)e^{-ikx}dk=\frac{1}{2\pi\lambda^{\alpha/2}}W\left(-\frac{|x|}{\lambda^{\alpha/2}}\left(\frac{\Gamma(\alpha+1)}{2\beta}\right)^{-1/2};-\frac{\alpha}{2};1-\frac{\alpha}{2}\right) \tag{35} $$

For this equation, their second order moment is

$$ <X^2>=\int_{-\infty}^{\infty}x^2\varphi(t,x)dx=\frac{1}{\beta}\lambda^\alpha=\tau_a^{-1} \tag{36} $$

where the parameter of derivation α is related to the size of the aerosol particles, and the coefficient $1/\beta$, related to the content of the aerosol, defines the diffusion coefficient. Also, we can obtain their high order moments (which could be measured) as:

$$ <X^{2n}>=\int_{-\infty}^{\infty}x^{2n}\varphi(\lambda,x)dx=\frac{\Gamma(2n+1)}{\Gamma(\alpha n+1)}\left(\frac{\Gamma(\alpha+1)}{2\beta}\lambda^\alpha\right)^n, \qquad n=0,1,2... \tag{37} $$

4.1. 3D Diffusion Model

The 1D model can be extended to full 3D:

$$ {}^{C}D^{\alpha}_{\lambda,0+}\varphi=\frac{\Gamma(\alpha+1)}{2\beta}\left(c_1\frac{\partial^2\varphi}{\partial x^2}+c_2\frac{\partial^2\varphi}{\partial y^2}+c_3\frac{\partial^2\varphi}{\partial z^2}\right), \qquad \alpha\in(0,1)\cup(1,2) \tag{38} $$

with an initial value problem and boundary conditions given by:

$$ \begin{cases} \lim\limits_{\|\vec{x}\|\to\infty}\varphi(\lambda,\vec{x})=0, & \lambda>0 \\ \varphi(0^+,\vec{x})=g(\vec{x}), & \vec{x}\in\mathbb{R}^3 \\ \dfrac{\partial}{\partial\lambda}\varphi(\lambda,\vec{x})\Big|_{\lambda=0}=0 & (\text{additional condition when } 1<\alpha<2) \end{cases} \tag{39} $$

where $\vec{x}=(x,y,z)$, and coefficients c_j, $j=1,2,3$, taken as constants, correspond to possible anisotropies along the three spatial directions. The initial profile $g(\vec{x})$ may correspond, for instance, to the incoming solar irradiance reaching the top of the atmosphere.

4.1.1. Radial Symmetry Case

Whenever the dust layers are stratified radially and the spatial dependence of φ is just with the distance and not the directions, we may consider the radial symmetry case. If the strata are homogeneous such that $c_1 = c_2 = c_3 = 1$, we perform a standard change of the function, defining $u(t,r) = r\varphi(t,r)$, and we have:

$$ {}^{C}D^{\alpha}_{\lambda,0+}u = \frac{\Gamma(\alpha+1)}{2\beta}\frac{\partial^2 u}{\partial r^2}\,, \tag{40} $$

with:

$$ \begin{cases} \lim\limits_{r\to\infty} u(\lambda,r) = 0, & \lambda > 0 \\ u(0^+,r) = f(r), & r \in \mathbb{R}^+ \\ \dfrac{\partial}{\partial\lambda}u(\lambda,r)\Big|_{\lambda=0} = 0 & \text{(additional condition when } 1<\alpha<2) \end{cases} \tag{41} $$

Formally this is a 1D problem that we solve using the same techniques as in [7]:

$$ u(\lambda,r) = r\varphi(\lambda,r) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_\alpha\left(-\frac{\Gamma(\alpha+1)}{2\beta}k^2\lambda^\alpha\right)F(k)e^{-ikr}dk \tag{42} $$

$$ = \sum_{j=0}^{\infty}\frac{f^{(2j)}(r)}{\Gamma(\alpha j+1)}\left(\frac{\Gamma(\alpha+1)}{2\beta}\lambda^\alpha\right)^j \tag{43} $$

where F is the Fourier transform of f and $f^{(2j)}$ is the $2j$-order derivative of f with respect to r.

If instead of a problem defined for $r \in \mathbb{R}^+$ we consider a finite region $r \in [0,R]$ with, for instance, fixed-end null boundary conditions, the solution using the standard separation of variables technique is given by a Fourier series instead of by the Fourier transform, as in (42):

$$ r\varphi(\lambda,r) = \sum_{k=1}^{\infty} c_k E_\alpha\left(-\frac{ck^2\pi^2}{R^2}\lambda\right)\sin\left(\frac{k\pi r}{R}\right) \tag{44} $$

where:

$$ c_k = \frac{2}{R}\int_0^R \sin\left(\frac{k\pi r}{R}\right)f(r)\,dr\,, \qquad E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)} \quad (\text{Mittag}-\text{Leffler functions}) \tag{45} $$

5. Numerical Methods

We have constructed two numerical methods for the 3D problem using two different approaches to the Caputo operator given by the Diethelm [13] and the Odibat [14] representations.

We define a discrete mesh with step size h for λ: $\lambda_n = nh$, and discrete meshes with a common step $\triangle\ell$ for each of the spatial variables: $x_i = i\triangle\ell$, $y_j = j\triangle\ell$, $z_k = k\triangle\ell$; $i,j,k \in \mathbb{Z}$. To represent the second order spatial derivatives we use standard, second order, centered, finite differences.

5.1. Diethelm Representation

In the case $0 < \alpha < 1$, the numerical scheme is given by:

$$ \begin{aligned} &\frac{1}{h^\alpha\Gamma(2-\alpha)}\left(\varphi_{n;\vec\ell} - \varphi_{0;\vec\ell} + \sum_{m=1}^{n-1} d_{mn}\left(\varphi_{n-m;\vec\ell} - \varphi_{0;\vec\ell}\right)\right) \\ = &\frac{\Gamma(\alpha+1)}{2\beta}\left(c_1\frac{\varphi_{n;i+1} - 2\varphi_{n;\vec\ell} + \varphi_{n;i-1}}{\triangle\ell^2} + c_2\frac{\varphi_{n;j+1} - 2\varphi_{n;\vec\ell} + \varphi_{n;j-1}}{\triangle\ell^2}\right. \\ &\left. + c_3\frac{\varphi_{n;k+1} - 2\varphi_{n;\vec\ell} + \varphi_{n;k-1}}{\triangle\ell^2}\right) \end{aligned} \tag{46} $$

with: $d_{mn} = (m+1)^{1-\alpha} - 2m^{1-\alpha} + (m-1)^{1-\alpha}$, $0 < m < n$. In order to avoid a cumbersome index notation, we use the following:

$$\varphi_{n;\vec{\ell}} = \varphi(\lambda_n; x_i, y_j, z_k), \qquad \varphi_{n;i\pm1} = \varphi(\lambda_n; x_{i\pm1}, y_j, z_k) \tag{47}$$

$$\varphi_{n;j\pm1} = \varphi(\lambda_n; x_i, y_{j\pm1}, z_k), \qquad \varphi_{n;k\pm1} = \varphi(\lambda_n; x_i, y_j, z_{k\pm1}) \tag{48}$$

In this way, when all indexes are at nominal values, i, j, k, we indicate that with $\vec{\ell}$, and, otherwise, only present the indexes that are incremented o decremented from the nominal values.

In the case $1 < \alpha < 2$ an additional auxiliary discrete variable $\theta_{n;\vec{\ell}}$ must be defined [15]. The numerical scheme has then two discrete equations:

$$\theta_{n;\vec{\ell}} = \frac{\varphi_{n+1;\vec{\ell}} - \varphi_{n-1;\vec{\ell}}}{2h} \tag{49}$$

$$\begin{aligned} &\frac{1}{h^{\alpha-1}\Gamma(3-\alpha)}\left(\theta_{n;\vec{\ell}} - \theta_{0;\vec{\ell}} + \sum_{m=1}^{n-1} e_{mn}\left(\theta_{n-m;\vec{\ell}} - \theta_{0;\vec{\ell}}\right)\right) \\ = \; &\frac{\Gamma(\alpha+1)}{2\beta}\left(c_1\frac{\varphi_{n;i+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;i-1}}{\triangle\ell^2} + c_2\frac{\varphi_{n;j+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;j-1}}{\triangle\ell^2}\right. \\ &\left. + c_3\frac{\varphi_{n;k+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;k-1}}{\triangle\ell^2}\right) \end{aligned} \tag{50}$$

with $e_{mn} = (m+1)^{2-\alpha} - 2m^{2-\alpha} + (m-1)^{2-\alpha}$, $0 < m < n$.

In both cases the numerical scheme has a local truncation error for the solution $\mathcal{O}(\triangle\ell^2 h^{2-\alpha})$.

5.2. Odibat Representation

The numerical scheme is given by:

$$\begin{aligned} &\frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left[\varphi_{n+1,\vec{\ell}} - \varphi_{n-1,\vec{\ell}} + \sum_{m=1}^{n-1} C_{nm}\left(\varphi_{m+1,\vec{\ell}} - \varphi_{m-1,\vec{\ell}}\right)\right] \\ = \; &\frac{\Gamma(\alpha+1)}{2\beta}\left(c_1\frac{\varphi_{n;i+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;i-1}}{\triangle\ell^2} + c_2\frac{\varphi_{n;j+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;j-1}}{\triangle\ell^2}\right. \\ &\left. + c_3\frac{\varphi_{n;k+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;k-1}}{\triangle\ell^2}\right) \end{aligned} \tag{51}$$

in the case $0 < \alpha < 1$, and by:

$$\begin{aligned} &\frac{h^{-\alpha}}{\Gamma(4-\alpha)}\left[\varphi_{n+1,\vec{\ell}} - 2\varphi_{n,\vec{\ell}} + \varphi_{n-1,\vec{\ell}} + \sum_{m=1}^{n-1} C_{nm}\left(\varphi_{m+1,\vec{\ell}} - 2\varphi_{m,\vec{\ell}} + \varphi_{m-1,\vec{\ell}}\right)\right] \\ = \; &\frac{\Gamma(\alpha+1)}{2\beta}\left(c_1\frac{\varphi_{n;i+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;i-1}}{\triangle\ell^2} + c_2\frac{\varphi_{n;j+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;j-1}}{\triangle\ell^2}\right. \\ &\left. + c_3\frac{\varphi_{n;k+1} - 2\varphi_{n;\vec{\ell}} + \varphi_{n;k-1}}{\triangle\ell^2}\right) \end{aligned} \tag{52}$$

in the case $1 < \alpha < 2$. In both cases $C_{nm} = (n-m+1)^{p-\alpha+1} - 2(n-m)^{p-\alpha+1} + (n-m-1)^{p-\alpha+1}$, $0 < m < n$.

The local truncation error for the solution is $\mathcal{O}(\triangle\ell^2 h^2)$, which is better than for the Diethelm approach. On the other hand, the computation of the coefficients C_{nm} is much more costly than in the Diethelm approach.

6. Conclusions

Fractional operators are an important tool for the modeling of complex diffusion processes. In particular, a wavelength-fractional diffusion model is used to obtain a more accurate model of the attenuation of the solar radiation traversing the atmosphere. With this fractional model, a generalization of the classical Angstrom law is obtained through the second order moment of the fractional equation.

The innovation of this work is that the framework of Fractional Calculus has been applied to the dynamic and behavior of the solar radiation in a medium with atmospheric dust for the first time and we are able to reproduce in 3D the observed dependence of Angstrom law, arising directly from the extended fractional modeling.

We have implemented the numerical schemes for the 3D problem. The computational resources are demanding and we are performing the ongoing simulations through cloud computing. Preliminary computational cases for the 1D problem was obtained in [7], where the efficiency of the numerical schemes was verified, and we are working presently with cloud computing in order to obtain results in the 3D problem shortly.

Acknowledgments: The authors thank to the support of MINECO of Spain, through the project ESP2016-79135-R.

Author Contributions: All authors contributed equally to this work.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cachorro, V.E.; de Frutos, A.M.; Casanova, J.L. Determination of the Angstrom turbidity parameters. *Appl. Opt.* **1987**, *26*, 3069–3076.
2. Córdoba, C.; Vázquez, L. Characterization of atmospheric aerosols by an in-situ photometric technique in planetary environments. In *First Jet Propulsion Laboratory In Situ Instruments Workshop*; Bearman, G.H., Beauchamp, P.M., Eds.; SPIE: Bellingham, WA, USA, 2003.
3. Ångström, A. On the atmospheric transmission of sun radiation and on dust in the air. *Geograf. Ann.* **1929**, *11*, 156–166, doi:10.2307/519399.
4. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
5. Zaslavsky, G.M.; Baleanu, D.; Tenreiro, J.A. *Fractional Differentiation and Its Applications*; Physica Scripta: Bristol, UK, 2009.
6. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus. Models and Numerical Methods*; World Scientific: Singapore, 2012.
7. Vázquez, L.; Velasco, M.P.; Vázquez-Poletti, J.L.; Llorente, I.M.; Usero, D.; Jiménez, S. Modeling and simulation of the atmospheric dust dynamics: Fractional calculus and cloud computing. *Int. J. Numer. Anal. Model.* **2017**, *15*, 74–85.
8. Diethelm, K. *The Analysis of Fractional Differential Equations*; Springer: New York, NY, USA, 2010.
9. Sun, H.; Chen, W.; Li, C.; Chen, Y. Fractional differential models for anomalous diffusion. *Phys. A Stat. Mech. Appl.* **2010**, *389*, 2719–2724.
10. Chen, W.; Sun, H.; Zhang, X.; Korošak, D. Anomalous diffusion modeling by fractal and fractional derivatives. *Comput. Math. Appl.* **2010**, *59*, 1754–1758
11. Chen, W. Time-space fabric underlying anomalous diffusion. *Chaos Solitons Fractals* **2006**, *28*, 923–929.
12. Velasco, M.P.; Usero, D.; Jiménez, S.; Aguirre, C.; Vázquez, L. Mathematics and Mars Exploration. *Pure Appl. Geophys.* **2015**, *172*, 33–47.
13. Diethelm, K.; Ford, N.J.; Freed, A.D.; Luchko, Y. Algorithms for the fractional calculus: A selection of numerical methods. *Comput. Methods Appl. Mech. Eng.* **2005**, *194*, 743–773.

14. Odibat, Z.M. Computational algorithms for computing the fractional derivatives of functions. *Math. Comput. Simul.* **2009**, *79*, 2013–2020.
15. Diethelm, K. Multi-term fractional differential equations, multi-order fractional differential systems and their numerical solution. *J. Eur. Syst. Autom.* **2008**, *42*, 665–676, doi:10.3166/jesa.42.665-676.