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Power Laws in Fractionally Electronic Elements

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Abstract: The highlight presented in this short article is about the power laws with respect to fractional capacitance and fractional inductance in terms of frequency.

Keywords: fractional capacitor; fractional inductor; power laws

1. Introduction

Let i_c and u_c be the current and voltage through and over a capacitor C, with the constant capacitance denoted by C again. Then, one says that C_f stands for a pseudo-capacitance in the sense that

$$i_c = C_f \frac{d^\alpha u_c(t)}{dt^\alpha} = C\frac{du_c(t)}{dt} \text{ for } 0 < \alpha < 1, \tag{1}$$

where $\frac{d^\alpha u_c(t)}{dt^\alpha} = u_c^{(\alpha)}(t)$ denotes the fractional derivative of order α of u_c [1]. One calls C_f the pseudo-capacitance of a capacitor because its unit is Farad $\times$ $s^{1-\alpha}$ instead of Farad [1]. In this article, we call it fractional capacitance of order α of a capacitor. Similarly, the fractional inductance of order β, denoted by L_f, is in the sense that

$$u_L = L_f \frac{d^\beta i_L(t)}{dt^\beta} = L\frac{di_L(t)}{dt} \text{ for } 0 < \beta < 1, \tag{2}$$

where u_L and i_L are the voltage and current over and through an inductor L with the constant inductance denoted again by L. The unit of L_f is Henry $\times$ $s^{1-\beta}$. It is also called the pseudo-inductance [1,2].

Fractional elements, including a fractional capacitor and a fractional inductor, attract research interests in engineering. The literature about their analysis and applications is rich, see References [1–10], referring [11–14] to some recent work on fractional calculus. However, reports about power laws that fractional elements follow are rarely seen. This short article aims at expounding the power laws that fractional elements follow.

In the rest of this article, we present the results in Section 2, which is followed by concluding remarks.

2. Results

Denoted by $X(\omega)$ the Fourier transform of $x(t)$. Then, one has, for $\alpha > 0$,

$$F\left[x^{(\alpha)}(t)\right] = \int_{-\infty}^{\infty} x^{(\alpha)}(t)e^{-j\omega t}dt = (j\omega)^\alpha X(\omega), \tag{3}$$

where $j = \sqrt{-1}$. Consequently,

$$x^{(\alpha)}(t) = F^{-1}\left[(j\omega)^{\alpha} X(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)^{\alpha} X(\omega) e^{j\omega t} d\omega, \tag{4}$$

see Miller and Ross [15], Uchaikin [16] (Section 4.5.3) and Lavoie [17] (p. 246).

Following Miller and Ross [15], Raina and Koul [18], we explain our research in the domain of generalized functions. Thus, any function considered in this article is differentiable of any times and its Fourier transform exists (Gelfand and Vilenkin [19]).

Theorem 1. *The fractional capacitance C_f may be expressed by*

$$C_f = (j\omega)^{1-\alpha} C. \tag{5}$$

Proof. The Fourier transform of $C_f \frac{d^{\alpha} u_c(t)}{dt^{\alpha}}$ in Equation (1) is given by

$$F\left[C_f \frac{d^{\alpha} u_c(t)}{dt^{\alpha}}\right] = (j\omega)^{\alpha} C_f U_c(\omega), \tag{6}$$

where $U_c(\omega) = F[u_c(t)]$. On the other hand, doing the Fourier transform of $C\frac{du_c(t)}{dt}$ in Equation (1) produces

$$F\left[C\frac{du_c(t)}{dt}\right] = j\omega C U_c(\omega). \tag{7}$$

Thus, according to Equation (1) and from Equations (6) and (7), we have $(j\omega)^{\alpha} C_f U_c(\omega) = j\omega C U_c(\omega)$. Therefore, we have $C_f = (j\omega)^{1-\alpha} C$. Hence, Theorem 1 holds. □

Note 1. *C_f reduces to C if $\alpha \to 1$. We use the symbol C_f to represent either fractional capacitance or fractional capacitor.*

Corollary 1. *Denote the capacitance ratio by*

$$Rc = C/C_f. \tag{8}$$

Then, Rc follows the power law in the form

$$Rc = Rc(f, \alpha) = (j2\pi f)^{\alpha-1}. \tag{9}$$

Proof. From Equation (2.3), we have $Rc = \frac{C}{C_f} = (j\omega)^{\alpha-1} = (j2\pi f)^{\alpha-1}$. The proof completes. □

Corollary 1 suggests a power law of Rc in terms of frequency with respect to the fractional capacitor C_f. The unit of Rc is Hertz$^{\alpha-1}$. Figure 1 shows the plots of $|Rc(f, \alpha)| = (2\pi f)^{\alpha-1}$.

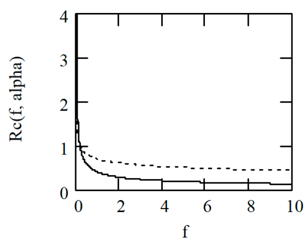

Figure 1. The plots of $|Rc(f, \alpha)|$. Solid line: $\alpha = 0.5$. Dot line: $\alpha = 0.8$.

Theorem 2. *The fractional inductance L_f may be in the form*

$$L_f = (j\omega)^{1-\beta}L. \tag{10}$$

Proof. The Fourier transform of $L_f \frac{d^\beta i_L(t)}{dt^\beta}$ in Equation (2) is in the form

$$F\left[L_f i_L^{(\beta)}(t)\right] = (j\omega)^\beta L_f I_L(\omega), \tag{11}$$

where $I_L(\omega) = F[i_L(t)]$. On the other side, in Equation (2), we have

$$F\left[L\frac{i_L(t)}{dt}\right] = (j\omega)LI_L(\omega). \tag{12}$$

From Equation (2) and according to Equations (11) and (12), we have $(j\omega)^\beta L_f I_L(\omega) = j\omega L I_L(\omega)$. Thus, we have $L_f = (j\omega)^{1-\beta}L$. This completes the proof. □

Note 2. *The fractional inductance L_f degenerates to L when $\beta \to 1$. The symbol L_f stands for both fractional inductance and fractional inductor.*

Corollary 2. *Let Rl be the inductance ratio in the form*

$$Rl = L/L_f. \tag{13}$$

Then, it follows the power law in the form

$$Rl = Rl(f,\beta) = (j2\pi f)^{\beta-1}. \tag{14}$$

Proof. From Equation (10), we have $Rl = \frac{L}{L_f} = (j\omega)^{\beta-1} = (j2\pi f)^{\beta-1}$. This completes the proof. □

Corollary 2 exhibits a power law of Rl in terms of frequency with respect to L_f. The unit of Rl is Hertz$^{\beta-1}$.

3. Concluding Remarks

We have presented Theorems 1 and 2 to express the fractional capacitance and fractional inductance, respectively. In addition, power laws in terms of frequency with respect to fractional capacitance and fractional inductance have been given in Corollaries 1 and 2. To be precise, for a fractional capacitor (inductor) of order α, the ratio of C (L) to C_f (L_f) obeys $(j2\pi f)^{\alpha-1}$ with the unit Hertz$^{\alpha-1}$. Specifically for a fractional capacitor, due to $0 < \alpha < 1$, the power law described by Corollary 1 reveals that $C_f \to \infty$ when $f \to 0$. Note that a key property of a supercapacitor or an ultracapacitor utilized in batteries is that it has an infinitely large capacitance for $f \to 0$ [20–22]. Therefore, the power law presented in Corollary 1 provides a new explanation about that as an application in the case of supercapacitors.

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