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Briot–Bouquet Differential Subordinations for Analytic Functions Involving the Struve Function

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Abstract: We define a new class of exponential starlike functions constructed by a linear operator involving normalized form of the generalized Struve function. Making use of a technique of differential subordination introduced by Miller and Mocanu, we investigate several new results related to the Briot–Bouquet differential subordinations for the linear operator involving the normalized form of the generalized Struve function. We also obtain univalent solutions to the Briot–Bouquet differential equations and observe that these solutions are the best dominant of the Briot–Bouquet differential subordinations for the exponential starlike function class. Moreover, we give an application of fractional integral operator for a complex-valued function associated with the generalized Struve function. The significance of this paper is due to the technique employed in proving the results and novelty of these results for the Struve functions. The approach used in this paper can lead to several new problems in geometric function theory associated with special functions.

Keywords: analytic function; Briot–Bouquet differential subordination; Struve function

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1. Introduction

Special functions occur in the solution to various problems of applied mathematics, physics, and engineering sciences. The theory of special functions has gained the attention of many researchers during the last two centuries due to its usefulness in the mathematical sciences. Among the special functions, the Struve function is a convenient function that provides solutions to various problems such as water-wave problems [1], unsteady aerodynamics [2], particle quantum dynamical studies of spin decoherence [3], and fractional-order differential equation or fractional-order integral equations [4,5]. The Struve function and its generalizations were studied in many respects. For instance, in [6], the authors studied k-fractional operators by using a k-Struve function. For more details on special functions and Struve functions, we refer to [7–11] and references therein.

The first impact of special functions in geometric function theory was by Brown [12], who studied the univalence of Bessel functions in 1960; in the same year, Kreyszig and Todd [13] determined the radius of univalence of Bessel functions. After Louis de Branges proved the Bieberbach Conjecture by using the generalized hypergeometric function in 1984, special functions became popular in studies of geometric function theory. Recently, there has been great interest dealing with various geometric properties of special functions such as the Mittag–Leffler function and Bessel, Struve, and Lommel functions of the first kind [14–16].

We denote by $\mathcal{A}$ the class of holomorphic (or analytic) functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ having the form

$$f(z) = z + \sum_{\ell=2}^{\infty} a_\ell z^\ell, \quad (z \in \mathbb{D}). \tag{1}$$

The subclass $\mathcal{S}$ of the class $\mathcal{A}$ contains all the univalent functions. We denote by Ω the class of Schwarz functions ω, which are holomorphic in $\mathbb{D}$ fulfilling $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{D}$. For analytic functions f_1 and f_2 in $\mathbb{D}$, we state that f_1 is subordinate to f_2, denoted by $f_1 \prec f_2$, if there exists a function ω such that $f_1 = f_2 \circ \omega$ (see [17]).

In [18], Ma and Minda considered the class of analytic functions ϕ with a positive real part in $\mathbb{D}$, which maps the disk $\mathbb{D}$ onto regions symmetric with respect to the real axis, starlike with respect to $\phi(0) = 1$ such that $\phi'(0) > 0$, and introduced

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}.$$

When $\phi(z) = (1 + Pz)/(1 + Rz)$, $(-1 \le R < P \le 1)$ is set, the class $\mathcal{S}^*(\phi)$ gives the class of Janowski starlike functions denoted by $\mathcal{S}^*[P, R]$ (see [19]). The special case $P = 1 - 2\beta$ and $R = -1$ with $0 \le \beta < 1$ gives the class $\mathcal{S}^*(\beta)$ of starlike functions of order β. Recently, various authors have introduced and studied several Ma–Minda-type classes of starlike functions [20–22]. Mendiretta et al. [23] considered the class $\mathcal{S}^*_e := \mathcal{S}(e^z)$ of exponential starlike functions, which involve the function $\phi(z) = e^z$. This function has a positive real part in $\mathbb{D}$ and is starlike with respect to 1 and $\phi'(0) > 0$, and $\phi(\mathbb{D}) = \{\omega \in \mathbb{C} : |\log \omega| < 1\}$ is symmetric with respect to the real axis. We denote by $\mathcal{P}_e$ the class of holomorphic functions $\tilde{p}$ in $\mathbb{D}$ with $\tilde{p}(0) = 1$ and $\tilde{p}(z) \prec e^z$ for every $z \in \mathbb{D}$. A function $f \in \mathcal{A}$ is said to be exponential starlike if $zf'(z)/f(z)$ belongs to $\mathcal{P}_e$.

Consider the second-order inhomogeneous Bessel differential equation (see [24]);

$$z^2\psi''(z) + z\psi'(z) + (z^2 - \nu^2)\psi(z) = \frac{4(z/2)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu + 1/2)}, \quad (z, \nu \in \mathbb{C}) \tag{2}$$

where $\Gamma(.)$ is the gamma function. The particular solution to Equation (2), which is referred to as the Struve function of the first kind of order ν, has the form

$$H_\nu(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\Gamma(\ell + 3/2)\Gamma(\nu + \ell + 3/2)} \left(\frac{z}{2}\right)^{2\ell+\nu+1}.$$

The differential equation

$$z^2\psi''(z) + z\psi'(z) - (z^2 + \nu^2)\psi(z) = \frac{4(z/2)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu + 1/2)}, \quad (z, \nu \in \mathbb{C}) \tag{3}$$

differs from Equation (2) in the coefficients of ψ. The particular solution to Equation (3), which is referred to as the modified Struve function of the first kind of order ν, has the representation (see [24]);

$$L_\nu(z) = -ie^{-i\nu\pi/2}H_\nu(iz) = \sum_{\ell=0}^{\infty} \frac{1}{\Gamma(\ell + 3/2)\Gamma(\nu + \ell + 3/2)} \left(\frac{z}{2}\right)^{2\ell+\nu+1}.$$

Consider the generalized second-order inhomogeneous differential equation (see [25]);

$$z^2\psi''(z) + dz\psi'(z) + (cz^2 - \nu^2 + (1 - d)\nu)\psi(z) = \frac{4(z/2)^{\nu+1}}{\sqrt{\pi}\Gamma(\nu + d/2)}, \quad (c, d, z, \nu \in \mathbb{C}). \tag{4}$$

The case $d = 1$ and $c = 1$ in Equation (4) gives the Equation (2), while the case $d = 1$ and $c = -1$ in Equation (4) leads to the Equation (3). The particular solution to Equation (4) has the form

$$M_{\nu,c,d}(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell c^\ell}{\Gamma(\ell + 3/2)\Gamma(\nu + \ell + (d + 2)/2)} \left(\frac{z}{2}\right)^{2\ell+\nu+1}, \quad (c, d, z, \nu \in \mathbb{C}) \tag{5}$$

and is called as the generalized Struve function of the first kind of order ν. This series is convergent in the whole complex plane, but is not univalent in $\mathbb{D}$. We take the transformation

$$v_{\nu,c,d}(z) = 2^{\nu}\sqrt{\pi}\Gamma(\nu + (d+2)/2)z^{-(\nu+1)/2}M_{\nu,c,d}(\sqrt{z}) = \sum_{\ell=0}^{\infty} \frac{(-c/4)^{\ell} z^{\ell}}{(3/2)_{\ell}(\kappa)_{\ell}}, \tag{6}$$

where $\kappa = \nu + (d+2)/2 \neq 0, -1, -2, \ldots$ and, where

$$(\alpha)_{\ell} = \frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+\ell-1), \quad (\alpha)_0 = 1$$

is the Pochhammer symbol. The entire function $v_{\nu,c,d}$ given by Equation (6) has the representation given by

$$4z^2 v''_{\nu,c,d}(z) + 2(2\nu+d+3)zv'_{\nu,c,d}(z) + (cz+2\nu+d)v_{\nu,c,d}(z) = 2\nu+d.$$

Starlikeness, convexity, univalency, and close-to-convexity of the function $v_{\nu,c,d}$ were studied by Orhan and Yagmur [25], Yagmur and Orhan [26]. Noreen et al. [27] explored a relationship between the function $v_{\nu,c,d}$ and the Janowski class. Recently, Naz et al. [28] introduced that the Struve function $v_{\nu,c,d}$ belongs to the class $\mathcal{P}_e$.

Using the convolution of the functions f given by Equation (1) and $g_{\nu,c,d} = zv_{\nu,c,d}$, Raza and Yagmur [29] introduced the linear operator $\mathcal{S}^c_{\kappa} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{S}^c_{\kappa} f(z) = g_{\nu,c,d}(z) * f(z) = z + \sum_{\ell=1}^{\infty} \frac{(-c/4)^{\ell}}{(3/2)_{\ell}(\kappa)_{\ell}} a_{\ell+1} z^{\ell+1}, \tag{7}$$

where $c, d, \nu \in \mathbb{C}$ and $\kappa = \nu + (d+2)/2 \notin \mathbb{Z}_0^-$. From this operator, the following can be obtained:

$$z(\mathcal{S}^c_{\kappa+1} f(z))' = \kappa \mathcal{S}^c_{\kappa} f(z) - (\kappa-1)\mathcal{S}^c_{\kappa+1} f(z). \tag{8}$$

Two special cases of the operator $\mathcal{S}^c_{\kappa}$ were presented by Raza and Yagmur [29] as follows:

(i) Choosing $d = 1, c = 1$ in Equation (7), we obtain the operator $\mathcal{S}_{\nu} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{S}_{\nu} f(z) = g_{\nu,1,1} * f(z) = z + \sum_{\ell=1}^{\infty} \frac{(-1/4)^{\ell}}{(3/2)_{\ell}(\nu+3/2)_{\ell}} a_{\ell+1} z^{\ell+1}.$$

This operator satisfies the recurrence formula

$$z(\mathcal{S}_{\nu+1} f(z))' = (\nu+3/2)\mathcal{S}_{\nu} f(z) - (\nu+1/2)\mathcal{S}_{\nu+1} f(z). \tag{9}$$

(ii) Choosing $d = 1, c = -1$ in Equation (7), we obtain the operator $\mathcal{H}_{\nu} : \mathcal{A} \to \mathcal{A}$ given by

$$\mathcal{H}_{\nu} f(z) = g_{\nu,-1,1} * f(z) = z + \sum_{\ell=1}^{\infty} \frac{(1/4)^{\ell}}{(3/2)_{\ell}(\nu+3/2)_{\ell}} a_{\ell+1} z^{\ell+1}.$$

This operator satisfies the recurrence formula

$$z(\mathcal{H}_{\nu+1} f(z))' = (\nu+3/2)\mathcal{H}_{\nu} f(z) - (\nu+1/2)\mathcal{H}_{\nu+1} f(z). \tag{10}$$

By using the operator $\mathcal{S}^c_{\kappa}$ and the concept of subordination, we define the new class of exponential starlike functions given below.

Definition 1. *Let $f \in \mathcal{A}$. A function f is a member of the class $\mathcal{S}^c_{\kappa,e}$ if it satisfies the condition*

$$\frac{z(\mathcal{S}^c_{\kappa} f(z))'}{\mathcal{S}^c_{\kappa} f(z)} \prec e^z, \tag{11}$$

where $\mathcal{S}_\kappa^c f(z)$ is given by Equation (7) for $c, d, \nu \in \mathbb{C}$, $\kappa = \nu + (d+2)/2 \notin \mathbb{Z}_0^-$ and $z \in \mathbb{D}$.

This paper deals with novel applications of Briot–Bouquet differential subordination given by

$$\varphi(z) + \frac{z\varphi'(z)}{\eta\varphi(z) + \mu} \prec \phi(z), \quad (\eta, \mu \in \mathbb{C}, \eta \neq 0) \tag{12}$$

with $\varphi(0) = \phi(0) = 1$. If the univalent function $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$ has the feature $\varphi \prec q$ for all analytic functions φ, then it is called a dominant of (12). If $\tilde{q} \prec q$ for all dominants q, then a dominant $\tilde{q}$ is said to be the best dominant (see [30,31]).

Recently, several researchers have studied some strong differential subordination, superordination, and sandwich-type results for analytic functions associated with the normalized form of the generalized Struve function (see [32]). In this paper, we employ a different method based upon the Briot–Bouquet differential subordination, which was investigated by Miller and Mocanu [31], and establish several new Briot–Bouquet differential subordination results for the linear operator $\mathcal{S}_\kappa^c$, which involves normalized form of the generalized Struve function. We also obtain univalent solutions to the Briot–Bouquet differential equations and observe that these solutions are the best dominant of the Briot–Bouquet differential subordinations for function class $\mathcal{S}_{\kappa,e}^c$. To prove our main results, we need to give the following lemmas.

Lemma 1 ([33])**.** *Let ϕ ($\phi(0) = 1$) be convex univalent in $\mathbb{D}$, and let φ of the form $\varphi(z) = 1 + b_1 z + b_2 z^2 + \ldots$ ($\varphi(0) = 1$) be analytic in $\mathbb{D}$. If*

$$\varphi(z) + \frac{1}{\mu} z\varphi'(z) \prec \phi(z), \quad (\mu \neq 0,\ \Re\mu \geq 0)$$

then

$$\varphi(z) \prec \tilde{\phi}(z) = \frac{\mu}{z^\mu} \int_0^z t^{\mu-1}\phi(t)dt \prec \phi(z), \tag{13}$$

and $\tilde{\phi}$ is the best dominant of (13).

Lemma 2 ([31])**.** *Let η ($\eta \neq 0$) and μ be complex constants, and let ϕ ($\phi(0) = 1$) be a convex univalent function in $\mathbb{D}$ with $\Re(\eta\phi(z) + \mu) > 0$. Let φ be analytic in $\mathbb{D}$ and satisfy Equation (12). If the Briot–Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = \phi(z), \quad (q(0) = 1) \tag{14}$$

has a univalent solution q, then

$$\varphi(z) \prec q(z) \prec \phi(z),$$

and q is the best dominant of (12). The solution to Equation (14) is

$$q(z) = z^\mu [H(z)]^\eta \left(\eta \int_0^z [H(t)]^\eta t^{\mu-1} dt \right)^{-1} - \mu/\eta, \tag{15}$$

where

$$H(z) = z \exp \int_0^z \frac{\phi(t) - 1}{t} dt.$$

2. Subordination Properties for the Operator $\mathcal{S}_\kappa^c$

In this section, we first obtain Briot–Bouquet differential subordination for the operator $\mathcal{S}_\kappa^c$.

Theorem 1. *Let $\lambda > 0$ and $\zeta \geq 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} + \lambda\frac{\mathcal{S}^c_{\kappa}f(z)}{z} \prec e^z, \tag{16}$$

then

$$\Re\left\{\left(\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{\kappa}{\lambda}\int_0^1 u^{\frac{\kappa}{\lambda}-1}e^{-u}du\right)^{1/\zeta}. \tag{17}$$

The result is sharp.

Proof. Consider the analytic function

$$\varphi(z) = \frac{\mathcal{S}^c_{\kappa+1}f(z)}{z}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$ in $\mathbb{D}$. Now, differentiating the above equality and using Equation (8), we obtain

$$\frac{\mathcal{S}^c_{\kappa}f(z)}{z} = \varphi(z) + \frac{1}{\kappa}z\varphi'(z),$$

and by applying the subordination condition (16), we arrive at

$$(1-\lambda)\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} + \lambda\frac{\mathcal{S}^c_{\kappa}f(z)}{z} = \varphi(z) + \frac{\lambda}{\kappa}z\varphi'(z) \prec e^z.$$

By using Lemma 1 on the right-hand side of above equation, we obtain

$$\varphi(z) \prec \frac{\kappa}{\lambda}z^{-\kappa/\lambda}\int_0^z t^{\frac{\kappa}{\lambda}-1}e^t dt,$$

or

$$\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} = \frac{\kappa}{\lambda}\int_0^1 u^{\frac{\kappa}{\lambda}-1}e^{u\omega(z)}du, \tag{18}$$

where ω is a Schwarz function. Using the result given by Mendiratti et al. [23] as

$$\Re e^{u\omega(z)} > e^{-ur}, \quad r \in (0,1)$$

and letting $r \to 1^-$ in Equation (18), we arrive at

$$\Re\left(\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z}\right) > \frac{\kappa}{\lambda}\int_0^1 u^{\frac{\kappa}{\lambda}-1}e^{-u}du > 0, \quad (z \in \mathbb{D}) \tag{19}$$

where $\lambda > 0$. Since $\Re(\omega^{1/\zeta}) \geq \Re(\omega)^{1/\zeta}$ for $\Re(\omega) > 0$ and $\zeta \geq 1$, from inequality (19), we prove the inequality (17). To prove sharpness, we take $f \in \mathcal{A}$ defined by

$$\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} = \frac{\kappa}{\lambda}\int_0^1 u^{\frac{\kappa}{\lambda}-1}e^{uz}du.$$

For this function, we find that

$$(1-\lambda)\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} + \lambda\frac{\mathcal{S}^c_{\kappa}f(z)}{z} = e^z,$$

and

$$\frac{\mathcal{S}^c_{\kappa+1}f(z)}{z} \to \frac{\kappa}{\lambda}\int_0^1 u^{\frac{\kappa}{\lambda}-1}e^{-u}du$$

as $z \to 1^-$. Thus, the proof is completed. □

Next corollaries are generated from the operators $\mathcal{S}_\nu$ and $\mathcal{H}_\nu$, respectively. These results are special cases of Theorem 1.

Corollary 1. *Let $\lambda > 0$ and $\zeta \geq 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{S}_{\nu+1}f(z)}{z} + \lambda\frac{\mathcal{S}_\nu f(z)}{z} \prec e^z, \tag{20}$$

then

$$\Re\left\{\left(\frac{\mathcal{S}_{\nu+1}f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{2\nu+3}{2\lambda}\int_0^1 u^{\frac{2\nu+3}{2\lambda}-1}e^{-u}du\right)^{1/\zeta}. \tag{21}$$

The result is sharp.

Proof. Suppose that

$$\varphi(z) = \frac{\mathcal{S}_{\nu+1}f(z)}{z}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$ in $\mathbb{D}$. Differentiating the above equality, using Equation (9) and (20), applying Lemma 1, we obtain the inequality (21). □

Corollary 2. *Let $\lambda > 0$ and $\zeta \geq 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{H}_{\nu+1}f(z)}{z} + \lambda\frac{\mathcal{H}_\nu f(z)}{z} \prec e^z, \tag{22}$$

then

$$\Re\left\{\left(\frac{\mathcal{H}_{\nu+1}f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{2\nu+3}{2\lambda}\int_0^1 u^{\frac{2\nu+3}{2\lambda}-1}e^{-u}du\right)^{1/\zeta}. \tag{23}$$

The result is sharp.

Proof. Considering the analytic function

$$\varphi(z) = \frac{\mathcal{H}_{\nu+1}f(z)}{z}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$ in $\mathbb{D}$, using Equations (10) and (22), applying Lemma 1, we obtain the inequality (23). □

For a function $f \in \mathcal{A}$, we consider the Bernardi–Libera–Livingston integral operator $\mathcal{L}_\sigma : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{L}_\sigma f(z) = \frac{\sigma+1}{z^\sigma}\int_0^z t^{\sigma-1}f(t)dt, \quad (\sigma > -1). \tag{24}$$

From this operator, the following recurrence formula can easily be obtained:

$$z(\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z))' = (\sigma+1)\mathcal{S}_\kappa^c f(z) - \sigma\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z). \tag{25}$$

Next, we obtain the Briot–Bouquet differential subordination for the operator $\mathcal{S}_\kappa^c$ associated with the Bernardi–Libera–Livingston integral operator $\mathcal{L}_\sigma f(z)$.

Theorem 2. *Let $\zeta \geq 1$ and $0 < \lambda < 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{S}_\kappa^c f(z)}{z} + \lambda\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z} \prec e^z, \tag{26}$$

then

$$\Re\left\{\left(\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{\sigma+1}{1-\lambda}\int_0^1 u^{\frac{\sigma+1}{1-\lambda}-1}e^{-u}du\right)^{1/\zeta}. \tag{27}$$

The result is sharp.

Proof. Suppose that

$$\varphi(z) = \frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$ in $\mathbb{D}$. Differentiating on both sides of the above equality, using Equation (25) and the subordination condition (26), we obtain

$$(1-\lambda)\frac{\mathcal{S}_\kappa^c f(z)}{z} + \lambda\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z} = \varphi(z) + \frac{1-\lambda}{\sigma+1}z\varphi'(z) \prec e^z.$$

Applying Lemma 1 and the same method as given in Theorem 1, we obtain the proof. For sharpness, let

$$\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z} = \frac{\sigma+1}{1-\lambda}\int_0^1 u^{\frac{\sigma+1}{1-\lambda}-1}e^{uz}du.$$

For this function, we find that

$$(1-\lambda)\frac{\mathcal{S}_\kappa^c f(z)}{z} + \lambda\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z} = e^z,$$

and

$$\frac{\mathcal{S}_\kappa^c\mathcal{L}_\sigma f(z)}{z} \to \frac{\sigma+1}{1-\lambda}\int_0^1 u^{\frac{\sigma+1}{1-\lambda}-1}e^{-u}du$$

as $z \to 1^-$. Thus, the proof of the inequality (27) is completed. □

The following corollaries are generated from the operators $\mathcal{S}_\nu$ and $\mathcal{H}_\nu$, respectively, and special cases of Theorem 2.

Corollary 3. *Let $\zeta \geq 1$ and $0 < \lambda < 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{S}_\nu f(z)}{z} + \lambda\frac{\mathcal{S}_\nu\mathcal{L}_\sigma f(z)}{z} \prec e^z,$$

then

$$\Re\left\{\left(\frac{\mathcal{S}_\nu\mathcal{L}_\sigma f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{\sigma+1}{1-\lambda}\int_0^1 u^{\frac{\sigma+1}{1-\lambda}-1}e^{-u}du\right)^{1/\zeta}.$$

The result is sharp.

Corollary 4. *Let $\zeta \geq 1$ and $0 < \lambda < 1$. If the function $f \in \mathcal{A}$ satisfies the subordination condition*

$$(1-\lambda)\frac{\mathcal{H}_\nu f(z)}{z} + \lambda\frac{\mathcal{H}_\nu\mathcal{L}_\sigma f(z)}{z} \prec e^z,$$

then

$$\Re\left\{\left(\frac{\mathcal{H}_\nu\mathcal{L}_\sigma f(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{\sigma+1}{1-\lambda}\int_0^1 u^{\frac{\sigma+1}{1-\lambda}-1}e^{-u}du\right)^{1/\zeta}.$$

The result is sharp.

3. Subordination Properties for the Class $\mathcal{S}^{c}_{\kappa,e}$

In the first theorem of this section, we find a univalent solution to the Briot–Bouquet differential equation, and we observe that this solution is the best possible solution to the Briot–Bouquet differential subordination for the class $\mathcal{S}^{c}_{\kappa,e}$.

Theorem 3. *If the function f belongs to the class $\mathcal{S}^{c}_{\kappa,e}$ such that $\mathcal{S}^{c}_{\kappa+1}f(z) \neq 0$ for all $z \in \mathbb{D}$, $\Re(\kappa) \geq 1$, and*

$$\Re\big(e^{z} + \kappa - 1\big) > 0, \quad (z \in \mathbb{D})$$

then

$$\frac{z\big(\mathcal{S}^{c}_{\kappa+1}f(z)\big)'}{\mathcal{S}^{c}_{\kappa+1}f(z)} \prec q(z) \prec e^{z}, \tag{28}$$

where

$$q(z) = z^{\kappa-1}e^{Chi(z)+Shi(z)-\gamma}\left(\int_0^z t^{\kappa-2}e^{Chi(t)+Shi(t)-\gamma}dt\right)^{-1} - \kappa + 1, \tag{29}$$

and q is the best dominant of (28).

Proof. Consider the analytic function

$$\varphi(z) = \frac{z(\mathcal{S}^{c}_{\kappa+1}f(z))'}{\mathcal{S}^{c}_{\kappa+1}f(z)}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$. By using Equation (8), we attain

$$\kappa\frac{\mathcal{S}^{c}_{\kappa}f(z)}{\mathcal{S}^{c}_{\kappa+1}f(z)} = \varphi(z) + \kappa - 1,$$

and logarithmic differentiation with respect to z and routine calculations give

$$\frac{z(\mathcal{S}^{c}_{\kappa}f(z))'}{\mathcal{S}^{c}_{\kappa}f(z)} = \varphi(z) + \frac{z\varphi'(z)}{\varphi(z)+\kappa-1} \prec \phi(z) = e^{z}. \tag{30}$$

Now, let us consider the Briot–Bouquet differential equation

$$q(z) + \frac{zq'(z)}{q(z)+\kappa-1} = \phi(z) = e^{z}, \tag{31}$$

where q $(q(0)=1)$ is analytic and $\phi(z) = e^{z}$ is convex univalent with $\phi(0) = 1$ in $\mathbb{D}$, and let $P(z) = \eta\phi(z) + \mu$. In view of Equation (31), we observe that $\eta = 1$, $\mu = \kappa - 1$ and

$$P(z) = e^{z} + \kappa - 1.$$

For proving $\Re(P(z)) > 0$, it is enough to set $z = e^{it},\ t \in [0,\pi]$ under the condition $\Re(\kappa) \geq 1$. Furthermore, $P(z)$ and $1/P(z)$ are convex. Hence, there is a univalent solution to Equation (31), and we obtain this solution by using the steps given in Lemma 2. Since $\phi(z) = e^{z}$, in view of Lemma 2, we find

$$\begin{aligned} H(z) &= z\exp\int_0^z \tfrac{\phi(t)-1}{t}dt \\ &= z\exp\int_0^z \tfrac{e^{t}-1}{t}dt =: e^{\mathrm{Chi}(z)+\mathrm{Shi}(z)-\gamma}, \end{aligned}$$

where $\mathrm{Chi}(z)$ is the cosh integral, $\mathrm{Shi}(z)$ is the sinh integral, and γ is the Euler's constant. Setting this result together with $\eta = 1$ and $\mu = \kappa - 1$ into the formula (15), we obtain

Equation (29) which is the univalent solution to Equation (31). Since φ is analytic and satisfies (30), we then derive

$$\varphi(z) \prec q(z) \prec \phi(z) =: e^z$$

and q is the best dominant of (28). □

In the same manner, we find the following result for the class $\mathcal{S}^c_{\kappa,e}$ associated with the the Bernardi–Libera–Livingston integral operator $\mathcal{L}_\sigma f(z)$.

Theorem 4. *If the function f belongs to the class $\mathcal{S}^c_{\kappa,e}$ such that $\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z) \neq 0$ for all $z \in \mathbb{D}$, $\Re(\sigma) \geq 0$ and*

$$\Re\big(e^z + \sigma\big) > 0, \quad (z \in \mathbb{D})$$

then $\mathcal{L}_\sigma f \in \mathcal{S}^c_{\kappa,e}$ where the operator $\mathcal{L}_\sigma$ is given by (24). Moreover, if $f \in \mathcal{S}^c_{\kappa,e}$ then

$$\frac{z\big(\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z)\big)'}{\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z)} \prec q_1(z) \prec e^z, \tag{32}$$

where

$$q_1(z) = z^\sigma e^{Chi(z)+Shi(z)-\gamma}\left(\int_0^z t^{\sigma-1} e^{Chi(t)+Shi(t)-\gamma} dt\right)^{-1} - \sigma, \tag{33}$$

and q_1 is the best dominant of (32).

Proof. Consider the analytic function

$$\varphi(z) = \frac{z\big(\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z)\big)'}{\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z)}, \quad (z \in \mathbb{D})$$

with $\varphi(0) = 1$. Using Equation (25), we obtain

$$(\sigma+1)\frac{\mathcal{S}^c_\kappa f(z)}{\mathcal{S}^c_\kappa \mathcal{L}_\sigma f(z)} = \varphi(z) + \sigma,$$

and applying logarithmic differentiation with respect to z, we arrive at

$$\frac{z\big(\mathcal{S}^c_\kappa f(z)\big)'}{\mathcal{S}^c_\kappa f(z)} = \varphi(z) + \frac{z\varphi'(z)}{\varphi(z)+\sigma} \prec e^z. \tag{34}$$

Further, let us consider the Briot–Bouquet differential equation

$$q_1(z) + \frac{zq_1'(z)}{q_1(z)+\sigma} = \phi(z) = e^z, \tag{35}$$

where q_1 is analytic with $q_1(0) = 1$ and $\phi(z) = e^z$ is convex univalent with $\phi(0) = 1$ in $\mathbb{D}$. From the similar technique applied in Theorem 3 and Lemma 2, the differential equation given by (35) has a univalent solution defined by Equation (33). Since φ satisfies the subordination in (34), we then find that

$$\varphi(z) \prec q_1(z) \prec \phi(z) := e^z,$$

and q_1 is the best dominant of (32). □

4. An Application of Fractional Calculus

The fractional calculus operators have been extensively used in solving various problems in applied sciences and also in geometric function theory. Specially, Srivastava et al. [34] defined the fractional integral operator for a complex-valued function as below.

Definition 2 ([34]). *For real numbers α, β, η with $\alpha > 0$, the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by*

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{z}\right) f(t)dt, \tag{36}$$

where ${}_2F_1$ is the Gauss hypergeometric function, and where f is an analytic function in a simply connected region of the z-plane containing the origin with the order

$$f(z) = O(|z|^{\varepsilon}),\ z \to 0$$

where $\varepsilon > \max\{0, \beta-\eta\} - 1$ and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\ln(z-t)$ to be real when $(z-t) > 0$.

We now give an example satisfying a link between the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta} f(z)$ and the generalized Struve function $M_{\nu,c,d}(z)$ given by Equation (5).

Example 1. *Let $\alpha, \beta, \eta, k, \nu, d, c \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\nu+k+1) > 0$ and $\nu + (d+2)/2 \neq 0, -1, -2, \ldots$. Then*

$$\begin{aligned}\left(I_{0,z}^{\alpha,\beta,\eta} t^{k-1} M_{\nu,c,d}(t)\right)(z) \\ &= \frac{z^{\nu+k-\beta}}{2^{\alpha-\beta+\eta+2k+3\nu+1}} \pi\Gamma(\nu+k+1)\Gamma(\nu+k-\beta+\eta+1) \\ \times {}_5F_6\left(\begin{matrix} 1, \frac{1}{2}(\nu+k+1), \frac{1}{2}(\nu+k+2), \frac{1}{2}(\nu+k-\beta+\eta+1), \frac{1}{2}(\nu+k-\beta+\eta+2) \\ \frac{3}{2}, \nu+\frac{d+2}{2}, \frac{1}{2}(\nu+k-\beta+1), \frac{1}{2}(\nu+k-\beta+2), \frac{1}{2}(\nu+k+\alpha+\eta+1), \frac{1}{2}(\nu+k+\alpha+\eta+2) \end{matrix}\middle| -\frac{cz^2}{4}\right).\end{aligned}$$

Proof. Using Equations (5) and (36) together, we observe that

$$\left(I_{0,z}^{\alpha,\beta,\eta} t^{k-1} M_{\nu,c,d}(t)\right)(z) = \left(I_{0,z}^{\alpha,\beta,\eta} t^{k-1} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} c^{\ell}}{\Gamma(\ell+3/2)\Gamma(\nu+\ell+(d+2)/2)} \left(\frac{t}{2}\right)^{2\ell+\nu+1}\right)(z),$$

and, on interchanging the integration and summation, we obtain

$$\left(I_{0,z}^{\alpha,\beta,\eta} t^{k-1} M_{\nu,c,d}(t)\right)(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} c^{\ell} 2^{-(2\ell+\nu+1)}}{\Gamma(\ell+3/2)\Gamma(\nu+\ell+(d+2)/2)} \left(I_{0,z}^{\alpha,\beta,\eta} t^{2\ell+\nu+k}\right)(z). \tag{37}$$

Furthermore, Srivastava et al. [34] proved that

$$I_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}, \tag{38}$$

where $\alpha > 0$ and $k > \beta - \eta - 1$. If we use Equation (38) in the Equation (37) for all $\ell \geq 0$ and $\Re(2\ell+\nu+k+1) \geq \Re(\nu+k+1) > \max\{0, \beta-\eta\}$, we obtain

$$\begin{aligned}\left(I_{0,z}^{\alpha,\beta,\eta} t^{k-1} M_{\nu,c,d}(t)\right)(z) &= \frac{z^{\nu+k-\beta}}{2^{\nu+1}} \\ \times \sum_{\ell=0}^{\infty} &\frac{\Gamma(2\ell+\nu+k+1)\Gamma(2\ell+\nu+k-\beta+\eta+1)}{\Gamma(\ell+3/2)\Gamma(\nu+\ell+(d+2)/2)\Gamma(2\ell+\nu+k-\beta+1)\Gamma(2\ell+\nu+k+\alpha+\eta+1)} (-\frac{cz^2}{4})^{\ell}.\end{aligned}$$

The last expression gives us the desired proof. □

5. Concluding Remarks

The Struve function has been successfully applied in many areas of science and engineering. Although there are many papers on applications of the Struve function in several areas, the impact of this function in geometric function theory is extremely new and there are very few data. Recently, several researchers have established some geometric properties of the Struve function, such as starlikeness, convexity, and univalency criteria. Motivated by recent applications of the Struve function, and applying a paper by Miller and Mocanu, we established novel results by using the technique of Briot–Bouquet differential subordination. We defined the exponential starlike function class constructed by the linear operator involving the normalized form of the generalized Struve function. In Section 2, we established the Briot–Bouquet differential subordination results for the operator $\mathcal{S}_{\kappa}^{c}$, and presented special cases of these results. In Section 3, we determined univalent solutions of Briot–Bouquet differential equations and concluded that these solutions are the best dominant of the Briot–Bouquet differential subordinations for the class $\mathcal{S}_{\kappa,e}^{c}$. We further presented an example that gives a link between the fractional integral operator for a complex-valued function and the generalized Struve function. The current work contains significant results because the idea of this paper can inspire several new problems in geometric function theory associated with special functions.

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