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Article

Certain Weighted Fractional Inequalities via the Caputo–Fabrizio Approach

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Abstract: The Caputo–Fabrizio fractional integral operator is one of the important notions of fractional calculus. It is involved in numerous illustrative and practical issues. The main goal of this paper is to investigate weighted fractional integral inequalities using the Caputo–Fabrizio fractional integral operator with non-singular $e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-s)}$, $0 < \delta < 1$. Furthermore, based on a family of n positive functions defined on $[0,\infty)$, we investigate some new extensions of weighted fractional integral inequalities.

Keywords: inequalities; Caputo–Fabrizio fractional integral operator; weighted fractional inequalities

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1. Introduction

Fractional calculus is the extension of conventional calculus into the non-integer differential and integral orders. It has found significant importance due to its application in various fields of science and engineering, such as life sciences, chemical science, and physical sciences. It naturally appears in numerous branches of science and engineering, including the physical, chemical, and life sciences. Numerous mathematicians have spent the last two decades studying fractional integral inequalities and applications using the Riemann–Liouville, Saigo, Hadamard, Marichev–Saigo-Maeda, generalized Katugampola, and generalized k-fractional integral operators, see Refs. [1–10].

Some recent developments in this field will now be presented. Houas M. looked at certain weighted integral inequalities involving fractional hypergeometric operators in Ref. [11]. Marichev [12] (see also Ref. [10]) introduced the generalization of the hypergeometric fractional integral, including the Saigo operator. Caputo and Fabrizio proposed a novel fractional derivative and application of a new time and spatial fractional derivative with exponential kernels (see Refs. [13,14]). Recently, certain fractional integral inequalities utilizing the Caputo–Fabrizio fractional integral were proposed by Gustava Nchama and et al. in Ref. [15]. In Refs. [16,17], the authors proposed some fractional inequalities for h-convex functions and preinvex functions involving the Caputo–Fabrizio operator. Gürbüz and et al. [18] obtained the Hermite–Hadamard inequality and related inequalities for fractional integrals of Caputo–Fabrizio type. The main motivation is the Caputo–Fabrizio integral and its derivative operator, which are general fractional integral and derivative, respectively (see Refs. [19,20]). Additionally, it has a non-singular kernel, which is a real power that has been transformed into an integral using the Laplace transform. As a result, many problems have an easy time finding an exact solution.

These days, fractional integrals and derivatives play a significant role in modeling a variety of physical phenomena. Due to the singular kernel, the Riemann–Liouville and Caputo fractional derivatives cannot adequately represent several phenomena relating to material heterogeneities. It is the result of Caputo and Fabrizio's suggestion of a novel fractional integral involving the nonsingular kernel $e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-s)}$, where the parameter δ

satisfies $0 < \delta < 1$. Furthermore, the Caputo–Fabrizio fractional integral operator has been increasingly popular among applied scientists' mathematicians as a modeling tool (see Refs. [21–23]). In Refs. [15,24], the authors developed fractional integral inequalities using the Caputo–Fabrizio operator. For further information, see Refs. [10,12]. As a concrete physical example, a heat conduction equation with the Caputo–Fabrizio derivative and an investigation of a line segment were utilized to provide the fundamental solutions to the Cauchy and Dirichlet problems in Ref. [25]. The major advantages of the Caputo–Fabrizio integral operator is that it allows the boundary condition of fractional differential equations with Caputo–Fabrizio derivatives to admit the same form as for the differential equations of integer order. A few studies using the Caputo and Caputo–Fabrizio integral operators to solve fractional integral inequalities have been published (see Refs. [15,26,27]). Our goal is to create some novel weighted fractional integral inequalities involving Caputo–Fabrizio fractional integral operators, which are inspired by Refs. [20,22,23,28,29].

The following paragraph describes how the paper has organized. In Section 2, basic definitions for the Caputo–Fabrizio fractional derivatives and integrals are provided. In Section 3, by utilizing the Caputo–Fabrizio fractional integral operator, we provide weighted fractional integral inequalities. A conclusion is given in Section 4.

2. Preliminaries

Here, the Caputo–Fabrizio fractional integral operator is discussed along with some fundamental definitions of fractional calculus.

Definition 1 ([3,15,26])**.** *Let $\delta, a \in \mathbb{R}$ such that $0 < \delta < 1$. The Caputo–Fabrizio fractional derivative of order δ of a function ϕ is defined by*

$$\mathcal{J}^{\delta}_{a,\varkappa}[\phi(\varkappa)] = \frac{1}{1-\delta}\int_a^{\varkappa} e^{-\left(\frac{\delta}{1-\delta}\right)(\varkappa-s)}\phi'(s)ds. \tag{1}$$

Definition 2 ([3,15,26])**.** *Let $\delta \in \mathbb{R}$ such that $0 < \delta \leq 1$. The Caputo–Fabrizio fractional integral of order δ of a function ϕ is defined by*

$$\mathcal{I}^{\delta}_{0,\varkappa}[\phi(\varkappa)] = \frac{1}{\delta}\int_0^{\varkappa} e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-s)}\phi(s)ds. \tag{2}$$

For the special case $\delta = 1$, it is reduced to the following integral:

$$\mathcal{I}^{1}_{0,\varkappa}[\phi(\varkappa)] = \int_0^{\varkappa} \phi(s)ds.$$

The above definition may be extended to any $\delta > 0$.

With the help of the Caputo–Fabrizio fractional integral operator, various novel inequalities will be demonstrated in this study.

3. Weighted Fractional Integral Inequalities

Using the Caputo–Fabrizio fractional integral operator, we establish a few weighted fractional integral inequalities in this section.

Theorem 1. *Let f be a positive continuous function on $[0,\infty)$, and $\varrho > 0$, $\varpi \geq \lambda > 0$, such that, for any $\mu, \zeta > 0$,*

$$(\zeta^{\varrho}\mathrm{f}^{\varrho}(\mu) - \mu^{\varrho}\mathrm{f}^{\varrho}(\zeta))(\mathrm{f}^{\varpi-\lambda}(\mu) - \mathrm{f}^{\varpi-\lambda}(\zeta)) \geq 0. \tag{3}$$

In addition, let w be a positive continuous function on $[0,\infty)$. Then, for all $\varkappa, \delta > 0$, we have

$$\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\,\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)] \leq \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]\,\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]. \tag{4}$$

Proof. From (3), we have

$$\zeta^{\varrho}\mathrm{f}^{\varpi-\lambda}(\zeta)\mathrm{f}^{\varrho}(\mu)+\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)\mathrm{f}^{\varrho}(\zeta)\leq\zeta^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\mu)+\mu^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\zeta). \tag{5}$$

Then, multiplying both sides of the inequality (5) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\mathrm{f}^{\lambda}(\mu)$, $\mu\in(0,\varkappa)$, $\varkappa>0$, and integrating the resultant identity with respect to μ from 0 to $\varkappa$, we get

$$\begin{aligned}&\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\zeta^{\varrho}\mathrm{f}^{\varpi-\lambda}(\zeta)w(\mu)\mathrm{f}^{\varrho+\lambda}(\mu)d\mu+\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi}(\mu)\mathrm{f}^{\varrho}(\zeta)w(\mu)d\mu\\&\leq\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\zeta^{\varrho}\mathrm{f}^{\varpi+\varrho}(\mu)w(\mu)d\mu+\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\zeta)w(\mu)\mathrm{f}^{\lambda}(\mu)d\mu,\end{aligned} \tag{6}$$

consequently,

$$\begin{aligned}&\zeta^{\varrho}\ \mathrm{f}^{\varpi-\lambda}(\zeta)\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]+\mathrm{f}^{\varrho}(\zeta)\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]\\&\leq\zeta^{\varrho}\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]+\mathrm{f}^{\varrho+\varpi-\lambda}(\zeta)\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)].\end{aligned} \tag{7}$$

Multiplying both sides of (7) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\lambda}(\zeta),\zeta\in(0,\varkappa),\varkappa>0$, then integrating the resulting identity with respect to ζ from 0 to $\varkappa$, we obtain

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]+\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\\&\leq\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]+\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)],\end{aligned} \tag{8}$$

which brings the proof to a close. □

The condition (3) is not severe; it can be demonstrated that it is satisfied for any decreasing function f, or any function f such that f and $\mathrm{f}(\varkappa)/\varkappa$ are increasing. We can notice that the inequality (4) becomes an equality for the basic function $f(\varkappa)=\varkappa$.

We will now present our major finding.

Theorem 2. *Let* f *be a positive continuous function on* $[0,\infty)$ *and* $\varrho>0$, $\varpi\geq\lambda>0$, *which satisfy (3). In addition, let* w *be a positive continuous function on* $[0,\infty)$. *Then, for all* $\varkappa,\delta>0$, *we have*

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]+\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\\&\leq\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]+\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)].\end{aligned} \tag{9}$$

Proof. Let us multiply both sides of (5) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\lambda}(\zeta)$, $(\zeta\in(0,\varkappa),\ \varkappa>0)$, under the circumstances specified in the theorem (this function remaining positive). Then, by integrating the result with respect to ζ from 0 to $\varkappa$, we obtain

$$\begin{aligned}&\mathrm{f}^{\varrho}(\mu)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\zeta)\zeta^{\varrho}\mathrm{f}^{\varpi}(\zeta)d\zeta+\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\zeta)\mathrm{f}^{\varrho+\lambda}(\zeta)d\zeta\\&\leq\mathrm{f}^{\varpi+\varrho-\lambda}(\mu)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\zeta)\zeta^{\varrho}\mathrm{f}^{\lambda}(\zeta)d\zeta+\mu^{\varrho}\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\zeta)\mathrm{f}^{\varpi+\varrho}(\zeta)d\zeta,\end{aligned} \tag{10}$$

therefore

$$\begin{aligned}&\mathrm{f}^{\varrho}(\mu)\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]+\mu^{\varrho}\ \mathrm{f}^{\varpi-\lambda}(\mu)\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\\&\leq\mathrm{f}^{\varrho+\varpi-\lambda}(\mu)\ \ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]+\mu^{\varrho}\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)].\end{aligned} \tag{11}$$

Again, multiplying both sides of (11) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\mathrm{f}^{\lambda}(\mu),\mu\in(0,\varkappa),\varkappa>0$, then integrating the obtained result with respect to ζ from 0 to $\varkappa$, we get

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\varpi}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\mathrm{f}^{\varrho+\lambda}(\mu)d\mu+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi}(\mu)w(\mu)d\mu\\&\leq\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mathrm{f}^{\varpi+\varrho}(\mu)w(\mu)d\mu+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\lambda}(\mu)w(\mu)d\mu.\end{aligned}\tag{12}$$

This completes the proof of the theorem. □

Theorem 3. *Let* f *and* g *be positive continuous functions on* $[0,\infty)$*, and* $\varrho>0$*,* $\varpi\geq\lambda>0$*, such that, for any* $\mu,\zeta>0$*, we have*

$$(\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varrho}(\mu)-\mathrm{g}^{\varrho}(\mu)\mathrm{f}^{\varrho}(\zeta))(\mathrm{f}^{\varpi-\lambda}(\mu)-\mathrm{f}^{\varpi-\lambda}(\zeta))\geq 0.\tag{13}$$

In addition, let w *be a positive continuous function on* $[0,\infty)$*. Then, for all* $\varkappa,\delta>0$*, we get*

$$\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\;\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\leq\;\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)].\tag{14}$$

Proof. From (13), we have

$$\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi-\lambda}(\zeta)\mathrm{f}^{\varrho}(\mu)+\mathrm{g}^{\varrho}(\mu)\mathrm{f}^{\varrho}(\zeta)\mathrm{f}^{\varpi-\lambda}(\mu)\leq\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi+\varrho-\lambda}(\mu)+\mathrm{g}^{\varrho}(\mu)\mathrm{f}^{\varpi+\varrho-\lambda}(\zeta).\tag{15}$$

Multiplying both sides of (15) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\mathrm{f}^{\lambda}(\mu)$, $\mu\in(0,\varkappa)$, $\varkappa>0$, then integrating the resulting identity with respect to μ from 0 to $\varkappa$, we obtain

$$\begin{aligned}&\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi-\lambda}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)[w(\mu)\mathrm{f}^{\varrho+\lambda}(\mu)]d\mu\\&+\mathrm{f}^{\varrho}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)[w(\mu)\mathrm{g}^{\varrho}(\mu)\mathrm{f}^{\varpi}(\mu)]d\mu\\&\leq\;\mathrm{g}^{\varrho}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)[w(\mu)\mathrm{f}^{\varpi+\varrho}(\mu)]d\mu\\&+\mathrm{f}^{\varrho+\varpi-\lambda}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}\mu^{\varrho}\mathrm{f}^{\varpi-\lambda}(\mu)[w(\mu)\mathrm{g}^{\varrho}(\mu)\mathrm{f}^{\lambda}(\mu)]d\mu.\end{aligned}\tag{16}$$

Thus, we establish that

$$\begin{aligned}&\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi-\lambda}(\zeta)\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]+\mathrm{f}^{\varrho}(\zeta)\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\\&\leq\mathrm{g}^{\varrho}(\zeta)\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)]+\mathrm{f}^{\varrho+\varpi-\lambda}(\zeta)\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)].\end{aligned}\tag{17}$$

Multiplying both sides of (17) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\lambda}(\zeta)$, then integrating the resulting inequality with respect to ζ over $(0,\varkappa)$, we obtain

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi}(\zeta)d\zeta\\&+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}\mathrm{f}^{\varrho+\lambda}(\zeta)w(\zeta)d\zeta\\&\leq\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}\mathrm{f}^{\lambda}(\zeta)w(\zeta)\mathrm{g}^{\varrho}(\zeta)d\zeta\\&+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\varpi+\varrho}(\zeta)d\zeta,\end{aligned}\tag{18}$$

which implies that

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[\mathrm{f}^{\varrho+\lambda}(\varkappa)w(\varkappa)]\\&\leq\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[\mathrm{f}^{\lambda}(\varkappa)w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)]+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)],\end{aligned}\tag{19}$$

that concludes the proof. □

It is worth noting that the condition (13) is satisfied for a wide range of function classes. It is satisfied in particular by any functions f and g such that either $\mathrm{f}(\varkappa)/\mathrm{g}(\varkappa)$ and $\mathrm{f}(\varkappa)$ are increasing (simultaneously), or $\mathrm{f}(\varkappa)/\mathrm{g}(\varkappa)$ and $\mathrm{f}(\varkappa)$ are decreasing (simultaneously).

Theorem 4. *Let* f *and* g *be two positive continuous functions on* $[0,\infty)$ *and* $\varrho > 0$, $\varpi \geq \lambda > 0$, *which satisfy* (13). *In addition, let* w *be a positive continuous function on* $[0,\infty)$. *Then, for all* $\varkappa, \delta, \beta > 0$, *we have*

$$\begin{aligned}&\mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)] \ + \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\\ &\leq \ \mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)] + \mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi-\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)].\end{aligned} \tag{20}$$

Proof. Multiplying the inequality (17) by $\frac{1}{\beta}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\lambda}(\zeta)$, $\zeta \in (0,\varkappa)$, $\varkappa > 0$ (this function remains positive under the conditions stated with the theorem), then integrating the obtained result with respect to ζ from 0 to $\varkappa$, we get

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa} e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{g}^{\varrho}(\zeta)\mathrm{f}^{\varpi}(\zeta)d\zeta\\ &+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa} e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}\mathrm{f}^{\varrho+\lambda}(\zeta)w(\zeta)d\zeta\\ &\leq \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa} e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}\mathrm{f}^{\lambda}(\zeta)w(\zeta)\mathrm{g}^{\varrho}(\zeta)d\zeta\\ &+\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\beta}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa} e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}^{\varpi+\varrho}(\zeta)d\zeta,\end{aligned} \tag{21}$$

which implies that

$$\begin{aligned}&\mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)] \ + \mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\varpi}(\varkappa)]\\ &\leq \ \mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}\mathrm{f}^{\lambda}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varpi+\varrho}(\varkappa)] + \mathcal{I}^{\beta}_{0,\varkappa}[w(\varkappa)\mathrm{f}^{\varrho+\varpi}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}^{\lambda}(\varkappa)].\end{aligned} \tag{22}$$

Thus, the proof is completed. □

In the sequel of the study, we will provide a new extension of weighted fractional integral inequalities based on a family of n positive functions defined on $[0,\infty)$.

Theorem 5. *Let* $\mathrm{f}_i, i = 1,\ldots,n$ *be* n *positive continuous functions on* $[0,\infty)$ *and* $\varrho > 0, \varpi \geq \lambda_r > 0, r = 1,\ldots,n$, *such that, for any* $\mu,\zeta > 0$,

$$(\zeta^{\varrho}\mathrm{f}_r^{\varrho}(\mu) - \mu^{\varrho}\mathrm{f}_r^{\varrho}(\zeta))(\mathrm{f}_r^{\varpi-\lambda_r}(\mu) - \mathrm{f}_r^{\varpi-\lambda_r}(\zeta)) \geq 0. \tag{23}$$

In addition, let w *be a positive continuous function on* $[0,\infty)$. *Then, for all* $\varkappa, \delta > 0$, *the following inequality is valid:*

$$\begin{aligned}&\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\ &\leq \ \mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\mathcal{I}^{\delta}_{0,\varkappa}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].\end{aligned} \tag{24}$$

Proof. From (23), we have

$$\zeta^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\mathrm{f}_r^{\varrho}(\mu) + \mu^{\varrho}\mathrm{f}_r^{\varrho}(\zeta)\mathrm{f}_r^{\varpi-\lambda_r}(\mu) \leq \zeta^{\varrho}\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\mu) + \mu^{\varrho}\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta). \tag{25}$$

Multiplying both sides of (25) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)$, then integrating the resulting inequality with respect to μ over $(0,\varkappa)$, we obtain

$$\begin{aligned}
&\zeta^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}[w(\mu)\mathrm{f}_r^{\varrho}(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)]d\mu\\
&+\mathrm{f}_r^{\varrho}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}[w(\mu)\mu^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\mu)]d\mu\\
&\le\zeta^{\varrho}\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}[w(\mu)\mathrm{f}_r^{\varpi+\varrho}(\mu)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\mu)]d\mu\\
&+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta)\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}[w(\mu)\mu^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)]d\mu.
\end{aligned}\tag{26}$$

Consequently,

$$\begin{aligned}
&\zeta^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\beta_i}(\varkappa)]+\mathrm{f}_r^{\varrho}(\zeta)\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&\le\sigma^{\varrho}\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta)\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned}\tag{27}$$

Again, multiplying the inequality (27) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta),\zeta\in(0,\varkappa),\varkappa>0$, this function remaining positive under the conditions stated with the theorem, then integrating the obtained result with respect to ζ from 0 to $\varkappa$, we have

$$\begin{aligned}
&\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\zeta^{\varrho}\mathrm{f}_r^{\varpi}(\zeta)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta\\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}_r^{\varrho}(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta\\
&\le\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\zeta^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta\\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\delta}\int_0^{\varkappa}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}_r^{\varpi+\varrho}(\zeta)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta,
\end{aligned}\tag{28}$$

therefore,

$$\begin{aligned}
&\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[\varkappa^{\varrho}\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&\le\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned}\tag{29}$$

This completes the inequality (24). □

The flexibility of the condition (23) can be commented as the condition (3), but for $\mathrm{f}_i, i=1,\dots,n$ instead of f.

Theorem 6. *Let* $\mathrm{f}_i, i=1,\dots,n$ *be* n *positive continuous functions on* $[0,\infty)$ *and* $\varrho>0$, $\varpi\ge\lambda_r>0, r=1,\dots,n$, *which satisfy (23). In addition, let* w *be a positive continuous function on* $[0,\infty)$. *Then, for all* $\varkappa,\delta,\beta>0$, *the following inequality holds:*

$$\begin{aligned}
&\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\beta_i}(\varkappa)]\\
&\le\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\\
&+\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned}\tag{30}$$

Proof. Multiplying the inequality (27) by $\frac{1}{\beta}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta), \zeta \in (0,\varkappa), \varkappa > 0$ (under the circumstances outlined in the theorem, this function remains positive), then integrating the result with respect to ζ from 0 to $\varkappa$, we get

$$\begin{aligned}
&\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\zeta^{\varrho}\mathrm{f}_r^{\varpi}(\zeta)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta \\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}_r^{\varrho}(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta \\
&\leq \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\zeta^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta \\
&+\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\varkappa^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\frac{1}{\beta}\int_0^{\varkappa}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\mathrm{f}_r^{\varpi+\varrho}(\zeta)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)d\zeta,
\end{aligned} \tag{31}$$

which reflects the inequality (30). □

Theorem 7. *Let* $\mathrm{f}_i, i = 1,\ldots,n$ *and* g *be positive continuous functions on* $[0,\infty)$*, and* $\varrho > 0$*,* $\varpi \geq \lambda_r > 0, r = 1,\ldots,n$*, such that, for any* $\mu,\zeta > 0$*,*

$$(\mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varrho}(\mu) - \mathrm{g}^{\varrho}(\mu)\mathrm{f}_r^{\varrho}(\zeta))(\mathrm{f}_r^{\varpi-\lambda_r}(\mu) - \mathrm{f}_r^{\varpi-\lambda_r}(\zeta)) \geq 0, \tag{32}$$

In addition, let w *be a positive continuous function on* $[0,\infty)$*. Then, for all* $\varkappa,\delta > 0$*, we have*

$$\begin{aligned}
&\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\
&\leq \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned} \tag{33}$$

Proof. From (32), it is clear that

$$\mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\mathrm{f}_r^{\varrho}(\mu) + \mathrm{f}_r^{\varrho}(\zeta)\mathrm{g}^{\varrho}(\mu)\mathrm{f}_r^{\varpi-\lambda_r}(\mu) \leq \mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\mu) + \mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta)\mathrm{g}^{\varrho}(\mu). \tag{34}$$

Multiplying both sides of (34) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)$, then integrating the resulting inequality with respect to μ over $(0,\varkappa)$, we obtain

$$\begin{aligned}
&\mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}[w(\mu)\mathrm{f}_r^{\varrho}(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)] \\
&+\mathrm{f}_r^{\varrho}(\zeta)\ \frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}[w(\mu)\mathrm{g}^{\varrho}(\mu)\mathrm{f}_r^{\varpi}(\mu)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\mu)] \\
&\leq \mathrm{g}^{\varrho}(\zeta)\ \frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}[w(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)] \\
&+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta)[w(\mu)\mathrm{f}_r^{\varpi+\varrho}(\mu)\mathrm{g}^{\varrho}(\mu)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\mu)].
\end{aligned} \tag{35}$$

So, we can write

$$\begin{aligned}
&\mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varpi-\lambda_r}(\zeta)\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] + \mathrm{f}_r^{\varrho}(\zeta)\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\
&\leq \mathrm{g}^{\varrho}(\zeta)\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] + \mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\zeta)\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned} \tag{36}$$

When we multiply both sides of (36) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\zeta)}w(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)$, and then integrate the resulting inequality with respect to ζ over $(0,\varkappa)$, we get the following result:

$$\begin{aligned}
&2\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\
&\leq 2\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\,\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)].
\end{aligned} \tag{37}$$

This completes the proof of Theorem 7. □

Theorem 8. *Let* $\mathrm{f}_i, i = 1, \ldots, n$ *and* g *be positive continuous functions on* $[0, \infty)$ *and* $\varrho > 0$, $\varpi \geq \lambda_r > 0, r = 1, \ldots, n$, *such that, for any* $\mu, \zeta > 0$,

$$(\mathrm{g}^{\varrho}(\zeta)\mathrm{f}_r^{\varrho}(\mu) - \mathrm{g}^{\varrho}(\mu)\mathrm{f}_r^{\varrho}(\zeta))(\mathrm{f}_r^{\varpi-\lambda_r}(\mu) - \mathrm{f}_r^{\varpi-\lambda_r}(\zeta)) \geq 0. \tag{38}$$

In addition, let w *be a positive continuous function on* $[0, \infty)$. *Then, for all* $\varkappa, \delta, \beta > 0$, *the following inequality is valid:*

$$\begin{aligned} &\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]+ \\ &\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\ &\leq\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\ &+\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]. \end{aligned} \tag{39}$$

Proof. Multiplying both sides of (34) by $\frac{1}{\beta}e^{-\left(\frac{1-\beta}{\beta}\right)(\varkappa-\zeta)}w(\zeta)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\zeta)$, then integrating the resulting inequality with respect to ζ over $(0, \varkappa)$, we obtain

$$\begin{aligned} &\mathrm{f}_r^{\varrho}(\mu)\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ +\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi-\lambda_r}(\mu)\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\ &\leq\ \mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\mu)\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] + \mathrm{g}^{\varrho}(\mu)\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]. \end{aligned} \tag{40}$$

Multiplying both sides of (40) by $\frac{1}{\delta}e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-\mu)}w(\mu)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\mu)$, then integrating the resulting inequality with respect to μ over $(0, \varkappa)$, we arrive at the following result:

$$\begin{aligned} &\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]+ \\ &\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\mathrm{f}_r^{\varpi}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\ \mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)] \\ &\leq\ \mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]+ \\ &\mathcal{I}_{0,\varkappa}^{\beta}[w(\varkappa)\mathrm{f}_r^{\varpi+\varrho}(\varkappa)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathcal{I}_{0,\varkappa}^{\delta}[w(\varkappa)\mathrm{g}^{\varrho}(\varkappa)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\varkappa)]. \end{aligned} \tag{41}$$

Thus, the proof is completed. □

4. Conclusions

Numerous mathematicians continue to be interested in the use of various fractional operators in the study of integral inequalities. Refs. [6,7,30] investigated weighed fractional integral inequalities employing, respectively, the Hadamard, Marichev–Saigo–Maeda, and generalized Katugampola fractional integral operators. While using the Caputo–Fabrizio fractional integral operators, we looked at several new fractional integral inequalities in this research. Due to its nonsingular kernel, i.e., $e^{-\left(\frac{1-\delta}{\delta}\right)(\varkappa-s)}$ with $0 < \delta < 1$, it reveals to be a more powerful fractional operator. By taking into account the Caputo–Fabrizio fractional integral operator, we investigated the newly weighted fractional integral inequalities in this study. Using the inequalities discussed in this paper, future research should be able to prove the existence and uniqueness of a number of ordinary differential equations as well as initial and boundary value problems using Caputo–Fabrizio fractional operators. It is also possible that the inequalities found in this study provide access to various relationships between fractals and Caputo–Fabrizio fractionals.

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