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Simpson's Second-Type Inequalities for Co-Ordinated Convex Functions and Applications for Cubature Formulas

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Abstract: Inequality theory has attracted considerable attention from scientists because it can be used in many fields. In particular, Hermite–Hadamard and Simpson inequalities based on convex functions have become a cornerstone in pure and applied mathematics. We deal with Simpson's second-type inequalities based on coordinated convex functions in this work. In this paper, we first introduce Simpson's second-type integral inequalities for two-variable functions whose second-order partial derivatives in modulus are convex on the coordinates. In addition, similar results are acquired by considering that powers of the absolute value of second-order partial derivatives of these two-variable functions are convex on the coordinates. Finally, some applications for Simpson's 3/8 cubature formula are given.

Keywords: coordinated convex functions; Simpson's type inequality

MSC: 26D15; 26D10; 90C23

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1. Introduction and Preliminaries

Simpson's rules (Thomas Simpson 1710–1761) are well-known methods in numerical analysis for the purpose of numerical integration and the numerical approximation of definite integrals. Two famous Simpson rules are known in the literature, and one of them is the following estimation known as Simpson's second-type (Simpson's $\frac{3}{8}$) inequality.

Theorem 1. *Let $\mathcal{F} : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a four-time continuously differentiable mapping on $[a,b]$ and $\parallel \mathcal{F}^{(4)} \parallel_\infty = \sup_{\varkappa_1 \in [a,b]} |\mathcal{F}^{(4)}| < \infty$. Then, the following inequality holds:*

$$\left|\frac{1}{8}\left[\mathcal{F}(a) + 3\mathcal{F}\left(\frac{2a+b}{3}\right) + 3\mathcal{F}\left(\frac{a+2b}{3}\right) + \mathcal{F}(b)\right] - \frac{1}{b-a}\int_a^b \mathcal{F}(\varkappa_1)\mathrm{d}\varkappa_1\right| \\ \leq \frac{1}{6480} \parallel \mathcal{F}^{(4)} \parallel_\infty (b-a)^5.$$

This result is also named a Newton-type inequality in the literature. Simpson- and Newton-type inequalities have attracted remarkable attention from the related researchers because these results have wide application areas in the applied sciences of mathematics. New Newton-type inequalities based on three-step quadratic kernels for various classes of functions have been developed by many authors. For illustration purposes, some Simpson-type inequalities for s-convex functions were provided by Alomari et al. in [1]. In [2], Sarikaya et al gave some inequalities of Simpson's type based on s-convexity and their applications for special means of real numbers. What is more, some Hadamard- and Simpson-type results for functions' second derivatives of which are s-convex in the second sense were deduced by Park in [3]. In addition, Gao and Shi obtained new inequalities of Newton's type for functions whose absolute values of second derivatives are convex in [4]. Afterwards, Hermite–Hadamard-, Simpson- and Newton-type inequalities for harmonically convex mappings have been observed by some researchers. As an example, authors have examined Newton-type results for harmonic and p-harmonic convex functions in [5,6].

Dragomir introduced the concept of coordinated convex functions in [7] as follows:

Definition 1. *A function $\mathcal{F} : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is said to be coordinated convex on the rectangle Δ if*

$$\begin{aligned}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)) \le{}& (1-\tau_1)(1-\tau_2)\mathcal{F}(a,c)+(1-\tau_1)\tau_2\mathcal{F}(a,d)\\ &+\tau_1(1-\tau_2)\mathcal{F}(b,c)+\tau_1\tau_2\mathcal{F}(b,d),\end{aligned}$$

for all (a,b), $(c,d) \in \Delta$ and $\tau_1,\tau_2 \in [0,1]$.

Recently, several papers have been written on convex functions and their variant forms on the coordinates. For example, Sarikaya et al. [2] proved some new trapezoidal-type inequalities for differentiable coordinated convex functions on a rectangle from the plane $\mathbb{R}^2$. Later, Latif et al. [8] established some new midpoint-type inequalities for differentiable coordinated convex functions with two variables. The authors provided some Hermite–Hadamard-type inequalities for coordinated convex functions in [9–11]. Alomari et al. [12] obtained the Hermite–Hadamard-type inequality for s-convex functions on the coordinates. Latif et al. [13] proved the analogous results for h-convex functions on the coordinates. Alomari et al. established some Hadamard-type inequalities for coordinated log-convex functions in [14]. Simpson-type inequalities on coordinates were introduced by Özdemir et al. [15]. For more recent developments and generalizations, see [8,16–20]. Chen [21] introduced the following lemma which generalized the previously known results; see [2,8,15]. For the appropriate and suitable choices of λ, he obtained several new and known midpoints, trapezoidal and Simpson's $\frac{1}{3}$-type inequalities for differentiable coordinated convex and concave functions in two variables.

Lemma 1. *Let $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on the rectangle $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 \mathcal{F}}{\partial\tau_1\partial\tau_2} \in L(\Delta)$ and $\lambda \in [0,1]$, then for any $\tau_1,\tau_2 \in [0,1]$ and $(\varkappa_1,\varkappa_2) \in \Delta$, we possess the inequality*

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \mathcal{F}(\varkappa_1,\varkappa_2)\mathrm{d}\varkappa_2\mathrm{d}\varkappa_1 \qquad (1)\\ &\quad+(1-\lambda)^2\mathcal{F}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &\quad+\frac{\lambda(1-\lambda)}{2}\left[\mathcal{F}\left(a,\frac{c+d}{2}\right)+\mathcal{F}\left(b,\frac{c+d}{2}\right)+\mathcal{F}\left(\frac{a+b}{2},c\right)+\mathcal{F}\left(\frac{a+b}{2},d\right)\right]\\ &\quad+\frac{\lambda^2}{4}[\mathcal{F}(a,c)+\mathcal{F}(a,d)+\mathcal{F}(b,c)+\mathcal{F}(b,d)]\\ &\quad-\frac{1}{2(b-a)}\int_a^b\left(\lambda\mathcal{F}(\varkappa_1,c)+2(1-\lambda)\mathcal{F}\left(\varkappa_1,\frac{c+d}{2}\right)+\lambda\mathcal{F}(\varkappa_1,d)\right)\mathrm{d}\varkappa_1\\ &\quad-\frac{1}{2(d-c)}\int_c^d\left(\lambda\mathcal{F}(a,\varkappa_2)+2(1-\lambda)\mathcal{F}\left(\frac{a+b}{2},\varkappa_2\right)+\lambda\mathcal{F}(b,\varkappa_2)\right)\mathrm{d}\varkappa_2\\ &=(b-a)(d-c)\int_0^1\int_0^1 M(\tau)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\mathrm{d}\tau_1,\end{aligned}$$

where

$$M(\tau) = \begin{cases} (\tau_1 - \frac{\lambda}{2})(\tau_2 - \frac{\lambda}{2}), & (\tau_1, \tau_2) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\ (\tau_1 - \frac{\lambda}{2})(\tau_2 - (1 - \frac{\lambda}{2})), & (\tau_1, \tau_2) \in [0, \frac{1}{2}] \times (\frac{1}{2}, 1] \\ (\tau_1 - (1 - \frac{\lambda}{2}))(\tau_2 - \frac{\lambda}{2}), & (\tau_1, \tau_2) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\ (\tau_1 - (1 - \frac{\lambda}{2}))(\tau_2 - (1 - \frac{\lambda}{2})), & (\tau_1, \tau_2) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]. \end{cases}$$

Inspired and motivated by the ongoing research on coordinates, in this paper we establish an auxiliary result to obtain new Simpson second-type inequalities for coordinated convex functions. With the help of this result, Simpson's second-type integral inequalities for mappings whose second-order partial derivatives in modulus are convex on the coordinates on the rectangle from the plane are given. Additionally, new estimations for Simpson's 3/8 cubature formula are presented via the results developed in this study.

2. Main Results

In this section, we obtain the Simpson's second-type integral inequalities based on the following lemma in two variables.

Lemma 2. *Suppose the function $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is partial differentiable on the rectangle $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$. If $\frac{\partial^2 \mathcal{F}}{\partial \tau_1 \partial \tau_2} \in L(\Delta)$, then for any $\tau_1, \tau_2 \in [0, 1]$ and $(x, y) \in \Delta$, we have*

$$\begin{aligned} &\mathfrak{S}(a, b, x; c, d, y) \\ &= (b-a)(d-c) \int_0^1 \int_0^1 \mathcal{K}(\tau_1)\mathcal{K}(\tau_2) \frac{\partial^2}{\partial \tau_1 \partial \tau_2} \mathcal{F}((1-\tau_1)a + \tau_1 b, (1-\tau_2)c + \tau_2 d) \mathrm{d}\tau_2 \mathrm{d}\tau_1, \end{aligned} \tag{2}$$

where $\mathcal{K}(\tau)$ and $\mathfrak{S}(a, b, x; c, d, y)$ are defined by

$$\mathcal{K}(\tau) = \begin{cases} \tau - \frac{1}{8}, & \tau \in [0, \frac{1}{3}) \\ \tau - \frac{1}{2}, & \tau \in [\frac{1}{3}, \frac{2}{3}) \\ \tau - \frac{7}{8}, & \tau \in [\frac{2}{3}, 1]. \end{cases}$$

and

$$\begin{aligned} &\mathfrak{S}(a, b, x; c, d, y) \\ &= \frac{\mathcal{F}(a,c) + \mathcal{F}(a,d) + \mathcal{F}(b,c) + \mathcal{F}(b,d)}{64} \\ &+ \frac{3}{64}\left\{ \mathcal{F}\left(a, \frac{2c+d}{3}\right) + \mathcal{F}\left(a, \frac{c+2d}{3}\right) + \mathcal{F}\left(b, \frac{2c+d}{3}\right) + \mathcal{F}\left(b, \frac{c+2d}{3}\right) \right. \\ &\left. + \mathcal{F}\left(\frac{2a+b}{3}, c\right) + \mathcal{F}\left(\frac{a+2b}{3}, c\right) + \mathcal{F}\left(\frac{2a+b}{3}, d\right) + \mathcal{F}\left(\frac{a+2b}{3}, d\right) \right\} \\ &+ \frac{9}{64}\left\{ \mathcal{F}\left(\frac{2a+b}{3}, \frac{2c+d}{3}\right) + \mathcal{F}\left(\frac{2a+b}{3}, \frac{c+2d}{3}\right) \right. \\ &\left. + \mathcal{F}\left(\frac{a+2b}{3}, \frac{2c+d}{3}\right) + \mathcal{F}\left(\frac{a+2b}{3}, \frac{c+2d}{3}\right) \right\} \\ &- \frac{1}{8}\frac{1}{b-a}\int_a^b \left[\mathcal{F}(x,c) + 3\mathcal{F}\left(x, \frac{2c+d}{3}\right) + 3\mathcal{F}\left(x, \frac{c+2d}{3}\right) + \mathcal{F}(x,d) \right] \mathrm{d}x \\ &- \frac{1}{8}\frac{1}{d-c}\int_c^d \left[\mathcal{F}(a,y) + 3\mathcal{F}\left(\frac{2a+b}{3}, y\right) + 3\mathcal{F}\left(\frac{a+2b}{3}, y\right) + \mathcal{F}(b,y) \right] \mathrm{d}y \\ &+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \mathcal{F}(x,y) \mathrm{d}y \mathrm{d}x, \end{aligned} \tag{3}$$

respectively.

Proof. We consider the double integral

$$\begin{aligned}
&\int_0^1\int_0^1 \mathcal{K}(\tau_1)\mathcal{K}(\tau_2)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&=\int_0^1 \mathcal{K}(\tau_1)\left\{\int_0^1 \mathcal{K}(\tau_2)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\right\}\mathrm{d}\tau_1.
\end{aligned} \tag{4}$$

Now, if we handle the integral inside the bracket, then we possess

$$\begin{aligned}
&\int_0^1 \mathcal{K}(\tau_2)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2 \\
&=\int_0^{\frac{1}{3}}\left(\tau_2-\frac{1}{8}\right)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2 \\
&\quad+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\tau_2-\frac{1}{2}\right)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2 \\
&\quad+\int_{\frac{2}{3}}^{1}\left(\tau_2-\frac{7}{8}\right)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2.
\end{aligned} \tag{5}$$

Calculating the first integral in the right-side of (5) by using integration by parts, we find that

$$\begin{aligned}
&\int_0^{\frac{1}{3}}\left(\tau_2-\frac{1}{8}\right)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2 \\
&=\frac{1}{d-c}\left\{\frac{5}{24}\frac{\partial}{\partial\tau_1}\mathcal{F}\left((1-\tau_1)a+\tau_1 b,\frac{2c+d}{3}\right)\right. \\
&\quad+\frac{1}{8}\frac{\partial}{\partial\tau_1}\mathcal{F}((1-\tau_1)a+\tau_1 b,c) \\
&\quad\left.-\int_0^{\frac{1}{3}}\frac{\partial}{\partial\tau_1}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\right\}.
\end{aligned}$$

Adding the resulting equalities side by side after having calculated the other integrals in the right-side of (5), one has the identity

$$\begin{aligned}
&\int_0^1 \mathcal{K}(\tau_2)\frac{\partial}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2 \\
&=\frac{1}{d-c}\left\{\frac{1}{8}\frac{\partial}{\partial\tau_1}\mathcal{F}((1-\tau_1)a+\tau_1 b,c)+\frac{1}{8}\frac{\partial}{\partial\tau_1}\mathcal{F}((1-\tau_1)a+\tau_1 b,d)\right. \\
&\quad+\frac{3}{8}\frac{\partial}{\partial\tau_1}\mathcal{F}\left((1-\tau_1)a+\tau_1 b,\frac{c+2d}{3}\right)+\frac{3}{8}\frac{\partial}{\partial\tau_1}\mathcal{F}\left((1-\tau_1)a+\tau_1 b,\frac{2c+d}{3}\right) \\
&\quad\left.-\int_0^1\frac{\partial}{\partial\tau_1}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\right\} \\
&:=\frac{1}{d-c}F(a,b,\tau_1;c,d,,\tau_2).
\end{aligned} \tag{6}$$

Substituting the equality (6) in (4), we find that

$$
\begin{aligned}
&(d-c)\int_0^1\int_0^1 \mathcal{K}(\tau_1)\mathcal{K}(\tau_2)\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&=\int_0^1 \mathcal{K}(\tau_1)F(a,b,\tau_1;c,d,,\tau_2)\mathrm{d}\tau_1\\
&=\int_0^{\frac{1}{3}}\left(\tau_1-\frac{1}{8}\right)F(a,b,\tau_1;c,d,,\tau_2)\mathrm{d}\tau_1+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\tau_1-\frac{1}{2}\right)F(a,b,\tau_1;c,d,,\tau_2)\mathrm{d}\tau_1\\
&+\int_{\frac{2}{3}}^1\left(\tau_1-\frac{7}{8}\right)F(a,b,\tau_1;c,d,,\tau_2)\mathrm{d}\tau_1.
\end{aligned}
$$

Computing these integrals and later using the change of the variable $x=(1-\tau_1)a+\tau_1 b$ and $y=(1-\tau_2)c+\tau_2 d$ for $\tau_1,\tau_2\in[0,1]$, we obtain the required equality. □

Theorem 2. *Let $\mathcal{F}:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}\right|$ is convex on the coordinates on Δ and $\tau_1,\tau_2\in[0,1]$, then, for $(x,y)\in\Delta$, the following inequality holds:*

$$
\begin{aligned}
|\mathfrak{S}(a,b,x;c,d,y)|\le(b-a)(d-c)\frac{625}{331776}\Bigg\{&\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|\\
&+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|\Bigg\},
\end{aligned}
$$

where $\mathfrak{S}(a,b,x;c,d,y)$ is defined as in (3).

Proof. Taking the absolute value in both sides of (2), due to the properties of modulus, we have the inequality

$$
\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)|\\
&\le(b-a)(d-c)\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\left|\frac{\partial^2\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)}{\partial\tau_1\partial\tau_2}\right|\mathrm{d}\tau_2\mathrm{d}\tau_1.
\end{aligned}
$$

On the grounds that $|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}|$ is convex function on the coordinates, one possesses

$$
\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)|\\
&\le(b-a)(d-c)\\
&\quad\times\Bigg\{\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&\qquad+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&\qquad+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c\right|\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1\\
&\qquad+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1\Bigg\}.
\end{aligned}
$$

On the other side, by fundamental integral calculation rules, we have

$$
\left.\begin{array}{l}
\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1\\
\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1\\
\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1\\
\int_0^1\int_0^1|\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1
\end{array}\right\}=\frac{625}{331776}. \tag{7}
$$

In the light of these results, the desired inequality can be readily attained. □

Theorem 3. *Let $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 \mathcal{F}}{\partial\tau_1\partial\tau_2}\right|^q$ is convex on the coordinates on Δ, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\tau_1, \tau_2 \in [0,1]$, then, for $(x,y) \in \Delta$, we have the inequality*

$$\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)| \\
&\leq \frac{(b-a)(d-c)}{4^{\frac{1}{q}}}\left(\frac{2}{p+1}\frac{3^{p+1}+4^{p+1}+5^{p+1}}{24^{p+1}}\right)^{\frac{2}{p}} \\
&\quad\times\left(\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

where $\mathfrak{S}(a,b,x;c,d,y)$ is defined as in (3).

Proof. Using the well-known Hölder inequality for double integrals after having taken the absolute value of both sides of (2), it is found that

$$\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)| \\
&\leq (b-a)(d-c)\left(\int_0^1\int_0^1 |\mathcal{K}(\tau_1)\mathcal{K}(\tau_2)|^p \mathrm{d}\tau_2\mathrm{d}\tau_1\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\right|^q \mathrm{d}\tau_2\mathrm{d}\tau_1\right)^{\frac{1}{q}}.
\end{aligned}$$

Inasmuch as $\left|\frac{\partial^2 \mathcal{F}}{\partial\tau_1\partial\tau_2}\right|^q$ is convex function on the coordinates, one has

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\frac{\partial^2\mathcal{F}}{\partial\tau_2\partial\tau_1}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\right|^q \mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&\leq \frac{1}{4}\left(\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|^q\right).
\end{aligned}$$

We also note that

$$\begin{aligned}
&\int_0^1\int_0^1 |\mathcal{K}(\tau_1)\mathcal{K}(\tau_2)|^p \mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&= \int_0^1 |\mathcal{K}(\tau_1)|^p \mathrm{d}\tau_1 \int_0^1 |\mathcal{K}(\tau_2)|^p \mathrm{d}\tau_2 = \left(\int_0^1 |\mathcal{K}(\tau)|^p \mathrm{d}\tau\right)^2 \\
&= \left(\frac{2}{p+1}\frac{3^{p+1}+4^{p+1}+5^{p+1}}{24^{p+1}}\right)^2.
\end{aligned}$$

Hence, the proof is completed. □

Theorem 4. *Let $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 \mathcal{F}}{\partial\tau_1\partial\tau_2}\right|^q$ is convex on the coordinates on Δ for $q \geq 1$ and $\tau_1, \tau_2 \in [0,1]$, then, for $(x,y) \in \Delta$, the following inequality holds:*

$$\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)| \\
&\leq \frac{625}{82944}\frac{(b-a)(d-c)}{4^{\frac{1}{q}}}\left(\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|^q\right. \\
&\qquad \left. + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|^q + \left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{8}$$

where $\mathfrak{S}(a,b,x;c,d,y)$ is defined as in (3).

Proof. From Lemma 2, we possess

$$\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)| \\
&\leq (b-a)(d-c)\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\left|\frac{\partial^2\mathcal{F}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)}{\partial\tau_1\partial\tau_2}\right|\mathrm{d}\tau_2\mathrm{d}\tau_1.
\end{aligned}$$

Using the well-known power mean inequality for double integrals, one has the inequality

$$\begin{aligned}
&|\mathfrak{S}(a,b,x;c,d,y)| \\
&\leq (b-a)(d-c)\left(\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\mathrm{d}\tau_2\mathrm{d}\tau_1\right)^{1-\frac{1}{q}} \\
&\qquad \times\left(\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\right|^q\mathrm{d}\tau_2\mathrm{d}\tau_1\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}\right|^q$ is a convex function on the coordinates, one can possess the result

$$\begin{aligned}
&\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\right|^q \\
&\leq (1-\tau_1)(1-\tau_2)\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|^q + (1-\tau_1)\tau_2\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|^q \\
&\qquad + \tau_1(1-\tau_2)\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|^q + \tau_1\tau_2\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|^q.
\end{aligned} \tag{9}$$

Due to the inequality (9), it follows that

$$\begin{aligned}
&\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}((1-\tau_1)a+\tau_1 b,(1-\tau_2)c+\tau_2 d)\right|^q\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
\leq\ &\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,c)\right|^q\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(a,d)\right|^q\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|(1-\tau_1)\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,c)\right|^q\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1(1-\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&+\left|\frac{\partial^2\mathcal{F}}{\partial\tau_1\partial\tau_2}(b,d)\right|^q\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\tau_1\tau_2\mathrm{d}\tau_2\mathrm{d}\tau_1.
\end{aligned}$$

If we use the identities given in (7) and the fact that

$$\int_0^1\int_0^1 |\mathcal{K}(\tau_1)||\mathcal{K}(\tau_2)|\mathrm{d}\tau_2\mathrm{d}\tau_1 = \frac{625}{82944},$$

then we readily obtain the inequality (8), which finishes the proof. □

3. Applications to Simpson's $\frac{3}{8}$ Cubature Formula

In this section, we handle applications of the integral inequalities developed in the main results section to obtain estimates of Simpson's Cubature formula. First of all, we recall Simpson's quadrature formula. Supposing that φ is a division of the interval $[a,b]$, i.e., $\varphi : a = x_0 < x_1 < x_2, \dots, < x_{n-1} < x_n = b$, $h_i = \frac{(x_{i+1}-x_i)}{3}$. The Simpson's $\frac{3}{8}$ quadrature formula is defined by

$$S(\mathcal{F},\varphi) = \sum_{i=0}^{n-1} \frac{\mathcal{F}(x_i) + 3\mathcal{F}(x_i+h_i) + 3\mathcal{F}(x_i+2h_i) + \mathcal{F}(x_{i+1})}{8}(x_{i+1}-x_i).$$

Now, we define Simpson's Cubature formula to derive new estimations. Assume that $I_m : a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b$ and $I_n : c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$ are divisions of the intervals $[a,b]$ and $[c,d]$. Then, we have the summation

$$\begin{aligned}
& C\big(\mathcal{F}, I_m, I_n\big) \qquad (10)\\
&:= \frac{1}{8}\sum_{j=0}^{n-1} k_j \int_{x_i}^{x_{i+1}} \Big[\mathcal{F}(t,y_j) + 3\mathcal{F}\big(t,y_j+k_j\big) + 3\mathcal{F}\big(t,y_j+2k_j\big) + \mathcal{F}\big(t,y_{j+1}\big)\Big]\mathrm{d}t \\
&\quad + \frac{1}{8}\sum_{i=0}^{m-1} h_i \int_{y_j}^{y_{j+1}} \Big[\mathcal{F}(x_i,s) + 3\mathcal{F}\big(x_i+h_i,s\big) + 3\mathcal{F}\big(x_i+2h_i,s\big) + \mathcal{F}\big(x_{i+1},s\big)\Big]\mathrm{d}s \\
&\quad - \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} h_i k_j \Bigg[\frac{\mathcal{F}(x_i,y_j) + \mathcal{F}(x_i,y_{j+1}) + \mathcal{F}(x_{i+1},y_j) + \mathcal{F}(x_{i+1},y_{j+1})}{64} \\
&\quad + \frac{3}{64}\Big\{\mathcal{F}\big(x_i,y_j+k_j\big) + \mathcal{F}\big(x_i,y_j+2k_j\big) + \mathcal{F}\big(x_{i+1},y_j+k_j\big) + \mathcal{F}\big(x_{i+1},y_j+2k_j\big) \\
&\quad + \mathcal{F}\big(x_i+h_i,y_j\big) + \mathcal{F}\big(x_i+2h_i,y_j\big) + \mathcal{F}\big(x_i+h_i,y_{j+1}\big) + \mathcal{F}\big(x_i+2h_i,y_{j+1}\big)\Big\} \\
&\quad + \frac{9}{64}\Big\{\mathcal{F}\big(x_i+h_i,y_j+k_j\big) + \mathcal{F}\big(x_i+2h_i,y_j+k_j\big) \\
&\quad + \mathcal{F}\big(x_i+h_i,y_j+k_j\big) + \mathcal{F}\big(x_i+h_i,y_j+2k_j\big)\Big\}\Bigg],
\end{aligned}$$

where $h_i = \frac{x_{i+1}-x_i}{3}$ and $k_j = \frac{y_{j+1}-y_j}{3}$ for $i = 0,1,2,\dots,m-1$; $j = 0,1,2,\dots,n-1$. So, we suppose that the interested integrals can be more easily calculated than the original integral

$$\int_a^b \int_c^d \mathcal{F}(t,s)\mathrm{d}s\mathrm{d}t.$$

We give new Simpson's cubature formulas in the following theorems.

Theorem 5. *Let $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be as in Theorem 2. If I_m and I_n divisions are defined as above, then we have the cubature formula*

$$\int_a^b \int_c^d \mathcal{F}(t,s)\mathrm{d}s\mathrm{d}t = C(\mathcal{F}, I_m, I_n) + R(\mathcal{F}, I_m, I_n)$$

where $C(\mathcal{F}, I_m, I_n)$ *is defined as in (10) and the remainder term* $R(\mathcal{F}, I_m, I_n)$ *satisfies the estimation:*

$$\begin{aligned}
&|R(\mathcal{F}, I_m, I_n)| \qquad (11)\\
&\leq \frac{625}{331776} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (x_{i+1} - x_i)^2 (y_{j+1} - y_j)^2 \\
&\times \left\{ \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_j) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_j) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_j) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_{j+1}) \right| \right\}.
\end{aligned}$$

Proof. Applying Theorem 2 to the interval $[x_i, x_{i+1}] \times [y_j, y_{j+1}], (i = 0, \ldots, m-1; j = 0, \ldots, n-1)$, we obtain

$$\begin{aligned}
&\Bigg| \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \mathcal{F}(t,s) \mathrm{d}s\mathrm{d}t + h_i k_j \Bigg[\frac{\mathcal{F}(x_i, y_j) + \mathcal{F}(x_i, y_{j+1}) + \mathcal{F}(x_{i+1}, y_j) + \mathcal{F}(x_{i+1}, y_{j+1})}{64} \\
&\quad + \frac{3}{64} \{ \mathcal{F}(x_i, y_j + 2k_j) + \mathcal{F}(x_i, y_j + k_j) + \mathcal{F}(x_{i+1}, y_j + 2k_j) + \mathcal{F}(x_{i+1}, y_j + k_j) \\
&\quad + \mathcal{F}(x_i + 2h_i, y_j) + \mathcal{F}(x_i + h_i, y_j) + \mathcal{F}(x_i + 2h_i, y_{j+1}) + \mathcal{F}(x_i + h_i, y_{j+1}) \} \\
&\quad + \frac{9}{64} \{ \mathcal{F}(x_i + 2h_i, y_j + 2k_j) + \mathcal{F}(x_i + 2h_i, y_j + k_j) \\
&\quad + \mathcal{F}(x_i + h_i, y_j + 2k_j) + \mathcal{F}(x_i + h_i, y_j + k_j) \} \Bigg] \\
&\quad - \frac{1}{8} k_j \int_{x_i}^{x_{i+1}} [\mathcal{F}(t, y_j) + \mathcal{F}(t, y_{j+1}) + 3\mathcal{F}(t, y_j + 2k_j) + 3\mathcal{F}(t, y_j + k_j)] \mathrm{d}t \\
&\quad - \frac{1}{8} h_i \int_{y_j}^{y_{j+1}} [\mathcal{F}(x_i, s) + \mathcal{F}(x_{i+1}, s) + 3\mathcal{F}(x_i + 2h_i, s) + 3\mathcal{F}(x_i + h_i, s)] \mathrm{d}s \Bigg] \Bigg| \\
&\leq \frac{625}{331776} (x_{i+1} - x_i)^2 (y_{j+1} - y_j)^2 \\
&\quad \times \left\{ \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_j) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_{j+1}) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_j) \right| + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_{j+1}) \right| \right\}
\end{aligned}$$

for all $i = 0, \ldots, m-1$; $j = 0, \ldots, n-1$ and where $h_i = \frac{x_{i+1} - x_i}{3}$ and $k_j = \frac{y_{j+1} - y_j}{3}$. Summing over i from 0 to $m-1$ and over j from 0 to $n-1$ by considering the generalized triangle inequality, the estimation (11) can be attained. □

Theorem 6. *Let* $\mathcal{F} : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ *be as in Theorem 3. If* I_m *and* I_n *divisions are defined as in above, then we have the cubature formula*

$$\int_a^b \int_c^d \mathcal{F}(t,s) \mathrm{d}s\mathrm{d}t = C(\mathcal{F}, I_m, I_n) + R(\mathcal{F}, I_m, I_n)$$

where $C(\mathcal{F}, I_m, I_n)$ *is defined as in (10) and the remainder term* $R(\mathcal{F}, I_m, I_n)$ *satisfies the estimation:*

$$\begin{aligned}
&|R(\mathcal{F}, I_m, I_n)| \qquad (12)\\
&\leq \frac{1}{4^{\frac{1}{q}}} \left(\frac{2}{p+1} \frac{3^{p+1} + 4^{p+1} + 5^{p+1}}{24^{p+1}} \right)^{\frac{2}{p}} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (x_{i+1} - x_i)^2 (y_{j+1} - y_j)^2 \\
&\times \left\{ \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_j) \right|^q + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_i, y_{j+1}) \right|^q + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_j) \right|^q + \left| \frac{\partial^2 \mathcal{F}}{\partial t_1 \partial t_2}(x_{i+1}, y_{j+1}) \right|^q \right\}^{\frac{1}{q}}.
\end{aligned}$$

Proof. Applying similar methods in the proof of Theorem 5 by considering the inequality given in the Theorem 3, the desired result can be readily obtained. □

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