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Controllability of Fractional Stochastic Delay Systems Driven by the Rosenblatt Process

Barakah Almarri [1] and Ahmed M. Elshenhab [2,3,*]

[1] Department of Mathematical Sciences, College of Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
[2] School of Mathematics, Harbin Institute of Technology, Harbin 150001, China
[3] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
[*] Correspondence: ahmedelshenhab@mans.edu.eg

Abstract: In this work, we consider linear and nonlinear fractional stochastic delay systems driven by the Rosenblatt process. With the aid of the delayed Mittag-Leffler matrix functions and the representation of solutions of these systems, we derive the controllability results as an application. By introducing a fractional delayed Gramian matrix, we provide sufficient and necessary criteria for the controllability of linear fractional stochastic delay systems. Furthermore, by employing Krasnoselskii's fixed point theorem, we establish sufficient conditions for the controllability of nonlinear fractional stochastic delay systems. Finally, an example is given to illustrate the main results.

Keywords: controllability; fractional stochastic delay system; Rosenblatt process; delayed Mittag-Leffler matrix function; fractional delayed Gramian matrix; Krasnoselskii's fixed point theorem

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1. Introduction

Due to its effective modeling in numerous fields of science and engineering, including economics, diffusion processes, control theory, viscoelastic systems, biology, physics, medicine, finance, fluid dynamics, and others, fractional functional differential equations and their applications have received a great deal of attention (see, for instance, [1–11]). In particular, the fractional derivative of an order α with $1 < \alpha \leq 2$ appears in several diffusion problems used in physical and engineering applications, such as in the mechanism of superdiffusion [12]. The typical variation in deterministic systems with environmental noise is considered to be random in nature. Stochastic differential equations can be used to simulate noise in financial mathematics, medicine, telecommunication networks, and other fields.

The concept of controllability of systems is one of the most fundamental and important concepts in contemporary control theory, which involves figuring out the control parameters that direct a control system's solutions from its initial state to its final state using the set of permissible controls, where the initial and final states may vary across the entire space. The representation of time delay system solutions has received recent attention. The seminal studies [13,14] in particular yielded several novel results in the representation of solutions, stability, and controllability of time delay systems (see, for instance, [15–23] and the references therein).

The Hermite process of an order of one is known as fractional Brownian motion, while the Hermite process of an order of two is known as the Rosenblatt process. Rosenblatt first proposed the following distribution for $x \geq 0$

$$Z_U(x) = D(U) \int_{\mathbb{R}^2} \left(\int_0^x (\vartheta - t_1)_+^{-(1+U)/2} (\vartheta - t_2)_+^{-(1+U)/2} \mathrm{d}\vartheta \right) \mathrm{d}J(t_1) \mathrm{d}J(t_2),$$

where $U \in \left(0, \frac{1}{2}\right)$, $D(U)$ is a positive normalization constant depending only on U, and $\{J(t),\ t \in U\}$ is a standard Brownian motion. The process of $Z_U(1)$ is known as the '1 non-Gaussian limiting distribution' (Rosenblatt distribution) (for more details, see [24]). The Rosenblatt process is a non-Gaussian process with many interesting properties, such as the stationary nature of the increments, long-range dependence, and self-similarity. Therefore, it seems interesting to study a new class of fractional stochastic differential equations driven by the Rosenblatt process. Shen and Ren [25] investigated the existence and uniqueness of the mild solution for neutral stochastic partial differential equations with finite delay driven by the Rosenblatt process in a real, separable Hilbert space. Maejima and Tudor [26] presented a technique for constructing self-similar processes in the second Wiener chaos using limit theorems. Shen et al. [27] used fixed point theory to examine controllability and stability analysis for functional nonlinear neutral fractional stochastic systems with delay driven by the Rosenblatt process (we refer the reader to [18,28–30] for further details on the Rosenblatt process).

Elshenhab and Wang [15] established a novel formula to solve the linear delay differential systems

$$\begin{aligned} &\left({}^{C}D^{\alpha}_{0^+}z\right)(x) + \Xi z(x-\omega) = g(x), \quad x \geq 0, \\ &z(x) \equiv \Pi(x),\ z'(x) \equiv \Pi'(x), \quad -\omega \leq x \leq 0, \end{aligned} \tag{1}$$

of the form

$$\begin{aligned} z(x) \; &= \mathcal{H}_{\omega,\alpha}\left(\Xi(x-\omega)^{\alpha}\right)\Pi(0) + \mathcal{M}_{\omega,\alpha}\left(\Xi(x-\omega)^{\alpha}\right)\Pi'(0) \\ &-\Xi \int_{-\omega}^{0} \mathcal{S}_{\omega,\alpha}\left(\Xi(x-2\omega-\vartheta)^{\alpha}\right)\Pi(\vartheta)\mathrm{d}\vartheta \\ &+ \int_{0}^{x} \mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)g(\vartheta)\mathrm{d}\vartheta, \end{aligned} \tag{2}$$

where $\mathcal{H}_{\omega,\alpha}(\Xi x^{\alpha})$, $\mathcal{M}_{\omega,\alpha}(\Xi x^{\alpha})$, and $\mathcal{S}_{\omega,\alpha}(\Xi x^{\alpha})$ are the delayed Mittag-Leffler type matrix functions defined by

$$\mathcal{H}_{\omega,\alpha}(\Xi x^{\alpha}) := \begin{cases} \not\leq, & -\infty < x < -\omega, \\ \mathbb{I}, & -\omega \leq x < 0, \\ \mathbb{I} - \Xi\frac{x^{\alpha}}{\Gamma(1+\alpha)}, & 0 \leq x < \omega, \\ \quad\vdots & \quad\vdots \\ \mathbb{I} - \Xi\frac{x^{\alpha}}{\Gamma(1+\alpha)} + \Xi^2\frac{(x-\omega)^{2\alpha}}{\Gamma(1+2\alpha)} & \\ +\cdots+(-1)^{\varsigma}\Xi^{\varsigma}\frac{(x-(\varsigma-1)\omega)^{\varsigma\alpha}}{\Gamma(1+\varsigma\alpha)}, & (\varsigma-1)\omega \leq x < \varsigma\omega, \end{cases} \tag{3}$$

$$\mathcal{M}_{\omega,\alpha}(\Xi x^{\alpha}) := \begin{cases} \not\leq, & -\infty < x < -\omega, \\ \mathbb{I}(x+\omega), & -\omega \leq x < 0, \\ \mathbb{I}(x+\omega) - \Xi\frac{x^{\alpha+1}}{\Gamma(2+\alpha)}, & 0 \leq x < \omega, \\ \quad\vdots & \quad\vdots \\ \mathbb{I}(x+\omega) - \Xi\frac{x^{\alpha+1}}{\Gamma(2+\alpha)} + \Xi^2\frac{(x-\omega)^{2\alpha+1}}{\Gamma(2+2\alpha)} & \\ +\cdots+(-1)^{\varsigma}\Xi^{\varsigma}\frac{(x-(\varsigma-1)\omega)^{\varsigma\alpha+1}}{\Gamma(2+\varsigma\alpha)}, & (\varsigma-1)\omega \leq x < \varsigma\omega, \end{cases} \tag{4}$$

and

$$\mathcal{S}_{\omega,\alpha}(\Xi x^{\alpha}) := \begin{cases} \nleq, & -\infty < x < -\omega, \\ \mathbb{I}\frac{(x+\omega)^{\alpha-1}}{\Gamma(\alpha)}, & -\omega \leq x < 0, \\ \mathbb{I}\frac{(x+\omega)^{\alpha-1}}{\Gamma(\alpha)} - \Xi\frac{x^{2\alpha-1}}{\Gamma(2\alpha)}, & 0 \leq x < \omega, \\ \vdots & \vdots \\ \mathbb{I}\frac{(x+\omega)^{\alpha-1}}{\Gamma(\alpha)} - \Xi\frac{x^{2\alpha-1}}{\Gamma(2\alpha)} + \Xi^2\frac{(x-\omega)^{3\alpha-1}}{\Gamma(3\alpha)} & \\ + \cdots + (-1)^{\varsigma}\Xi^{\varsigma}\frac{(x-(\varsigma-1)\omega)^{\alpha(\varsigma+1)-1}}{\Gamma(\alpha(\varsigma+1))}, & (\varsigma-1)\omega \leq x < \varsigma\omega, \end{cases} \quad (5)$$

respectively, where the notations $\nleq$ and $\mathbb{I}$ are the $n \times n$ null and identity matrix, respectively, Γ is a gamma function, and $\varsigma = 0, 1, 2, \ldots$.

Motivated by the aforementioned works, and based on [15], as an application, we investigate the controllability of fractional stochastic linear delay systems driven by the Rosenblatt process

$$\begin{aligned} \left({}^{C}D^{\alpha}_{0+}z\right)(x) + \Xi z(x-\omega) &= Bu(x) + \overline{\Delta}(x)\mathrm{d}Z_H(x), \;\; x \in \mp := [0, x_1], \\ z(x) \equiv \Pi(x),\; z'(x) &\equiv \Pi'(x), \;\; -\omega \leq x \leq 0, \end{aligned} \quad (6)$$

as well as the controllability of the corresponding fractional stochastic nonlinear delay systems driven by the Rosenblatt process

$$\begin{aligned} \left({}^{C}D^{\alpha}_{0+}z\right)(x) + \Xi z(x-\omega) &= Bu(x) + \Delta(x, z(x))\mathrm{d}Z_H(x), \;\; x \in \mp, \\ z(x) \equiv \Pi(x),\; z'(x) &\equiv \Pi'(x), \;\; -\omega \leq x \leq 0, \end{aligned} \quad (7)$$

where ${}^{C}D^{\alpha}_{0+}$ is called the Caputo fractional derivative of the order $\alpha \in (1, 2]$ with a lower index of zero, $\omega > 0$ is a delay, $x_1 > (n-1)\omega$, state vector $z(x) \in \mathbb{R}^n$, $\Pi \in C([-\omega, 0], \mathbb{R}^n)$, $\Xi \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are any matrices, $u(x) \in \mathbb{R}^m$ shows the control vector, and $\overline{\Delta} \in C(\mp, \mathcal{T}(\mathbb{R}^n))$, where the Thorin class, symbolized by $\mathcal{T}(\mathbb{R}^n)$, is the smallest distribution class on $\mathbb{R}^n$ that comprises all Gamma distributions and is closed under convolution and weak convergence. Let $z(\cdot)$ take a value in the separable Hilbert space $\mathbb{R}^n$ with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. $Z_H(x)$ is a Rosenblatt process with the parameter $H \in \left(\frac{1}{2}, 1\right)$ on an another real separable Hilbert space $(K, \|\cdot\|_K, \langle\cdot,\cdot\rangle_K)$. Moreover, assume $\Delta \in C(\mp \times \mathbb{R}^n, L^0_2)$, where $L^0_2 = L_2\left(Q^{\frac{1}{2}}K, \mathbb{R}^n\right)$.

The following is how the rest of this paper is structured. In Section 2, we provide some introductions, fundamental notation and definitions, as well as some relevant lemmas. In Section 3, using a fractional delayed Gramian matrix, we give sufficient and necessary conditions for the controllability of Equation (6). In Section 4, by applying Krasnoselskii's fixed point theorem, we estabilish sufficient conditions for the controllability of Equation (7). Finally, to illustrate the theoretical findings, we provide numerical examples.

2. Preliminaries

Throughout the paper, let $(\Omega, \mathfrak{F}, \mathbb{P})$ be the complete probability space with probability measure $\mathbb{P}$ on Ω with a filtration $\{\mathfrak{F}_x | \, x \in \mp\}$ generated by $\{Z_H(s) | \, s \in [0, x]\}$. Let $\mathcal{D}, \mathcal{C}$ be two Banach spaces and $L_b(\mathcal{D}, \mathcal{C})$ be the space of the bounded linear operators from $\mathcal{D}$ to $\mathcal{C}$, while $Q \in L_b(\mathcal{D}, \mathcal{D})$ represents a nonnegative self-adjoint trace class operator on $\mathcal{D}$. Let $L^0_2 = L_2\left(Q^{\frac{1}{2}}\mathcal{D}, \mathcal{C}\right)$ be the space of all Q Hilbert–Schmidt operators from $Q^{\frac{1}{2}}\mathcal{D}$ into $\mathcal{C}$, equipped with the norm

$$\|\varphi\|^2_{L^0_2} = \left\|\varphi Q^{\frac{1}{2}}\right\|^2 = \mathrm{Tr}\left(\varphi Q \varphi^T\right).$$

Now, for some $1 < e < \infty$, let $L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{R}^n)$ be the Hilbert space of all $\mathfrak{F}_{x_1}$-measurable, eth-integrable variables with values in $\mathbb{R}^n$ with the norm $\|z\|_{L^e}^e = \mathbf{E}\|z(x)\|^e$, where the expectation $\mathbf{E}$ is defined by $\mathbf{E}z = \int_\Omega z \mathrm{d}\mathbb{P}$. Let $L^e_{\mathfrak{F}}(\mp, \mathbb{R}^n)$ be the Banach space of all functions $g : \mp \longrightarrow \mathbb{R}^n$ that are Bochner integrable, normed by $\|g\|_{L^e_{\mathfrak{F}}(\mp,\mathbb{R}^n)}$, and $\mathfrak{F}_{x_1}$-measurable processes with values in $\mathbb{R}^n$. Let $\mathcal{F} := C([-\omega, 0], L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n))$ be the Banach space of all eth-integrable and $\mathfrak{F}_{x_1}$-adapted processes ϕ endowed with the norm $\|\phi\|_C = \left(\sup_{x\in[-\omega,0]} \mathbf{E}\|\phi(x)\|^e\right)^{1/e}$. Additionally, we denote $C(\mp, L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n))$ as the Banach space of continuous function from $\mp \longrightarrow L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n)$ endowed with the norm $\|z\|_{C(\mp)} = (\sup_{x\in\mp} \mathbf{E}\|z(x)\|^e)^{1/e}$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$ and let the matrix norm (column sum)

$$\|\Xi\| = \max\left\{\sum_{i=1}^n |a_{i1}|, \sum_{i=1}^n |a_{i2}|, \ldots, \sum_{i=1}^n |a_{in}|\right\},$$

where $\Xi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We define a space

$$\begin{aligned} &C^1(\mp, L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n)) \\ &= \left\{z \in C(\mp, L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n)) : z' \in C(\mp, L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{P}, \mathbb{R}^n))\right\}. \end{aligned}$$

Furthermore, we let

$$\|\Pi\|_C = \left(\sup_{s\in[-\omega,0]} \mathbf{E}\|\Pi(s)\|^e\right)^{1/e} \text{ and } \|\Pi'\|_C = \left(\sup_{s\in[-\omega,0]} \mathbf{E}\|\Pi'(s)\|^e\right)^{1/e}.$$

The Wiener–Ito multiple integral of an order k with respect to the standard Wiener process $(\mathcal{G}(\rho))_{\rho\in\mathbb{R}}$ is given by

$$Z_H^k(x) = c(H,k) \int_{R^k} \left(\int_0^x \prod_{j=1}^k (\vartheta - \rho_j)_+^{-\left(\frac{1}{2} + \frac{1-H}{k}\right)} \mathrm{d}\vartheta \right) \mathrm{d}\mathcal{G}(\rho_1) \ldots \mathrm{d}\mathcal{G}(\rho_k), \tag{8}$$

where $c(H,k)$ is a normalizing constant such that $\mathbf{E}\left(Z_H^k(1)\right)^2 = 1$ and $\rho_+ = \max(\rho, 0)$. The process $\left(Z_H^k(x)\right)_{x\geq 0}$ is called the Hermite process. If $k = 1$, then the Hermite process given by Equation (8) is the fBm with a Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. Furthermore, the process is not Gaussian for $k = 2$. Moreover, for $k = 2$, the Hermite process given by Equation (8) is called the Rosenblatt process.

We provide some fundamental concepts and lemmas used in this work:

Lemma 1 ([31]). *If $\sigma : \mp \longrightarrow L_2^0$ satisfies*

$$\int_0^{x_1} \|\sigma(\vartheta)\|_{L_2^0}^2 \mathrm{d}\vartheta < \infty,$$

then, for a, $b \in \mp$ with $b > a$, we have

$$\mathbf{E}\left\|\int_0^x \sigma(\vartheta) \mathrm{d}Z_H(\vartheta)\right\|^2 \leq 2Hx^{2H-1} \int_0^x \|\sigma(\vartheta)\|_{L_2^0}^2 \mathrm{d}\vartheta.$$

Definition 1 ([32]). *If there exists a control function $u \in L^2(\Omega, \mathbb{R}^m)$ such that Equation (6) or (7) has a solution $z : [-\omega, x_1] \to \mathbb{R}^n$ with $z(0) = z_0$, then $z'(0) = z_0'$ satisfies $z(x_1) = z_1$ for all z_0, z_0', $z_1 \in \mathbb{R}^n$, then the systems in Equation (6) or (7) are controllable on $\Omega = [0, x_1]$.*

Definition 2 ([5]). *The two-parameter Mittag-Leffler function is provided by*

$$\mathbb{E}_{\alpha,\gamma}(x) = \sum_{\varsigma=0}^{\infty} \frac{x^{\varsigma}}{\Gamma(\alpha\varsigma+\gamma)}, \quad \alpha,\ \gamma > 0,\ x \in \mathbb{C}.$$

In the case of $\gamma = 1$, then

$$\mathbb{E}_{\alpha,1}(x) = \mathbb{E}_{\alpha}(x) = \sum_{\varsigma=0}^{\infty} \frac{x^{\varsigma}}{\Gamma(\alpha\varsigma+1)}, \quad \alpha > 0.$$

Definition 3 ([5]). *The Caputo fractional derivative of the order $\alpha \in (1,2]$ with a lower index 0 of a function $z : [-\omega,\infty) \to \mathbb{R}^n$ is given by*

$$\left({}^{C}D_{0^+}^{\alpha} z\right)(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^x \frac{z''(\vartheta)}{(x-\vartheta)^{\alpha-1}} \mathrm{d}\vartheta, \quad x > 0.$$

Lemma 2 ([23]). *For any $x \in [(\varsigma-1)\omega, \varsigma\omega]$, $\varsigma = 1,2,...,$ we have*

$$\|\mathcal{H}_{\omega,\alpha}(\Xi x^{\alpha})\| \leq \mathbb{E}_{\alpha}(\|\Xi\| x^{\alpha}),$$

$$\|\mathcal{M}_{\omega,\alpha}(\Xi x^{\alpha})\| \leq (x+\omega)\mathbb{E}_{\alpha,2}\left(\|\Xi\|(x+\omega)^{\alpha}\right),$$

and

$$\|\mathcal{S}_{\omega,\alpha}(\Xi x^{\alpha})\| \leq (x+\omega)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x+\omega)^{\alpha}\right).$$

Lemma 3 (Krasnoselskii's fixed point theorem [33]). *Let M be a closed, bounded, and convex subset of a real Banach space $\mathcal{K}$, and let J_1 and J_2 be operators on M satisfying the following conditions:*

(1) $J_1 x + J_2 z \in M$ for $x, z \in M$;
(2) J_1 is compact and continuous;
(3) J_2 is a contraction mapping.

Then, there exists $m \in M$ such that $m = J_1 m + J_2 m$.

We define the operator $Q_{x_1} \in L_b\left(L^{e}_{\mathfrak{F}}(\mp,\mathbb{R}^m), L^{e}(\Omega,\mathfrak{F}_{x_1},\mathbb{R}^n)\right)$ as

$$Q_{x_1} u = \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^{\alpha}\right) B u(\vartheta) \mathrm{d}\vartheta,$$

In addition, its adjoint operator $Q^T_{x_1} \in L_b\left(L^{e}(\Omega,\mathfrak{F}_{x_1},\mathbb{R}^n), L^{e}_{\mathfrak{F}}(\mp,\mathbb{R}^m)\right)$ is defined as

$$Q^T_{x_1} u = B^T \mathcal{S}_{\omega,\alpha}\left(\Xi^T (x_1-\omega-x)^{\alpha}\right) \mathbf{E}\{u|\mathfrak{F}_x\}.$$

Consider the linear controllability operator

$$\begin{aligned}\Gamma^{x_1}_{\omega}\{\cdot\} &= Q_{x_1} Q^T_{x_1}\{\cdot\} \\ &= \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^{\alpha}\right) BB^T \mathcal{S}_{\omega,\alpha}\left(\Xi^T(x_1-\omega-\vartheta)^{\alpha}\right) \mathbf{E}\{\cdot|\mathfrak{F}_{\vartheta}\} \mathrm{d}\vartheta,\end{aligned}$$

as well as the fractional delayed Gramian matrix $W_{\omega,\alpha}[0,x_1] \in L_b(\mathbb{R}^n,\mathbb{R}^n)$ defined by

$$W_{\omega,\alpha}[0,x_1] = \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^{\alpha}\right) BB^T \mathcal{S}_{\omega,\alpha}\left(\Xi^T(x_1-\omega-\vartheta)^{\alpha}\right) \mathrm{d}\vartheta. \tag{9}$$

Here, T denotes the transpose.

3. Controllability of Linear Fractional Stochastic Delay Systems

In this section, we derive the controllability results for Equation (6) using the fractional delayed Gramian matrix $W_{\omega,\alpha}[0,x_1]$ defined by Equation (9):

Theorem 1. *The stochastic system in Equation (6) is controllable if and only if $W_{\omega,\alpha}[0,x_1]$ is positive definite.*

Proof. Sufficiency. Assuming that $W_{\omega,\alpha}[0,x_1]$ is positive definite, then it is invertible. Consequently, for any finite terminal conditions $z_1, z_1' \in \mathbb{R}^n$, we can derive the associated control input $u(x)$ as

$$u(x) = B^T \mathcal{S}_{\omega,\alpha}\left(\Xi^T(x_1-\omega-x)^\alpha\right) W_{\omega,\alpha}^{-1}[0,x_1]\beta, \tag{10}$$

where

$$\begin{aligned}\beta &= z_1 - \mathcal{H}_{\omega,\alpha}(\Xi(x-\omega)^\alpha)\Pi(0) - \mathcal{M}_{\omega,\alpha}(\Xi(x-\omega)^\alpha)\Pi'(0)\\ &+\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}(\Xi(x-2\omega-\vartheta)^\alpha)\Pi(\vartheta)\mathrm{d}\vartheta\\ &-\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x-\omega-\vartheta)^\alpha)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta).\end{aligned} \tag{11}$$

By applying Equation (2), the solution to Equation (6) can be expressed as

$$\begin{aligned}z(x) &= \mathcal{H}_{\omega,\alpha}(\Xi(x-\omega)^\alpha)\Pi(0) + \mathcal{M}_{\omega,\alpha}(\Xi(x-\omega)^\alpha)\Pi'(0)\\ &-\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}(\Xi(x-2\omega-\vartheta)^\alpha)\Pi(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^{x}\mathcal{S}_{\omega,\alpha}(\Xi(x-\omega-\vartheta)^\alpha)Bu(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^{x}\mathcal{S}_{\omega,\alpha}(\Xi(x-\omega-\vartheta)^\alpha)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta).\end{aligned} \tag{12}$$

From Equation (12), the solution $z(x_1)$ to Equation (6) can be given by

$$\begin{aligned}z(x_1) &= \mathcal{H}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi(0) + \mathcal{M}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi'(0)\\ &-\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}(\Xi(x_1-2\omega-\vartheta)^\alpha)\Pi(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^\alpha)Bu(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^\alpha)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta).\end{aligned} \tag{13}$$

By substituting Equation (10) into Equation (13), we obtain

$$\begin{aligned}z(x_1) &= \mathcal{H}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi(0) + \mathcal{M}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi'(0)\\ &-\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}(\Xi(x_1-2\omega-\vartheta)^\alpha)\Pi(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^\alpha)BB^T\mathcal{S}_{\omega,\alpha}(\Xi^T(x_1-\omega-\vartheta)^\alpha)\mathrm{d}\vartheta\\ &\times W_{\omega,\alpha}^{-1}[0,x_1]\beta\\ &+\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^\alpha)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta).\end{aligned} \tag{14}$$

From Equations (9), (11), and (14), we obtain

$$\begin{aligned}z(x_1) = {}& \mathcal{H}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi(0) + \mathcal{M}_{\omega,\alpha}(\Xi(x_1-\omega)^\alpha)\Pi'(0)\\ &-\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}(\Xi(x_1-2\omega-\vartheta)^\alpha)\Pi(\vartheta)\mathrm{d}\vartheta + \beta\\ &+\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^\alpha)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta).\\ &= z_1.\end{aligned}$$

We can see from Equations (3), (4), and (12) that the boundary conditions $z(x) \equiv \Pi(x)$, $z'(x) \equiv \Pi'(x)$, and $-\omega \leq x \leq 0$ hold. Thus, Equation (6) is controllable.

Necessity. Let Equation (6) be controllable. Assume for the sake of a contradiction that $W_{\omega,\alpha}[0,x_1]$ is not positive definite and there exists at least a nonzero vector $\rho \in \mathbb{R}^n$ such that $\rho^T W_{\omega,\alpha}[0,x_1]\rho = 0$, which implies that

$$\begin{aligned}
0 &= \rho^T W_{\omega,\alpha}[0,x_1]\rho \\
&= \int_0^{x_1} \rho^T \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) BB^T \mathcal{S}_{\omega,\alpha}\left(\Xi^T(x_1-\omega-\vartheta)^\alpha\right)\rho \mathrm{d}\vartheta \\
&= \int_0^{x_1} \left[\rho^T \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B\right]\left[\rho^T \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B\right]^T \mathrm{d}\vartheta \\
&= \int_0^{x_1} \left\|\rho^T \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B\right\| \mathrm{d}\vartheta.
\end{aligned}$$

Hence, we have

$$\rho^T \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B = (0,\ldots,0) := \mathbf{0}^T, \quad \text{for all } \vartheta \in \mp, \tag{15}$$

where $\mathbf{0}$ denotes the n dimensional zero vector. Since Equation (6) is controllable, from Definition 1, there exists a control function $u_1(x)$ that steers the initial state to $z_1 = 0$ at $x = x_1$. Then, we have

$$\begin{aligned}
z(x_1) \quad &= \mathcal{H}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi(0) + \mathcal{M}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi'(0) \\
&-\Xi \textstyle\int_{-\omega}^{0} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-2\omega-\vartheta)^\alpha\right)\Pi(\vartheta)\mathrm{d}\vartheta \\
&+ \textstyle\int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B u_1(\vartheta)\mathrm{d}\vartheta \\
&+ \textstyle\int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta) \\
&= \mathbf{0}.
\end{aligned} \tag{16}$$

Similarly, there is a control function $u_2(x)$ that steers the initial state to $z_1 = \rho$ at $x = x_1$. Then, we have

$$\begin{aligned}
z(x_1) \quad &= \mathcal{H}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi(0) + \mathcal{M}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi'(0) \\
&-\Xi \textstyle\int_{-\omega}^{0} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-2\omega-\vartheta)^\alpha\right)\Pi(\vartheta)\mathrm{d}\vartheta \\
&+ \textstyle\int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B u_2(\vartheta)\mathrm{d}\vartheta \\
&+ \textstyle\int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right)\overline{\Delta}(\vartheta)\mathrm{d}Z_H(\vartheta). \\
&= \rho.
\end{aligned} \tag{17}$$

By combining Equation (16) with Equation (17), we have

$$\rho = \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right) B[u_2(\vartheta) - u_1(\vartheta)]\mathrm{d}\vartheta \tag{18}$$

By multiplying Equation (18) by ρ^T and using Equation (15), we obtain $\rho^T\rho = 0$. This is a contradiction to $\rho \neq \mathbf{0}$. Thus, $W_{\omega,\alpha}[0,x_1]$ is positive definite. This completes the proof. □

4. Controllability of Nonlinear Fractional Stochastic Delay Systems

In this section, we present sufficient conditions for the controllability of Equation (7). The following hypotheses are made:

(J1) The function $\Delta : \mp \times \mathbb{R}^n \longrightarrow L_2^0$ is continuous, and there exists a constant $L_\Delta \in L^q(\mp, \mathbb{R}^+)$ where $q > 1$ such that

$$\mathbf{E}\|\Delta(x,z_1)-\Delta(x,z_2)\|^e_{L_2^0} \leq L_\Delta(x)\|z_1-z_2\|^e, \quad \text{for all } x \in \mp, z_1, z_2 \in \mathbb{R}^n.$$

Let $e \in [2,\infty)$ and $\sup_{x\in\mp} \mathbf{E}\|\Delta(x,0)\|^e_{L_2^0} = N_\Delta < \infty$.

(J2) The linear stochastic delay system in Equation (6) is controllable on $\mp$.

Under the assumption of (**J2**), for some $\eta > 0$, we have $\mathbf{E}\langle \Gamma_\omega^{x_1} z, z\rangle \geq \eta \mathbf{E}\|z\|^e$ for all $z \in L^e(\Omega, \mathfrak{F}_{x_1}, \mathbb{R}^n)$ (see [34], Lemma 2). Furthermore, $\left\|\left(\Gamma_\omega^{x_1}\right)^{-1}\right\|^e \leq 1/\eta := N_1$ (see [35]), and we set $N := \max\left\{\left\|W_\omega^{\mathcal{M}}[\vartheta, x_1]\right\|^e : \vartheta \in \mp\right\}$:

Theorem 2. *Let (J1) and (J2) be satisfied. Then, the nonlinear stochastic system in Equation (7) is controllable on* $\mp$ *if there exists a constant* $\tau_e > 0$ *such that*

$$N_2\left[1+5^{e-1}NN_1\right] < 1, \tag{19}$$

where

$$N_2 := \frac{5^{e-1}\tau_e(2H)^{e/2}x_1^{e(H+\alpha-1)-\frac{1}{q}}}{((\alpha-1)ep+1)^{\frac{1}{p}}}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^\alpha))^e\|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)},$$

and $\frac{1}{p}+\frac{1}{q}=1$, $p, q > 1$.

Proof. Before beginning to prove this theorem, we consider the set

$$\mathcal{B}_\lambda = \left\{z \in \mathcal{F} : \|z\|^e_{\mathcal{F}} = \sup_{x\in[-\omega,x_1]} \mathbf{E}\|z(x)\|^e \leq \lambda\right\},$$

for each postive number λ. Let $x \in [0, x_1]$. With the aid of Equation (2), the solution to Equation (7) can be expressed as

$$\begin{aligned} z(x) &= \mathcal{H}_{\omega,\alpha}\left(\Xi(x-\omega)^\alpha\right)\Pi(0) + \mathcal{M}_{\omega,\alpha}\left(\Xi(x-\omega)^\alpha\right)\Pi'(0) \\ &- \Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}\left(\Xi(x-2\omega-\vartheta)^\alpha\right)\Pi(\vartheta)\mathrm{d}\vartheta \\ &+ \int_0^x \mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^\alpha\right)Bu(\vartheta)\mathrm{d}\vartheta \\ &+ \int_0^x \mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^\alpha\right)\Delta(\vartheta, z(\vartheta))\mathrm{d}Z_H(\vartheta), \end{aligned}$$

In addition, its control function u_z is defined as

$$\begin{aligned} u_z(x) &= B^T\mathcal{S}_{\omega,\alpha}\left(\Xi^T(x_1-\omega-x)^\alpha\right) \\ &\times\mathbf{E}\left\{\left(\Gamma_\omega^{x_1}\right)^{-1}\left[z_1 - \mathcal{H}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi(0) - \mathcal{M}_{\omega,\alpha}\left(\Xi(x_1-\omega)^\alpha\right)\Pi'(0)\right.\right. \\ &+\Xi\int_{-\omega}^0 \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-2\omega-\vartheta)^\alpha\right)\Pi(\vartheta)\mathrm{d}\vartheta \\ &\left.\left.- \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\left(\Xi(x_1-\omega-\vartheta)^\alpha\right)\Delta(\vartheta, z(\vartheta))\mathrm{d}Z_H(\vartheta)\right]|\mathfrak{F}_x\right\} \end{aligned} \tag{20}$$

for $x \in \mp$. Additionally, we define the following operators $\mathcal{L}_1$, $\mathcal{L}_2$ on $\mathcal{B}_\lambda$ of the form

$$\begin{aligned}(\mathcal{L}_1 z)(x) \quad &= \mathcal{H}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\Pi(0) + \mathcal{M}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\Pi'(0)\\ &-\Xi\int_{-\omega}^{0}\mathcal{S}_{\omega,\alpha}\big(\Xi(x-2\omega-\vartheta)^{\alpha}\big)\Pi(\vartheta)\mathrm{d}\vartheta\\ &+\int_0^x \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)Bu_z(\vartheta)\mathrm{d}\vartheta,\end{aligned} \tag{21}$$

$$(\mathcal{L}_2 z)(x) = \int_0^x \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)\Delta(\vartheta, z(\vartheta))\mathrm{d}Z_H(\vartheta). \tag{22}$$

Now, we see that $\mathcal{B}_\lambda$ is a closed, bounded, and convex set of $\mathcal{F}$. Therefore, there are three essential steps to our proof:

Step 1. We prove that there exists a $\lambda > 0$ such that $\mathcal{L}_1 z + \mathcal{L}_2\rho \in \mathcal{B}_\lambda$ for all $z, \rho \in \mathcal{B}_\lambda$. Using Equations (21) and (22), we obtain

$$\begin{aligned}&\|\mathcal{L}_1 z + \mathcal{L}_2\rho\|_{\mathcal{F}}^e\\ &= \sup_{x\in[-\omega,x_1]}\mathbf{E}\|(\mathcal{L}_1 z + \mathcal{L}_2\rho)(x)\|^e\\ &\le 5^{e-1}\Big[\big\|\mathcal{H}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\big\|^e\mathbf{E}\|\Pi(0)\|^e + \big\|\mathcal{M}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\big\|^e\mathbf{E}\|\Pi'(0)\|^e\\ &+\|\Xi\|^e\mathbf{E}\Big\|\int_{-\omega}^{0}\mathcal{S}_{\omega,\alpha}\big(\Xi(x-2\omega-\vartheta)^{\alpha}\big)\Pi(\vartheta)\mathrm{d}\vartheta\Big\|^e\\ &+\mathbf{E}\Big\|\int_0^x\mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)Bu_z(\vartheta)\mathrm{d}\vartheta\Big\|^e\\ &+\mathbf{E}\Big\|\int_0^x\mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)\Delta(\vartheta,\rho(\vartheta))\mathrm{d}Z_H(\vartheta)\Big\|^e\Big]\\ &= \sum_{n=1}^{5}\mathbf{I}_n,\end{aligned} \tag{23}$$

for each $x \in \mp$ and $z, \rho \in \mathcal{B}_\lambda$. From Lemma 2, we have

$$\begin{aligned}\mathbf{I}_1 &= 5^{e-1}\big\|\mathcal{H}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\big\|^e\mathbf{E}\|\Pi(0)\|^e\\ &\le 5^{e-1}\big(\mathbb{E}_\alpha\big(\|\Xi\|(x-\omega)^{\alpha}\big)\big)^e\mathbf{E}\|\Pi\|_C^e,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_2 &= 5^{e-1}\big\|\mathcal{M}_{\omega,\alpha}\big(\Xi(x-\omega)^{\alpha}\big)\big\|^e\mathbf{E}\big\|\Pi'(0)\big\|^e\\ &\le 5^{e-1}(x\mathbb{E}_{\alpha,2}(\|\Xi\|x^{\alpha}))^e\mathbf{E}\big\|\Pi'\big\|_C^e,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_3 &= 5^{e-1}\|\Xi\|^e\mathbf{E}\bigg\|\int_{-\omega}^{0}\mathcal{S}_{\omega,\alpha}\big(\Xi(x-2\omega-\vartheta)^{\alpha}\big)\Pi(\vartheta)\mathrm{d}\vartheta\bigg\|^e\\ &\le 5^{e-1}\|\Xi\|^e\omega^{e-1}\mathbf{E}\|\Pi\|_C^e\int_{-\omega}^{0}\big\|\mathcal{S}_{\omega,\alpha}\big(\Xi(x-2\omega-\vartheta)^{\alpha}\big)\big\|^e\mathrm{d}\vartheta\\ &\le 5^{e-1}\|\Xi\|^e\omega^e\Big(x^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x^{\alpha})\Big)^e\mathbf{E}\|\Pi\|_C^e,\end{aligned}$$

$$\begin{aligned}\mathbf{I}_4 &= 5^{e-1}\mathbf{E}\bigg\|\int_0^x\mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)\Delta(\vartheta,\rho(\vartheta))\mathrm{d}Z_H(\vartheta)\bigg\|^e\\ &= 5^{e-1}\mathbf{E}\Bigg\{\bigg\|\int_0^x\mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^{\alpha}\big)\Delta(\vartheta,\rho(\vartheta))\mathrm{d}Z_H(\vartheta)\bigg\|^2\Bigg\}^{e/2},\end{aligned}$$

By employing Lemma 1, the Kahane–khintchine inequality, and Hölder's inequality, there exists a constant τ_e such that

$$\begin{aligned}
\mathbf{I}_4 &\leq 5^{e-1}\tau_e\left\{\mathbf{E}\left\|\int_0^x \mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)\Delta(\vartheta,\rho(\vartheta))\mathrm{d}Z_H(\vartheta)\right\|^2\right\}^{e/2} \\
&\leq 5^{e-1}\tau_e\left\{2Hx^{2H-1}\int_0^x \mathbf{E}\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)\Delta(\vartheta,\rho(\vartheta))\right\|^2_{L_2^0}\mathrm{d}\vartheta\right\}^{e/2} \\
&\leq 5^{e-1}\tau_e\left(2Hx^{2H-1}\right)^{e/2}\left\{\int_0^x \mathbf{E}\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)\Delta(\vartheta,\rho(\vartheta))\right\|^2_{L_2^0}\mathrm{d}\vartheta\right\}^{e/2} \\
&\leq 5^{e-1}\tau_e\left(2Hx^{2H-1}\right)^{e/2} \\
&\times\left\{\left(\int_0^x\left(\mathbf{E}\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)\Delta(\vartheta,\rho(\vartheta))\right\|^2_{L_2^0}\right)^{e/2}\mathrm{d}\vartheta\right)^{2/e}\left(\int_0^x \mathrm{d}\vartheta\right)^{\frac{e-2}{e}}\right\}^{e/2} \\
&\leq 5^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1}\int_0^x \mathbf{E}\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x-\omega-\vartheta)^{\alpha}\right)\Delta(\vartheta,\rho(\vartheta))\right\|^e_{L_2^0}\mathrm{d}\vartheta,
\end{aligned}$$

By employing Lemma 2 and (**J1**), we obtain

$$\begin{aligned}
\mathbf{I}_4 \quad &\leq 5^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1}\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e\mathbf{E}\|\Delta(\vartheta,\rho(\vartheta))\|^e_{L_2^0}\mathrm{d}\vartheta \\
&\leq 5^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1} \\
&\times 2^{e-1}\left\{\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e\mathbf{E}\|\Delta(\vartheta,\rho(\vartheta))-\Delta(\vartheta,0)\|^e_{L_2^0}\mathrm{d}\vartheta\right. \\
&\left.+\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e\mathbf{E}\|\Delta(\vartheta,0)\|^e_{L_2^0}\mathrm{d}\vartheta\right\} \\
&\leq 5^{e-1}2^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1}\left\{\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e L_\Delta(\vartheta)\|\rho(\vartheta)\|^e\mathrm{d}\vartheta\right. \\
&\left.+N_\Delta\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e\mathrm{d}\vartheta\right\} \\
&\leq 5^{e-1}2^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1}\left\{\|\rho\|^e_{\mathcal{F}}\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e L_\Delta(\vartheta)\mathrm{d}\vartheta\right. \\
&\left.+\frac{x_1^{e(\alpha-1)+1}N_\Delta}{e(\alpha-1)+1}\left(\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|x_1^{\alpha}\right)\right)^e\right\}.
\end{aligned} \tag{24}$$

Moreover, from (**J1**) and the Hölder inequality, we have

$$\begin{aligned}
&\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^e L_\Delta(\vartheta)\mathrm{d}\vartheta \\
&\leq\left(\int_0^x\left((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|(x-\vartheta)^{\alpha}\right)\right)^{ep}\mathrm{d}\vartheta\right)^{\frac{1}{p}}\left(\int_0^x L_\Delta^q(\vartheta)\mathrm{d}\vartheta\right)^{\frac{1}{q}} \\
&\leq\left(\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|x_1^{\alpha}\right)\right)^e\left(\int_0^x(x-\vartheta)^{(\alpha-1)ep}\mathrm{d}\vartheta\right)^{\frac{1}{p}}\left(\int_0^x L_\Delta^q(\vartheta)\mathrm{d}\vartheta\right)^{\frac{1}{q}} \\
&\leq\frac{x_1^{(\alpha-1)e+\frac{1}{p}}}{((\alpha-1)ep+1)^{\frac{1}{p}}}\left(\mathbb{E}_{\alpha,\alpha}\left(\|\Xi\|x_1^{\alpha}\right)\right)^e\|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)}.
\end{aligned} \tag{25}$$

By substituting Equation (25) into Equation (24), we find

$$\begin{aligned}
\mathbf{I}_4 &\leq 5^{e-1}2^{e-1}\tau_e(2H)^{e/2}x_1^{eH-1} \\
&\quad\times\left\{\frac{\lambda x_1^{(\alpha-1)e+\frac{1}{p}}}{((\alpha-1)ep+1)^{\frac{1}{p}}}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^{\alpha}))^e\|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)}\right. \\
&\qquad\left.+\frac{x_1^{e(\alpha-1)+1}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^{\alpha}))^e\right\} \\
&= 2^{e-1}N_2\lambda+\frac{(10)^{e-1}\tau_e(2H)^{e/2}x_1^{e(H+\alpha-1)}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^{\alpha}))^e.
\end{aligned}$$

Furthermore, using Equaiton (20), we obtain

$$\begin{aligned}
\mathbf{I}_5 &= 5^{e-1}\mathbf{E}\left\|\int_0^x\mathcal{S}_{\omega,\alpha}(\Xi(x-\omega-\vartheta)^{\alpha})Bu_z(\vartheta)\mathrm{d}\vartheta\right\|^e \\
&\leq 5^{e-1}\left\|W_\omega^{\mathcal{M}}[0,x_1]\right\|^e \\
&\quad\times\left\{\left\|(\Gamma_\omega^{x_1})^{-1}\right\|^e 5^{e-1}\left[\mathbf{E}\|z_1\|^e+\|\mathcal{H}_{\omega,\alpha}(\Xi(x_1-\omega)^{\alpha})\|^e\mathbf{E}\|\Pi(0)\|^e\right.\right. \\
&\quad+\|\mathcal{M}_{\omega,\alpha}(\Xi(x_1-\omega)^{\alpha})\|^e\mathbf{E}\|\Pi'(0)\|^e \\
&\quad+\|\Xi\|^e\mathbf{E}\left\|\int_{-\omega}^0\mathcal{S}_{\omega,\alpha}(\Xi(x_1-2\omega-\vartheta)^{\alpha})\Pi(\vartheta)\mathrm{d}\vartheta\right\|^e \\
&\quad\left.\left.+\mathbf{E}\left\|\int_0^{x_1}\mathcal{S}_{\omega,\alpha}(\Xi(x_1-\omega-\vartheta)^{\alpha})\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta)\right\|^e\right]\right\} \\
&\leq 5^{2(e-1)}NN_1\left[\mathbf{E}\|z_1\|^e+\theta(x_1)+\left(\frac{2}{5}\right)^{e-1}N_2\lambda\right],
\end{aligned}$$

where

$$\begin{aligned}
\theta(x) :=& \left(\mathbb{E}_\alpha\left(\|\Xi\|(x-\omega)^{\alpha}\right)\right)^e\mathbf{E}\|\Pi\|_C^e+(x\mathbb{E}_{\alpha,2}(\|\Xi\|x^{\alpha}))^e\mathbf{E}\|\Pi'\|_C^e \\
&+\|\Xi\|^e\omega^e\left(x^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x^{\alpha})\right)^e\mathbf{E}\|\Pi\|_C^e \\
&+\frac{2^{e-1}\tau_e(2H)^{e/2}x^{e(H+\alpha-1)}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x^{\alpha}))^e.
\end{aligned}$$

From $\mathbf{I}_1$ to $\mathbf{I}_5$, Equation (23) becomes

$$\begin{aligned}
&\|\mathcal{L}_1z+\mathcal{L}_2\rho\|_{\mathcal{F}}^e \\
&\leq 5^{e-1}\left\{\left(\mathbb{E}_\alpha\left(\|\Xi\|(x-\omega)^{\alpha}\right)\right)^e\mathbf{E}\|\Pi\|_C^e\right. \\
&+(x\mathbb{E}_{\alpha,2}(\|\Xi\|x^{\alpha}))^e\mathbf{E}\|\Pi'\|_C^e \\
&+\|\Xi\|^e\omega^e\left(x^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x^{\alpha})\right)^e\mathbf{E}\|\Pi\|_C^e \\
&+\left(\frac{2}{5}\right)^{e-1}N_2\lambda+\frac{2^{e-1}\tau_e(2H)^{e/2}x_1^{e(H+\alpha-1)}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^{\alpha}))^e \\
&\left.+5^{e-1}NN_1\left[\mathbf{E}\|z_1\|^e+\theta(x_1)+\left(\frac{2}{5}\right)^{e-1}N_2\lambda\right]\right\} \\
&\leq 5^{e-1}\left\{\theta(x_1)\left(1+5^{e-1}NN_1\right)\right. \\
&\left.+5^{e-1}NN_1\mathbf{E}\|z_1\|^e+\left(\frac{2}{5}\right)^{e-1}\lambda N_2\left(1+5^{e-1}NN_1\right)\right\}.
\end{aligned}$$

Thus, for some sufficiently large λ, and from Equation (19), we have $\mathcal{L}_1 z + \mathcal{L}_2 \rho \in \mathcal{B}_\lambda$.

Step 2. We prove $\mathcal{L}_1 : \mathcal{B}_\lambda \longrightarrow \mathcal{F}$ is a contraction. Using Equation (20), we obtain

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{L}_1 z)(x) - (\mathcal{L}_1 \rho)(x)\|^e \\
&= \mathbf{E}\left\| \int_0^x \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^\alpha\big) B\big[u_z(\vartheta) - u_\rho(\vartheta)\big] \mathrm{d}\vartheta \right\|^e \\
&\leq \left\| W_\omega^{\mathcal{M}}[0, x_1] \right\|^e \left\| (\Gamma_\omega^{x_1})^{-1} \right\|^e \\
&\times \mathbf{E}\left\| \int_0^{x_1} \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^\alpha\big)[\Delta(\vartheta, \rho(\vartheta)) - \Delta(\vartheta, z(\vartheta))] \mathrm{d}Z_H(\vartheta) \right\|^e \\
&\leq \tau_e N N_1 (2H)^{e/2} x_1^{eH-1} \\
&\times \int_0^x \mathbf{E}\left\| \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^\alpha\big)[\Delta(\vartheta, \rho(\vartheta)) - \Delta(\vartheta, z(\vartheta))] \right\|_{L_2^0}^e \mathrm{d}\vartheta \\
&\leq \tau_e N N_1 (2H)^{e/2} x_1^{eH-1} \mathbf{E}\|z-\rho\|_{\mathcal{F}}^e \int_0^x \Big((x-\vartheta)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(\|\Xi\|(x-\vartheta)^\alpha\big) \Big)^e L_\Delta(\vartheta) \mathrm{d}\vartheta \\
&\leq N N_1 \frac{\tau_e (2H)^{e/2} x_1^{e(H+\alpha-1)-\frac{1}{q}}}{((\alpha-1)ep+1)^{\frac{1}{p}}} \big(\mathbb{E}_{\alpha,\alpha}(\|\Xi\| x_1^\alpha)\big)^e \|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)} \mathbf{E}\|z-\rho\|_{\mathcal{F}}^e \\
&\leq \frac{N N_1 N_2}{5^{e-1}} \mathbf{E}\|z-\rho\|_{\mathcal{F}}^e \\
&\leq \mu \mathbf{E}\|z-\rho\|_{\mathcal{F}}^e,
\end{aligned}$$

for each $x \in \mp$ and $z, \rho \in \mathcal{B}_\lambda$, where $\mu := N N_1 N_2 / 5^{e-1}$. We may deduce from Equation (19) and, noting $\mu < 1$, that $\mathcal{L}_1$ is a contraction mapping.

Step 3. We prove $\mathcal{L}_2 : \mathcal{B}_\lambda \longrightarrow \mathcal{F}$ is a continuous compact operator.

First, we show that $\mathcal{L}_2$ is continuous. Let $\{z_n\}$ be a sequence such that $z_n \longrightarrow z$ as $n \longrightarrow \infty$ in $\mathcal{B}_\lambda$. Thus, for each $x \in \mp$, using Equation (22) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{L}_2 z_n)(x) - (\mathcal{L}_2 z)(x)\|^e \\
&\leq \tau_e (2H)^{e/2} x_1^{eH-1} \int_0^x \left\| \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^\alpha\big) \right\|^e \mathbf{E}\|\Delta(\vartheta, z_n(\vartheta)) - \Delta(\vartheta, z(\vartheta))\|_{L_2^0}^e \mathrm{d}\vartheta \\
&\leq \tau_e (2H)^{e/2} x_1^{eH-1} \int_0^x \Big((x-\vartheta)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(\|\Xi\|(x-\vartheta)^\alpha\big) \Big)^e L_\Delta(\vartheta) \\
&\qquad \mathbf{E}\|z_n(\vartheta) - z(\vartheta)\|^e \mathrm{d}\vartheta \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence, $\mathcal{L}_2 : \mathcal{B}_\lambda \longrightarrow \mathcal{F}$ is continuous.

After that, we prove that $\mathcal{L}_2$ is uniformly bounded on $\mathcal{B}_\lambda$. For each $x \in \mp$, $z \in \mathcal{B}_\lambda$, we obtain

$$\begin{aligned}\|\mathcal{L}_2 z\|^e_{\mathcal{F}} &= \sup_{x\in\mp} \mathbf{E}\|(\mathcal{L}_2 z)(x)\|^e \\ &\le \sup_{x\in\mp}\left\{\mathbf{E}\left\|\int_0^x \mathcal{S}_{\omega,\alpha}\big(\Xi(x-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta)\right\|^e\right\} \\ &\le \left(\frac{2}{5}\right)^{e-1} N_2\lambda + \frac{2^{e-1}\tau_e(2H)^{e/2}x_1^{e(H+\alpha-1)}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^\alpha))^e,\end{aligned}$$

which leads to $\mathcal{L}_2$ being uniformly bounded on $\mathcal{B}_\lambda$.

It remains to be proven that $\mathcal{L}_2$ is equicontinuous. For $x_2, x_3 \in \mp$, $0 < x_2 < x_3 \le x_1$, and $z \in \mathcal{B}_\lambda$, using Equation (22), we obtain

$$\begin{aligned}&(\mathcal{L}_2 z)(x_3) - (\mathcal{L}_2 z)(x_2) \\ &= \int_0^{x_3} \mathcal{S}_{\omega,\alpha}\big(\Xi(x_3-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta) \\ &- \int_0^{x_2} \mathcal{S}_{\omega,\alpha}\big(\Xi(x_2-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta) \\ &= \Psi_1 + \Psi_2,\end{aligned}$$

where

$$\Psi_1 = \int_{x_2}^{x_3} \mathcal{S}_{\omega,\alpha}\big(\Xi(x_3-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta),$$

and

$$\Psi_2 = \int_0^{x_2}\big[\mathcal{S}_{\omega,\alpha}\big(\Xi(x_3-\omega-\vartheta)^\alpha\big) - \mathcal{S}_{\omega,\alpha}\big(\Xi(x_2-\omega-\vartheta)^\alpha\big)\big]\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta).$$

Thus, we have

$$\begin{aligned}\mathbf{E}\|(\mathcal{L}_2 z)(x_3) - (\mathcal{L}_2 z)(x_2)\|^e &= \mathbf{E}\|\Psi_1+\Psi_2\|^e \\ &\le 2^{e-1}\big\{\mathbf{E}\|\Psi_1\|^e + \mathbf{E}\|\Psi_2\|^e\big\}.\end{aligned} \tag{26}$$

Now, we can check $\|\Psi_i\| \longrightarrow 0$ as $x_2 \longrightarrow x_3$, $i = 1, 2$. For Ψ_1, we obtain

$$\begin{aligned}\mathbf{E}\|\Psi_1\|^e &= \mathbf{E}\left\|\int_{x_2}^{x_3} \mathcal{S}_{\omega,\alpha}\big(\Xi(x_3-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\mathrm{d}Z_H(\vartheta)\right\|^e \\ &\le \tau_e(2H)^{e/2}(x_3-x_2)^{eH-1}\int_{x_2}^{x_3}\mathbf{E}\big\|\mathcal{S}_{\omega,\alpha}\big(\Xi(x_3-\omega-\vartheta)^\alpha\big)\Delta(\vartheta,z(\vartheta))\big\|^e_{L_2^0}\mathrm{d}\vartheta \\ &\le 2^{e-1}\tau_e(2H)^{e/2}(x_3-x_2)^{eH-1} \\ &\times\left\{\|z\|^e_{\mathcal{F}}\int_{x_2}^{x_3}\Big((x-\vartheta)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}\big(\|\Xi\|(x-\vartheta)^\alpha\big)\Big)^e L_\Delta(\vartheta)\mathrm{d}\vartheta\right. \\ &\left.+\frac{(x_3-x_2)^{e(\alpha-1)+1}N_\Delta}{e(\alpha-1)+1}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_3^\alpha))^e\right\} \longrightarrow 0, \text{ as } x_2 \longrightarrow x_3.\end{aligned}$$

For Ψ_2, we find

$$
\begin{aligned}
&\mathbf{E}\|\Psi_2\|^e \\
&= \mathbf{E}\left\| \int_0^{x_2} \left[\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right]\Delta(\vartheta, z(\vartheta))\mathrm{d}Z_H(\vartheta) \right\|^e \\
&\le \tau_e(2H)^{e/2} x_2^{eH-1} \\
&\times \int_0^{x_2} \mathbf{E}\left\| \left[\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right]\Delta(\vartheta, z(\vartheta))\right\|^e_{L_2^0} \mathrm{d}\vartheta \\
&\le 2^{e-1}\tau_e(2H)^{e/2} x_2^{eH-1} \\
&\times \left\{ \lambda \int_0^{x_2} \left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right\|^e L_\Delta(\vartheta)\mathrm{d}\vartheta \right. \\
&\left. + N_\Delta \int_0^{x_2} \left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right\|^e \mathrm{d}\vartheta \right\} \\
&\le 2^{e-1}\tau_e(2H)^{e/2} x_2^{eH-1} \\
&\times \left\{ \lambda \|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)} \right. \\
&\times \left(\int_0^{x_2} \left(\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right\|^e \right)^p \right)^{1/p} \mathrm{d}\vartheta \\
&\left. + N_\Delta \int_0^{x_2} \left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right\|^e \mathrm{d}\vartheta \right\}
\end{aligned}
$$

From Equation (4), we know that $\mathcal{S}_{\omega,\alpha}\left(\Xi(x)^\alpha\right)$ is uniformly continuous for $x \in \mp$. Hence, we have

$$
\left\|\mathcal{S}_{\omega,\alpha}\left(\Xi(x_3-\omega-\vartheta)^\alpha\right) - \mathcal{S}_{\omega,\alpha}\left(\Xi(x_2-\omega-\vartheta)^\alpha\right)\right\| \longrightarrow 0, \text{ as } x_2 \longrightarrow x_3.
$$

Therefore, we have $\|\Psi_i\| \longrightarrow 0$ as $x_2 \longrightarrow x_3$, $i = 1, 2$, which implies, using Equation (26), that

$$
\mathbf{E}\|(\mathcal{L}_2 z)(x_3) - (\mathcal{L}_2 z)(x_2)\|^e \longrightarrow 0, \quad \text{as } x_2 \longrightarrow x_3,
$$

for all $z \in \mathcal{B}_\lambda$. As a result, $\mathcal{L}_2$ is compact on $\mathcal{B}_\lambda$ by applying the Arzelà–Ascoli theorem. Thus, $\mathcal{L}_1 + \mathcal{L}_2$ has a fixed point z on $\mathcal{B}_\lambda$ using Krasnoselskii's fixed point theorem (Lemma 3). Moreover, z is also a solution to Equation (7), and $(\mathcal{L}_1 z + \mathcal{L}_2 z)(x_1) = z_1$. This indicates that u_z steers the system in Equation (7) from z_0 to z_1 in a finite time x_1, , implying that Equation (7) is controllable on $\mp$. This completes the proof. □

5. An Example

Consider the following linear delay fractional stochastic controlled system:

$$
\begin{aligned}
\left({}^C D_{0^+}^{1.5} z\right)(x) + \Xi z(x-0.5) &= Bu(x) + \overline{\Delta}(x)\mathrm{d}Z_H(x), \quad \text{for } x \in \Omega := [0,1], \\
z(x) \equiv \Pi(x),\ z'(x) &\equiv \Pi'(x) \quad \text{for } -0.5 \le x \le 0,
\end{aligned} \tag{27}
$$

where

$$
\Xi = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \overline{\Delta}(x) = \begin{pmatrix} \frac{\sqrt{x}e^{-x}}{4} \\ \frac{\sqrt{x}e^{-x}}{4} \end{pmatrix},
$$

and

$$
\Pi(x) = \begin{pmatrix} 2x \\ x \end{pmatrix},\ \Pi'(x) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
$$

By constructing the corresponding fractional delayed Gramian matrix of Equation (27) via Equation (9), we obtain

$$\begin{aligned} W_{0.5,1.5}[0,1] &= \int_0^1 \mathcal{S}_{0.5,1.5}\left(\Xi(0.5-\vartheta)^{1.5}\right)BB^T\mathcal{S}_{0.5,1.5}\left(\Xi^T(0.5-\vartheta)^{1.5}\right)\mathrm{d}\vartheta \\ &=: O_1 + O_2, \end{aligned}$$

where

$$O_1 = \int_0^{0.5} \mathcal{S}_{0.5,1.5}\left(\Xi(0.5-\vartheta)^{1.5}\right)BB^T\mathcal{S}_{0.5,1.5}\left(\Xi^T(0.5-\vartheta)^{1.5}\right)\mathrm{d}\vartheta,$$

for $(0.5-\vartheta) \in (0, 0.5)$,

$$O_2 = \int_{0.5}^{1} \mathcal{S}_{0.5,1.5}\left(\Xi(0.5-\vartheta)^{1.5}\right)BB^T\mathcal{S}_{0.5,1.5}\left(\Xi^T(0.5-\vartheta)^{1.5}\right)\mathrm{d}\vartheta,$$

for $(0.5-\vartheta) \in (-0.5, 0)$, and

$$\mathcal{H}_{0.5,1.5}\left(\Xi x^{1.5}\right) := \begin{cases} \not\leq, & -\infty < x < -0.5, \\ \mathbb{I}, & -0.5 \leq x < 0, \\ \mathbb{I} - \Xi\frac{x^{1.5}}{\Gamma(2.5)} & 0 \leq x < 0.5, \\ \mathbb{I} - \Xi\frac{x^{1.5}}{\Gamma(2.5)} + \Xi^2\frac{(x-0.5)^3}{\Gamma(4)}, & 0.5 \leq x < 1, \end{cases}$$

$$\mathcal{M}_{0.5,1.5}\left(\Xi x^{1.5}\right) := \begin{cases} \not\leq, & -\infty < x < -0.5, \\ \mathbb{I}(x+0.5), & -0.5 \leq x < 0, \\ \mathbb{I}(x+0.5) - \Xi\frac{x^{2.5}}{\Gamma(3.5)}, & 0 \leq x < 0.5, \\ \mathbb{I}(x+0.5) - \Xi\frac{x^{2.5}}{\Gamma(3.5)} + \Xi^2\frac{(x-0.5)^4}{\Gamma(5)}, & 0.5 \leq x < 1, \end{cases}$$

in addition to

$$\mathcal{S}_{0.5,1.5}\left(\Xi x^{1.5}\right) := \begin{cases} \not\leq, & -\infty < x < -0.5, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)}, & -0.5 \leq x < 0, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)} - \Xi\frac{x^2}{\Gamma(3)}, & 0 \leq x < 0.5, \\ \mathbb{I}\frac{(x+0.5)^{0.5}}{\Gamma(1.5)} - \Xi\frac{x^2}{\Gamma(3)} + \Xi^2\frac{(x-0.5)^{3.5}}{\Gamma(4.5)}, & 0.5 \leq x < 1. \end{cases}$$

Next, we can calculate that

$$O_1 = \begin{pmatrix} 0.1274 & 0.5036 \\ 0.5036 & -0.11406 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 0.15915 & 0.3183 \\ 0.3183 & 0.6366 \end{pmatrix}.$$

Then, we obtain

$$W_{0.5,1.5}[0,1] = O_1 + O_2 = \begin{pmatrix} 0.28655 & 0.8219 \\ 0.8219 & 0.52254 \end{pmatrix},$$

and

$$W^{-1}_{0.5,1.5}[0,1] = \begin{pmatrix} -0.99382 & 1.5632 \\ 1.5632 & -0.54500 \end{pmatrix}.$$

Therefore, we see that $W_{0.5,1.5}[0,1]$ is positive definite. Hence, the system in Equation (27) is controllable on $[0,1]$ by Theorem 1, which implies that the assumption (**J2**) is satisfied. Furthermore, consider the corresponding nonlinear fractional stochastic delay system of Equation (27) as follows:

$$\begin{aligned} \left({}^{C}D^{1.5}_{0^+}z\right)(x) + \Xi z(x-0.5) &= Bu(x) + \Delta(x, z(x))\mathrm{d}Z_H(x), \quad \text{for } x \in \mp := [0,1], \\ z(x) \equiv \Pi(x),\ z'(x) &\equiv \Pi'(x) \quad \text{for } -0.5 \leq x \leq 0, \end{aligned} \tag{28}$$

where

$$\Delta(x, z(x)) = \begin{pmatrix} \frac{\sqrt{x}e^{-x}}{4} z_1(x) \\ \frac{\sqrt{x}e^{-x}}{4} z_2(x) \end{pmatrix}.$$

Next, by selecting $e = p = q = 2$, we find

$$\begin{aligned} \mathbf{E}\|\Delta(x,z) - \Delta(x,\rho)\|^2_{L^0_2} &= \left(\frac{\sqrt{x}e^{-x}}{4}\right)^2 \left[(z_1(x) - \rho_1(x))^2 + (z_2(x) - \rho_2(x))^2\right] \\ &= \frac{xe^{-2x}}{16}\|z - \rho\|^2_{L^0_2}. \end{aligned}$$

for all $x \in \mp$, and $z(x)$, $\rho(x) \in \mathbb{R}^2$. We set $L_\Delta(x) = x\exp(-2x)/16$ such that $L_\Delta \in L^2(\mp, \mathbb{R}^+)$ in (**J1**), and we have

$$\|L_\Delta\|_{L^2(\mp,\mathbb{R}^+)} = \left(\int_0^1 \left[\frac{\vartheta\exp(-2\vartheta)}{16}\right]^2 \mathrm{d}\vartheta\right)^{\frac{1}{2}} = 0.00964.$$

Then, by choosing $\alpha = 1.5$, $\tau_e = 0.018$, and $H = 0.75$, we obtain

$$N_2 := \frac{5^{e-1}\tau_e(2H)^{e/2}x_1^{e(H+\alpha-1)-\frac{1}{q}}}{((\alpha-1)ep+1)^{\frac{1}{p}}}(\mathbb{E}_{\alpha,\alpha}(\|\Xi\|x_1^\alpha))^e\|L_\Delta\|_{L^q(\mp,\mathbb{R}^+)} = 0.01.$$

Furthermore, we have

$$\mathbf{E}\langle W_{0.5,1.5}[0,1]z, z\rangle = \begin{pmatrix} 0.28655z_1^2 & 0.8219z_2^2 \\ 0.8219z_1^2 & 0.52254z_2^2 \end{pmatrix} \geq \eta\mathbf{E}\|z\|^2,$$

where $0 < \eta \leq 0.28655$, and thus $N_1 = 3.4898$ and $N = 1.8075$. Finally, we calculate that

$$N_2\left[1 + 5^{e-1}NN_1\right] = 0.32539 < 1,$$

which implies that all the conditions of Theorem 2 are met. Therefore, the system in Equation (28) is controllable.

6. Conclusions

In this paper, using a fractional delayed Gramian matrix and the exact solutions of linear fractional stochastic delay systems, we derived the controllability results. Furthermore, by applying Krasnoselskii's fixed point theorem and the exact solutions of nonlinear fractional stochastic delay systems, we established the controllability results.

The results of this paper will be supplemented in the future to derive the Hyers–Ulam stability of fractional stochastic delay systems of the order $\alpha \in (1, 2]$.

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