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Article

Some New Fractal Milne-Type Integral Inequalities via Generalized Convexity with Applications

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Abstract: This study aims to construct some new Milne-type integral inequalities for functions whose modulus of the local fractional derivatives is convex on the fractal set. To that end, we develop a novel generalized integral identity involving first-order generalized derivatives. Finally, as applications, some error estimates for the Milne-type quadrature formula and new inequalities for the generalized arithmetic and p-Logarithmic means are derived. This paper's findings represent a significant improvement over previously published results. The paper's ideas and formidable tools may inspire and motivate further research in this worthy and fascinating field.

Keywords: Milneinequality; generalized convex functions; local fractional integrals; local fractional derivatives; fractal sets

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1. Introduction

Convexity plays a critical and important role in various disciplines, such as economics, finance, optimization, as well as the game theory. For sufficiently nice functions, one can easily determine convexity by looking at its second derivative. Due to its varies type of applications, this notion has been developed and generalized in numerous ways. The concept is intimately connected to the evolution of the theory of inequalities, which has several applications in differential and difference equations, as well as numerical analysis. See [1–5] and its cited sources for a list of articles on quadrature.

Fractional calculus(FC) is a significant part of mathematical analysis that arises from the classical definitions of integral and derivative operators of noninteger order. It is also an effective tool for describing the memory and inherited features of a wide variety of materials and processes. As a crucial instrument for scientists, fractional calculus is gaining prominence at present. It has been utilized successfully in a variety of scientific and engineering sectors, see [6,7].

Fractals have been observed in various scientific disciplines for approximately a century. However, they have only recently become a subject of mathematical study. Benoit Mandelbrot was the founder of the theory of fractals. Since then, lots of scholarly articles, surveys, popular papers, and books on fractals have been published. Mandelbrot, in [8], defined a fractal set which Hausdorff dimension stricly exceeds the topological dimension. In addition, Yang [9] established the numerical γ-sets, where γ is the fractal's dimension. For additional information on fractal sets, see [9–11] and their citations.

The notion of fractal calculus (often known as local fractional calculus) has received a great deal of interest from researchers recently. This concept has advanced rapidly due to its diverse and extensive applications, not only in mathematics but also in other scientific disciplines. In [12], the fractional-wave equation on Cantor sets was studied. The Cantor sets' heat conduction equation was presented in [13]. Ref. [14] provides the perturbation solution for the oscillator of free damped vibrations. Local fractional PDEs of elliptic, hyperbolic, and parabolic types were considered in [15].

In recent years, the relationship between fractal sets, integral inequalities, and convexity has attracted significant attention from researchers. Notably, a number of articles covering this context have been published. To see papers dealing inequalities in the fractal set, we refer readers to [16–28]. This work is focused on three-point Newton–Cotes formulas, examples of which are provided below.

In [29], the authors established the Simpson inequality via generalized quasi convexity as follows:

Theorem 1. *Consider the subset I of $\mathbb{R}$, $\Phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ (I° represents the interior of I) where $\Phi \in D_\gamma(I^\circ)$ and $\Phi^{(\gamma)} \in C_\gamma[\kappa,\tau]$ for $\kappa,\tau \in I^\circ$ with $\kappa < \tau$. If $\left|\Phi^{(\gamma)}\right|$ is the generalized quasi convex function; then, the inequality holds:*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\gamma\left(\Phi(\kappa)+4^\gamma\Phi\left(\frac{\kappa+\tau}{2}\right)+\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\,{}_\kappa I_\tau^{(\gamma)}\Phi(\chi)\right|\\ \le&(\tau-\kappa)^\gamma\left(\frac{5}{18}\right)^\gamma\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}\sup\left\{\left|\Phi^{(\gamma)}(\kappa)\right|,\left|\Phi^{(\gamma)}(\tau)\right|\right\}.\end{aligned}$$

Moreover, Sarikaya et al. [28] presented the following Simpson-type inequality via generalized convexity.

Theorem 2. *Consider the subset $I \subseteq$ of $\mathbb{R}$, $\Phi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ (I° represents the interior of I) where $\Phi \in D_\gamma(I^\circ)$ and $\Phi^{(\gamma)} \in C_\gamma[\kappa,\tau]$ for $\kappa,\tau \in I^\circ$ with $\kappa < \tau$. If $\left|\Phi^{(\gamma)}\right|$ is the generalized convex function; then, the inequality holds:*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\gamma\left(\Phi(\kappa)+4^\gamma\Phi\left(\frac{\kappa+\tau}{2}\right)+\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\,{}_\kappa I_\tau^{(\gamma)}\Phi(\chi)\right|\\ \le&\frac{(\tau-\kappa)^\gamma}{12^\gamma}\left(\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}+\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left(\left|\Phi^{(\gamma)}(\kappa)\right|+\left|\Phi^{(\gamma)}(\tau)\right|\right).\end{aligned}$$

This result has been extended for generalized (s,m)-convex functions by Abdeljawad et al. [16].

Theorem 3. *Cosidering $\Phi : I^\circ \to \mathbb{R}^\gamma$ is a differentiable on I° where $\Phi^{(\gamma)} \in C_\gamma[\kappa,m\tau]$ for $\kappa,\tau \in I^\circ$ with $\kappa < \tau$ and $s,m \in (0,1]$. If $\left|\Phi^{(\gamma)}\right|$ is generalized (s,m)-convex on I; then, we have*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\gamma\left(\Phi(\kappa)+4^\gamma\Phi\left(\frac{\kappa+m\tau}{2}\right)+\Phi(m\tau)\right)-\frac{\Gamma(\gamma+1)}{(m\tau-\kappa)^\gamma}\,{}_\kappa I_{m\tau}^{(\gamma)}\Phi(\chi)\right|\\ \le&(m\tau-\kappa)^\gamma\left(\frac{2^\gamma\left(5^{(s+2)\gamma}-3^{(s+1)\gamma}\right)-5^\gamma\left(6^{(s+1)\gamma}+3^{(s+1)\gamma}\right)}{6^{(s+2)\gamma}}\right)\\ &\times\left(\frac{\Gamma(1+s\gamma)}{\Gamma(1+(s+1)\gamma)}+\frac{\Gamma(1+(s+1)\gamma)}{\Gamma(1+(s+2)\gamma)}\right)\left(\left|\Phi^{(\gamma)}(\kappa)\right|+m\left|\Phi^{(\gamma)}(\tau)\right|\right).\end{aligned}$$

Furthermore, Meftah et al. [25] provided the following Maclaurin-type inequality for generalized convex functions.

Theorem 4. *Assume that $\Phi : [\kappa, \tau] \to \mathbb{R}^{\gamma}$ is a differentiable on (κ, τ) such that $\Phi \in D_{\gamma}[\kappa, \tau]$ and $\Phi^{(\gamma)} \in C_{\gamma}[\kappa, \tau]$ with $0 \leq \kappa < \tau$. If $\left|\Phi^{(\gamma)}\right|$ is a generalized convex on $[\kappa, \tau]$; then, we have*

$$\begin{aligned}
&\left|\frac{1}{8^{\gamma}}\left(3^{\gamma}\Phi\left(\frac{5\kappa+\tau}{6}\right)+2^{\gamma}\Phi\left(\frac{\kappa+\tau}{2}\right)+3^{\gamma}\Phi\left(\frac{\kappa+5\tau}{6}\right)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^{\gamma}}\,{}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{36^{\gamma}}\left(\left(\frac{\Gamma(\gamma+1)}{\Gamma(1+2\gamma)}-\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}(\kappa)\right|\right. \\
&+\left(\left(\frac{171}{64}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(2\gamma+1)}-\left(\frac{67}{64}\right)^{\gamma}\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{5\kappa+\tau}{6}\right)\right| \\
&+\left(\left(\frac{131}{32}\right)^{\gamma}\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}-\left(\frac{35}{32}\right)^{\gamma}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right| \\
&+\left(\left(\frac{171}{64}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(2\gamma+1)}-\left(\frac{67}{64}\right)^{\gamma}\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+5\tau}{6}\right)\right| \\
&\left.+\left(\frac{\Gamma(\gamma+1)}{\Gamma(1+2\gamma)}-\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}(\tau)\right|\right).
\end{aligned}$$

More recently, the corrected Dual–Simpson–Type inequalities for generalized functions was establish by Lakhdari et al. in [23].

Theorem 5. *Assume that $\Phi : [\kappa, \tau] \to \mathbb{R}^{\gamma}$ is a differentiable on (κ, τ) such that $\Phi \in D_{\gamma}[\kappa, \tau]$ and $\Phi^{(\gamma)} \in C_{\gamma}[\kappa, \tau]$ with $0 \leq \kappa < \tau$. If $\left|\Phi^{(\gamma)}\right|$ is generalized convex on $[\kappa, \tau]$, then we have*

$$\begin{aligned}
&\left|\frac{1}{(15)^{\gamma}}\left(8^{\gamma}\Phi\left(\frac{3\kappa+\tau}{4}\right)-\Phi\left(\frac{\kappa+\tau}{2}\right)+8^{\gamma}\Phi\left(\frac{\kappa+3\tau}{4}\right)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^{\gamma}}\,{}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{(16)^{\gamma}}\left(\left(\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}-\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left(\left|\Phi^{(\gamma)}(\kappa)\right|+\left|\Phi^{(\gamma)}(\tau)\right|\right)\right. \\
&+\left(\left(\frac{2}{15}\right)^{\gamma}\frac{\Gamma(\gamma+1)}{\Gamma(2\gamma+1)}+2^{\gamma}\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left(\left|\Phi^{(\gamma)}\left(\frac{3\kappa+\tau}{4}\right)\right|+\left|\Phi^{(\gamma)}\left(\frac{\kappa+3\tau}{4}\right)\right|\right) \\
&\left.+\left(\left(\frac{34}{15}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(2\gamma+1)}-2^{\gamma}\frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|\right).
\end{aligned}$$

Inspired by the preceding investigation, the present research paper purports to examine the Milne-type quadrature formula, which represents another type of three-point Newton–Cotes formulas involving a corrector term.

The Milne-type rule defined on the real line numbers can be stated as follows:

$$\int_{\kappa}^{\tau}\Phi(x)dx=\mathcal{M}(\Phi)+\mathcal{R}(\Phi),$$

where

$$\mathcal{M}(\Phi)=\frac{(\tau-\kappa)}{3}\left(2\Phi(\kappa)-\Phi\left(\frac{\kappa+\tau}{2}\right)+2\Phi(\tau)\right)$$

and $\mathcal{R}(\Phi)$ denotes the associated error.

By using new generalized integral identities, one can establish some new estimates of the Milne quadrature property on fractal set for functions having noninteger derivatives of order γ that are generalized convex. later some examples as applications of our findings will be discussed.

The remainder of the paper is structured as follows: Recalling some definitions and concepts related to fractal calculus in Section 2. Section 3 begins by presenting a new identity for the Milne quadrature formula involving the first-order noninteger derivatives, followed by some related inequalities via generalized convexity. Finally, we provide some applications to support our findings, followed by a brief conclusion.

2. Preliminaries

In this part of work, we review and recall some concepts from fractal theory; for details, see [9], which will be applied in the development of the study. The following γ-type sets are defined for $0 < \gamma \leq 1$:

The γ-type set of integer is defined as

$$\mathbb{Z}^{\gamma} := \{0^{\gamma}, \pm 1^{\gamma}, \pm 2^{\gamma}, \cdots, \pm n^{\gamma}, \cdots, \}.$$

The definition of the γ-type set of rational is

$$\mathbb{Q}^{\gamma} := \left\{ c^{\gamma} = \left(\frac{\tau}{\kappa}\right)^{\gamma} : \tau, \kappa \in \mathbb{Z} \text{ and } \kappa \neq 0 \right\}.$$

The γ-type irrational set is defined as

$$\mathbb{J}^{\gamma} := \{c^{\gamma} : c \in \mathbb{J}\} := \left\{ c^{\gamma} \neq \left(\frac{\tau}{\kappa}\right)^{\gamma} : \tau, \kappa \in \mathbb{Z} \text{ and } \kappa \neq 0 \right\}.$$

The definition of the γ-type set of the real numbers is given by

$$\mathbb{R}^{\gamma} := \mathbb{Q}^{\gamma} \cup \mathbb{J}^{\gamma}.$$

Note that according to Yang's definition of γ-type sets, the theory of operator algebras for the real line number on fractal sets is the theory of the real line number operations, and for all c^{γ}, d^{γ}, and e^{γ} in $\mathbb{R}^{\gamma}$, we have

1. $c^{\gamma} + d^{\gamma}$ and $c^{\gamma} d^{\gamma}$ are in the set $\mathbb{R}^{\gamma}$.
2. $c^{\gamma} + d^{\gamma} = d^{\gamma} + c^{\gamma} = (c+d)^{\gamma} = (d+c)^{\gamma}$.
3. $c^{\gamma} + (d^{\gamma} + e^{\gamma}) = (c+d)^{\gamma} + e^{\gamma}$.
4. $c^{\gamma} d^{\gamma} = d^{\gamma} c^{\gamma} = (cd)^{\gamma} = (dc)^{\gamma}$.
5. $c^{\gamma}(d^{\gamma} e^{\gamma}) = (c^{\gamma} d^{\gamma}) e^{\gamma}$.
6. $c^{\gamma}(d^{\gamma} + e^{\gamma}) = c^{\gamma} d^{\gamma} + c^{\gamma} e^{\gamma}$.
7. $c^{\gamma} + 0^{\gamma} = 0^{\gamma} + c^{\gamma} = c^{\gamma}$ and $c^{\gamma} 1^{\gamma} = 1^{\gamma} c^{\gamma} = c^{\gamma}$.

In [9,30], Gao-Yang-Kang introduced the local noninteger derivative and integrals as follows.

Definition 1 ([9])**.** *A non-differentiable function $\Phi : \mathbb{R} \to \mathbb{R}^{\gamma}$ is called a local fractional continuous at χ_0, if*

$$\forall \epsilon > 0, \ \ \exists \delta > 0 : |\Phi(\chi) - \Phi(\chi_0)| < \epsilon^{\gamma}$$

holds for $|\chi - \chi_0| < \delta$, where $\epsilon, \delta \in \mathbb{R}$ and we denote $C_{\gamma}(\kappa, \tau)$ as space of all locally fractional continuous functions on (κ, τ).

Definition 2 ([9])**.** *The local fractional derivative of $\Phi(\chi)$ of order γ at the point $\chi = \chi_0$ is defined as*

$$\Phi^{(\gamma)}(\chi_0) = \left.\frac{d^{\gamma}\Phi(\chi)}{d\chi^{\gamma}}\right|_{x=\chi_0} = \lim_{\chi \to \chi_0} \frac{\Delta^{\gamma}(\Phi(\chi) - \Phi(\chi_0))}{(\chi - \chi_0)^{\gamma}},$$

where

$$\Delta^{\gamma}(\Phi(\chi) - \Phi(\chi_0)) \cong \Gamma(\gamma+1)(\Phi(\chi) - \Phi(\chi_0)).$$

If

$$\Phi^{(k+1)\gamma}(\chi) = \overbrace{D^{\gamma} D^{\gamma} \cdots D^{\gamma}}^{(k+1) \text{ times}} \Phi(\chi)$$

exists $\forall\chi \in I \subseteq \mathbb{R}$; then, $\Phi \in D_{(k+1)\gamma}(I)$ for $k = 0, 1, 2, 3, \cdots$

Definition 3 ([9]). *Consider $\Phi(\chi) \in C_\gamma[\kappa, \tau]$. The fractional integral is defined as*

$$ {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi) = \frac{1}{\Gamma(\gamma+1)}\int_{\kappa}^{\tau}\Phi(\chi)(d\chi)^{\gamma} = \frac{1}{\Gamma(\gamma+1)}\lim_{\Delta\chi\to 0}\sum_{j=0}^{N-1}\Phi(\chi_j)(\Delta\chi_j)^{\gamma}, $$

where $\Delta\chi_j = \chi_{j+1} - \chi_j$ and $\Delta x = \max\{\Delta\chi_j\}$, for $j = 0, 1, \cdots, N-1$, and $\kappa = \chi_0 < \chi_1 < \cdots < \chi_N = \tau$ is partition of interval $[\kappa, \tau]$.

In this case, it is evident that

$$ {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi) = 0 \text{ if } \kappa = \tau \text{ and } {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi) = -{}_{\tau}I_{\kappa}^{\gamma}\Phi(\chi) \text{ if } \kappa < \tau. $$

We designated by $\Phi(\chi) \in I_x^{\gamma}[\kappa, \tau]$ if ${}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)$ exists, $\forall\chi \in [\kappa, \tau]$.

Lemma 1 ([9]). *Let $\Phi(\chi) = \Psi^{(\gamma)}(\chi) \in C_\gamma[\kappa, \tau]$; hence,*

$$ {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi) = \Psi(\tau) - \Psi(\kappa). $$

Integration by parts for local fractional *Considering $\Phi, \Psi \in D_\gamma[\kappa, \tau]$ and $\Phi^{(\gamma)}(\chi), \Psi^{(\gamma)}(\chi) \in C_\gamma[\kappa, \tau]$, we obtain*

$$ {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\Psi^{(\gamma)}(\chi) = \Phi(\chi)\Psi(\chi)|_{\kappa}^{\tau} - {}_{\kappa}I_{\tau}^{\gamma}\Phi^{(\gamma)}(\chi)\Psi(\chi). $$

Lemma 2 ([9]). *For $\Phi(\chi) = \chi^{k\gamma}$, we have*

$$ \frac{d^{\gamma}\Phi(\chi)}{d\chi^{\gamma}} = \frac{\Gamma(1+k\gamma)}{\Gamma(1+(k-1)\gamma)}\chi^{(k-1)\gamma}, \quad \forall k \in \mathbb{R}, $$
$$ \frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}\Phi(\chi)(d\chi)^{\gamma} = \frac{\Gamma(1+k\gamma)}{\Gamma(1+(k+1)\gamma)}\left(\tau^{(k+1)\gamma} - \kappa^{(k+1)\gamma}\right). $$

Lemma 3 ([9]). *Assuming $\Phi, \Psi \in C_\gamma[\kappa, \tau]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$; then,*

$$ \frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Phi(\chi)\Psi(\chi)|(d\chi)^{\gamma} $$
$$ \leq \left(\frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Phi(\chi)|^{p}(d\chi)^{\gamma}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Psi(\chi)|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}}. $$

Lemma 4 (Generalized power mean inequality [9]). *Let $\Phi, \Psi \in C_\gamma[\kappa, \tau]$, $q > 1$; then,*

$$ \frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Phi(\chi)\Psi(\chi)|(d\chi)^{\gamma} $$
$$ \leq \left(\frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Phi(\chi)|(d\chi)^{\gamma}\right)^{1-\frac{1}{q}}\left(\frac{1}{\Gamma(1+\gamma)}\int_{\kappa}^{\tau}|\Phi(\chi)||\Psi(\chi)|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}}. $$

Definition 4 (Generalized convex function [9]). *A function $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}^{\gamma}$ is said to be a generalized convex function on I if*

$$ \Phi(\lambda\chi_1 + (1-\lambda)\chi_2) \leq \lambda^{\gamma}\Phi(\chi_1) + (1-\lambda)^{\gamma}\Phi(\chi_2) $$

holds, $\forall \chi_1, \chi_2 \in I$ *and* $\lambda \in [0,1]$.

Two straightforward examples of generalized convex functions are provided below:

1. $\Phi(\chi) = \chi^{\gamma p}$, with $p > 1$ and $\chi \geq 0$.
2. $\Phi(\chi) = E_\gamma(\chi^\gamma)$, $\chi \in \mathbb{R}$ where $E_\gamma(\chi^\gamma) = \sum_{q=0}^{\infty} \frac{\chi^{\gamma q}}{\Gamma(1+q\gamma)}$ denotes the Mittag–Leffler function.

3. Main Results

To demonstrate our results, we require the following identity:

Lemma 5. *Let* $\Phi : I \to \mathbb{R}^\gamma$ *be a differentiable function on* I°, $\kappa, \tau \in I^\circ$ *with* $\kappa < \tau$, *and* $\Phi^{(\gamma)} \in C_\gamma[\kappa, \tau]$; *hence we get*

$$\begin{aligned}&\frac{1}{3^\gamma}\left(2^\gamma\Phi(\kappa) - \Phi\left(\frac{\kappa+\tau}{2}\right) + 2^\gamma\Phi(\tau)\right) - \frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\, {}_\kappa I_\tau^\gamma \Phi(\chi)\\ =&\frac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(u-\frac{4}{3}\right)^\gamma \Phi^{(\gamma)}\left((1-u)\kappa + u\frac{\kappa+\tau}{2}\right)(du)^\gamma\right.\\ &\left.+\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(u+\frac{1}{3}\right)^\gamma \Phi^{(\gamma)}\left((1-u)\frac{\kappa+\tau}{2} + u\tau\right)(du)^\gamma\right).\end{aligned}$$

Proof. Let

$$I_1 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(u-\frac{4}{3}\right)^\gamma \Phi^{(\gamma)}\left((1-u)\kappa + u\frac{\kappa+\tau}{2}\right)(du)^\gamma$$

and

$$I_2 = \frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(u+\frac{1}{3}\right)^\gamma \Phi^{(\gamma)}\left((1-u)\frac{\kappa+\tau}{2} + u\tau\right)(du)^\gamma.$$

Using LFI by parts for I_1, we obtain

$$\begin{aligned} I_1 &= \frac{2^\gamma}{(\tau-\kappa)^\gamma}\left(u-\frac{4}{3}\right)^\gamma \Phi\left((1-u)\kappa + u\tfrac{\kappa+\tau}{2}\right)\Big|_{u=0}^{u=1}\\ &\quad - \frac{2^\gamma}{(\tau-\kappa)^\gamma}\int_0^1 \Phi\left((1-u)\kappa + u\tfrac{\kappa+\tau}{2}\right)(du)^\gamma\\ &= \frac{(-2)^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi\left(\tfrac{\kappa+\tau}{2}\right) - \frac{(-8)^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\kappa) - \frac{2^\gamma}{(\tau-\kappa)^\gamma}\int_0^1 \Phi\left((1-u)\kappa + u\tfrac{\kappa+\tau}{2}\right)(du)^\gamma\\ &= \frac{8^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\kappa) - \frac{2^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi\left(\tfrac{\kappa+\tau}{2}\right) - \frac{4^\gamma}{(\tau-\kappa)^{2\gamma}}\int_\kappa^{\frac{\kappa+\tau}{2}} \Phi(\chi)(d\chi)^\gamma. \end{aligned} \tag{1}$$

Similarly, we obtain for I_2

$$\begin{aligned} I_2 &= \frac{2^\gamma}{(\tau-\kappa)^\gamma}\left(u+\frac{1}{3}\right)^\gamma \Phi\left((1-u)\tfrac{\kappa+\tau}{2} + u\tau\right)\Big|_{u=0}^{u=1}\\ &\quad - \frac{2^\gamma}{(\tau-\kappa)^\gamma}\int_0^1 \Phi\left((1-u)\tfrac{\kappa+\tau}{2} + u\tau\right)(du)^\gamma\\ &= \frac{8^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\tau) - \frac{2^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi\left(\tfrac{\kappa+\tau}{2}\right) - \frac{2^\gamma}{(\tau-\kappa)^\gamma}\int_0^1 \Phi\left((1-u)\tfrac{\kappa+\tau}{2} + u\tau\right)(du)^\gamma\\ &= \frac{8^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\tau) - \frac{2^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi\left(\tfrac{\kappa+\tau}{2}\right) - \frac{4^\gamma}{(\tau-\kappa)^{2\gamma}}\int_{\frac{\kappa+\tau}{2}}^{\tau} \Phi(\chi)(d\chi)^\gamma. \end{aligned} \tag{2}$$

Summing (1) and (2), we obtain

$$\begin{aligned}
I_1+I_2 =& \frac{8^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\kappa) - \frac{4^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi\left(\frac{\kappa+\tau}{2}\right) + \frac{8^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\Phi(\tau) \\
&- \frac{4^\gamma}{(\tau-\kappa)^{2\gamma}}\left(\int_\kappa^{\frac{\kappa+\tau}{2}}\Phi(\chi)(d\chi)^\gamma + \int_{\frac{\kappa+\tau}{2}}^{\tau}\Phi(\chi)(d\chi)^\gamma\right) \\
=& \frac{4^\gamma}{3^\gamma(\tau-\kappa)^\gamma}\left(2^\gamma\Phi(\kappa) - \Phi\left(\frac{\kappa+\tau}{2}\right) + 2^\gamma\Phi(\tau)\right) - \frac{4^\gamma}{(\tau-\kappa)^{2\gamma}}\int_\kappa^\tau \Phi(\chi)(d\chi)^\gamma.
\end{aligned}$$

Multiplying the above equality by $\frac{(\tau-\kappa)^\gamma}{4^\gamma}$, we achieve the intended outcome. □

Theorem 6. *Considering $\Phi : I \to \mathbb{R}^\gamma$ is a differentiable function on I°, $\kappa, \tau \in I^\circ$ with $\kappa < \tau$, where $\Phi \in D_\gamma[\kappa,\tau]$ and $\Phi^{(\gamma)} \in C_\gamma[\kappa,\tau]$. If $\left|\Phi^{(\gamma)}\right|$ is generalized convex on $[\kappa,\tau]$, then we have*

$$\begin{aligned}
&\left|\frac{1}{3^\gamma}\left(2^\gamma\Phi(\kappa) - \Phi\left(\frac{\kappa+\tau}{2}\right) + 2^\gamma\Phi(\tau)\right) - \frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\, {}_\kappa I_\tau^\gamma \Phi(\chi)\right| \\
\leq& \frac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\left(\left(\frac{1}{3}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(2\gamma+1)} + \frac{\Gamma(2\gamma+1)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}(\kappa)\right|\right. \\
&+ 2^\gamma\left(\left(\frac{4}{3}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} - \frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right| \\
&+ \left.\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)} + \left(\frac{1}{3}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left|\Phi^{(\gamma)}(\tau)\right|\right).
\end{aligned}$$

Proof. Based on Lemma 5, the generalized convexity of $\left|\Phi^{(\gamma)}\right|$ and the modulus properties, we have

$$\begin{aligned}
&\left|\tfrac{1}{3^\gamma}\left(2^\gamma\Phi(\kappa) - \Phi\left(\tfrac{\kappa+\tau}{2}\right) + 2^\gamma\Phi(\tau)\right) - \tfrac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\, {}_\kappa I_\tau^\gamma \Phi(\chi)\right| \\
\leq\ & \tfrac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\tfrac{4}{3}-\chi\right)^\gamma \left|\Phi^{(\gamma)}\left((1-\chi)\kappa + \chi\tfrac{\kappa+\tau}{2}\right)\right|(d\chi)^\gamma\right. \\
&+ \left.\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\tfrac{1}{3}\right)^\gamma \left|\Phi^{(\gamma)}\left((1-\chi)\tfrac{\kappa+\tau}{2} + \chi\tau\right)\right|(d\chi)^\gamma\right) \\
\leq\ & \tfrac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\tfrac{4}{3}-\chi\right)^\gamma \left((1-\chi)^\gamma\left|\Phi^{(\gamma)}(\kappa)\right| + \chi^\gamma\left|\Phi^{(\gamma)}\left(\tfrac{\kappa+\tau}{2}\right)\right|\right)(d\chi)^\gamma\right. \\
&+ \left.\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\tfrac{1}{3}\right)^\gamma \left((1-\chi)^\gamma\left|\Phi^{(\gamma)}\left(\tfrac{\kappa+\tau}{2}\right)\right| + \chi^\gamma\left|\Phi^{(\gamma)}(\tau)\right|\right)(d\chi)^\gamma\right) \\
=\ & \tfrac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\left|\Phi^{(\gamma)}(\kappa)\right|\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\tfrac{4}{3}-\chi\right)^\gamma (1-\chi)^\gamma(d\chi)^\gamma + \left|\Phi^{(\gamma)}\left(\tfrac{\kappa+\tau}{2}\right)\right|\right. \\
&\times\left(\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\tfrac{4}{3}-\chi\right)^\gamma \chi^\gamma(d\chi)^\gamma + \tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\tfrac{1}{3}\right)^\gamma (1-\chi)^\gamma(d\chi)^\gamma\right) \\
&+ \left.\left|\Phi^{(\gamma)}(\tau)\right|\tfrac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\tfrac{1}{3}\right)^\gamma \chi^\gamma(d\chi)^\gamma\right).
\end{aligned} \tag{3}$$

We obtain the following by applying Lemma 2 to the above integrals:

$$\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\frac{4}{3}-\chi\right)^{\gamma}(1-\chi)^{\gamma}(d\chi)^{\gamma} = \left(\frac{1}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} + \frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}, \tag{4}$$

$$\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\frac{4}{3}-\chi\right)^{\gamma}\chi^{\gamma}(d\chi)^{\gamma} = \left(\frac{4}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} - \frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}, \tag{5}$$

$$\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\frac{1}{3}\right)^{\gamma}(1-\chi)^{\gamma}(d\chi)^{\gamma} = \left(\frac{4}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} - \frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)} \tag{6}$$

and

$$\frac{1}{\Gamma(\gamma+1)}\int_0^1 \left(\chi+\frac{1}{3}\right)^{\gamma}\chi^{\gamma}(d\chi)^{\gamma} = \frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)} + \left(\frac{1}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}. \tag{7}$$

Combining (3)–(7), we obtain the desired outcome. □

Theorem 7. *Let $\Phi : I \to \mathbb{R}^{\gamma}$ be a differentiable function on I°, $\kappa, \tau \in I^{\circ}$ with $\kappa < \tau$, such that $\Phi \in D_{\gamma}[\kappa,\tau]$ and $\Phi^{(\gamma)} \in C_{\gamma}[\kappa,\tau]$. If $\left|\Phi^{(\gamma)}\right|^{q}$ is generalized convex on $[\kappa,\tau]$, where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}
&\left|\frac{1}{3^{\gamma}}\left(2^{\gamma}\Phi(\kappa) - \Phi\left(\frac{\kappa+\tau}{2}\right) + 2^{\gamma}\Phi(\tau)\right) - \frac{\Gamma(\gamma+1)}{(\tau-\kappa)^{\gamma}}\, {}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\right| \\
\leq &\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\left(\frac{4}{3}\right)^{(p+1)\gamma} - \left(\frac{1}{3}\right)^{(p+1)\gamma}\right)\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{\frac{1}{q}} \\
&\times\left(\left(\left|\Phi^{(\gamma)}(\kappa)\right|^{q} + \left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\right)^{\frac{1}{q}} + \left(\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q} + \left|\Phi^{(\gamma)}(\tau)\right|^{q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Proof. From Lemma 5, properties of modulus, the generalized power mean inequality, and the generalized convexity of $\left|\Phi^{(\gamma)}\right|^{q}$, we have

$$
\begin{aligned}
&\left|\frac{1}{3^{\gamma}}\left(2^{\gamma}\Phi(\kappa)-\Phi\left(\frac{\kappa+\tau}{2}\right)+2^{\gamma}\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^{\gamma}}\,{}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{p\gamma}(d\chi)^{\gamma}\right)^{\frac{1}{p}}\right. \\
&\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left|\Phi^{(\gamma)}\left((1-\chi)\kappa+\chi\frac{\kappa+\tau}{2}\right)\right|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{p\gamma}(d\chi)^{\gamma}\right)^{\frac{1}{p}} \\
&\left.\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left|\Phi^{(\gamma)}\left((1-\chi)\frac{\kappa+\tau}{2}+\chi\tau\right)\right|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}}\right) \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{p\gamma}(du)^{\gamma}\right)^{\frac{1}{p}}\right. \\
&\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left((1-\chi)^{\gamma}\left|\Phi^{(\gamma)}(\kappa)\right|^{q}+\chi^{\gamma}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\right)(d\chi)^{\gamma}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{p\gamma}(du)^{\gamma}\right)^{\frac{1}{p}} \\
&\left.\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left((1-\chi)^{\gamma}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}+\chi^{\gamma}\left|\Phi^{(\gamma)}(\tau)\right|^{q}\right)(d\chi)^{\gamma}\right)^{\frac{1}{q}}\right). \qquad (8)
\end{aligned}
$$

Using Lemma 2, (8) provides

$$
\begin{aligned}
&\left|\frac{1}{3^\gamma}\left(2^\gamma\Phi(\kappa)-\Phi\left(\frac{\kappa+\tau}{2}\right)+2^\gamma\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\,{}_\kappa I_\tau^\gamma\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\left(\frac{4}{3}\right)^{(p+1)\gamma}-\left(\frac{1}{3}\right)^{(p+1)\gamma}\right)\right)^{\frac{1}{p}}\right. \\
&\times\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left|\Phi^{(\gamma)}(\kappa)\right|^q+\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\left(\frac{4}{3}\right)^{(p+1)\gamma}-\left(\frac{1}{3}\right)^{(p+1)\gamma}\right)\right)^{\frac{1}{p}} \\
&\left.\times\left(\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q+\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left|\Phi^{(\gamma)}(\tau)\right|^q\right)\right)^{\frac{1}{q}}\right) \\
=&\frac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\left(\frac{4}{3}\right)^{p+1}-\left(\frac{1}{3}\right)^{p+1}\right)^\gamma\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{\frac{1}{q}} \\
&\times\left(\left(\left|\Phi^{(\gamma)}(\kappa)\right|^q+\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q+\left|\Phi^{(\gamma)}(\tau)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

□

Theorem 8. *Assuming $\Phi : I \to \mathbb{R}^\gamma$ is a differentiable function on I°, $\kappa,\tau \in I^\circ$ with $\kappa<\tau$, such that $\Phi \in D_\gamma[\kappa,\tau]$ and $\Phi^{(\gamma)} \in C_\gamma[\kappa,\tau]$. If $\left|\Phi^{(\gamma)}\right|^q$ is generalized convex on $[\kappa,\tau]$, where $q \geq 1$, then we have*

$$
\begin{aligned}
&\left|\frac{1}{3^\gamma}\left(2^\gamma\Phi(\kappa)-\Phi\left(\frac{\kappa+\tau}{2}\right)+2^\gamma\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^\gamma}\,{}_\kappa I_\tau^\gamma\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\left(\frac{4}{3}\right)^{2\gamma}-\left(\frac{1}{3}\right)^{2\gamma}\right)\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\left(\left(\frac{1}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}+\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}(\kappa)\right|^q\right.\right. \\
&\left.+\left(\left(\frac{4}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\left(\left(\frac{4}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^q\right. \\
&\left.\left.+\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}+\left(\frac{1}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left|\Phi^{(\gamma)}(\tau)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Proof. Based on Lemma 5, the generalized convexity of $\left|\Phi^{(\gamma)}\right|^q$, the generalized Hölder's inequality, and the modulus properties, we have

$$
\begin{aligned}
&\left|\frac{1}{3^{\gamma}}\left(2^{\gamma}\Phi(\kappa)-\Phi\left(\frac{\kappa+\tau}{2}\right)+2^{\gamma}\Phi(\tau)\right)-\frac{\Gamma(\gamma+1)}{(\tau-\kappa)^{\gamma}}\,{}_{\kappa}I_{\tau}^{\gamma}\Phi(\chi)\right| \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{\gamma}(d\chi)^{\gamma}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{\gamma}\left|\Phi^{(\gamma)}\left((1-\chi)\kappa+\chi\frac{\kappa+\tau}{2}\right)\right|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{\gamma}(d\chi)^{\gamma}\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{\gamma}\left|\Phi^{(\gamma)}\left((1-\chi)\frac{\kappa+\tau}{2}+\chi\tau\right)\right|^{q}(d\chi)^{\gamma}\right)^{\frac{1}{q}}\right) \\
\leq&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{\gamma}(du)^{\gamma}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{\gamma}\left((1-\chi)^{\gamma}\left|\Phi^{(\gamma)}(\kappa)\right|^{q}+\chi^{\gamma}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\right)(d\chi)^{\gamma}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{\gamma}(du)^{\gamma}\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{\gamma}\left((1-\chi)^{\gamma}\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}+\chi^{\gamma}\left|\Phi^{(\gamma)}(\tau)\right|^{q}\right)(d\chi)^{\gamma}\right)^{\frac{1}{q}}\right) \\
=&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{\gamma}(du)^{\gamma}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\left|\Phi^{(\gamma)}(\kappa)\right|^{q}\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{\gamma}(1-\chi)^{\gamma}(d\chi)^{\gamma}\right)\right. \\
&\left.+\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\frac{4}{3}-\chi\right)^{\gamma}\chi^{\gamma}(d\chi)^{\gamma}\right)\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(\gamma+1)}\int_{\frac{1}{3}}^{\frac{4}{3}}u^{\gamma}(du)^{\gamma}\right)^{1-\frac{1}{q}} \\
&\times\left(\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{\gamma}(1-\chi)^{\gamma}(d\chi)^{\gamma}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\Phi^{(\gamma)}(\tau)\right|^{q}\left(\frac{1}{\Gamma(\gamma+1)}\int_{0}^{1}\left(\chi+\frac{1}{3}\right)^{\gamma}\chi^{\gamma}(d\chi)^{\gamma}\right)\Bigg)^{\frac{1}{q}}\Bigg)\\
=&\frac{(\tau-\kappa)^{\gamma}}{4^{\gamma}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\left(\frac{4}{3}\right)^{2\gamma}-\left(\frac{1}{3}\right)^{2\gamma}\right)\right)^{1-\frac{1}{q}}\\
&\times\Bigg(\Bigg(\left(\left(\frac{1}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}+\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}(\kappa)\right|^{q}\\
&+\left(\left(\frac{4}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\Bigg)^{\frac{1}{q}}\\
&+\Bigg(\left(\left(\frac{4}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\kappa+\tau}{2}\right)\right|^{q}\\
&+\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}+\left(\frac{1}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left|\Phi^{(\gamma)}(\tau)\right|^{q}\Bigg)^{\frac{1}{q}}\Bigg),
\end{aligned}
$$

where we have used Lemma 2 and (4)–(7). The proof is completed. □

4. Applications

Milne's Quadrature Formula

Considering Υ is the partition of the points $\kappa=\chi_0<\chi_1<\ldots<\chi_n=\tau$ of the interval $[\kappa,\tau]$ and let

$$
\frac{1}{\Gamma(\gamma+1)}\int_{\kappa}^{\tau}\Phi(\chi)(d\chi)^{\gamma}=\lambda(\Phi,\Upsilon)+R(\Phi,\Upsilon),
$$

where

$$
\lambda(\Phi,\Upsilon)=\frac{1}{\Gamma(\gamma+1)}\sum_{i=0}^{n-1}\frac{(\chi_{i+1}-\chi_i)^{\gamma}}{3^{\gamma}}\left(2^{\gamma}\Phi(\chi_i)-\Phi\left(\frac{\chi_i+\chi_{i+1}}{2}\right)+2^{\gamma}\Phi(\chi_{i+1})\right)
$$

and $R(\Phi,\Upsilon)$ represents the corresponding approximation error.

Proposition 1. *Assuming $\Phi:[\kappa,\tau]\to\mathbb{R}^{\gamma}$ is a differentiable function on (κ,τ) with $0\le\kappa<\tau$ and $\Phi^{(\gamma)}\in C_{\gamma}[\kappa,\tau]$. If $\left|\Phi^{(\gamma)}\right|$ is genaralized convex function, we optain for $n\in\mathbb{N}$*

$$
\begin{aligned}
&|R(\Phi,\Upsilon)|\\
\le&\frac{1}{\Gamma(1+\gamma)}\sum_{i=0}^{n-1}\frac{(\chi_{i+1}-\chi_i)^{2\gamma}}{4^{\gamma}}\left(\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}+\left(\frac{1}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left(\left|\Phi^{(\gamma)}(\chi_i)\right|\right.\right.\\
&\left.\left.+\left|\Phi^{(\gamma)}(\chi_{i+1})\right|\right)+2^{\gamma}\left(\left(\frac{4}{3}\right)^{\gamma}\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\chi_i+\chi_{i+1}}{2}\right)\right|\right).
\end{aligned}
$$

Proof. Applying Theorem 6 on the subintervals $[\chi_i,\chi_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the partition Υ, we obtain

$$
\begin{aligned}
&\left|\frac{1}{3^\gamma}\left(2^\gamma\Phi(\chi_i)-\Phi\left(\frac{\chi_i+\chi_{i+1}}{2}\right)+2^\gamma\Phi(\chi_{i+1})\right)-\frac{\Gamma(\gamma+1)}{(\chi_{i+1}-\chi_i)^\gamma}{}_{\chi_i}I^{\gamma}_{\chi_{i+1}}\Phi(t)\right| \\
\leq&\frac{(\chi_{i+1}-\chi_i)^\gamma}{4^\gamma}\left(\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}+\left(\frac{1}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left|\Phi^{(\gamma)}(\chi_i)\right|\right. \\
&+2^\gamma\left(\left(\frac{4}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}-\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}\right)\left|\Phi^{(\gamma)}\left(\frac{\chi_i+\chi_{i+1}}{2}\right)\right| \\
&\left.+\left(\frac{\Gamma(1+2\gamma)}{\Gamma(3\gamma+1)}+\left(\frac{1}{3}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)\left|\Phi^{(\gamma)}(\chi_{i+1})\right|\right).
\end{aligned}
$$

The required result is produced by multiplying both sides of the inequality by $\frac{1}{\Gamma(1+\gamma)}(\chi_{i+1}-\chi_i)^\gamma$, summing the resulting inequalities for $i=0,1,\ldots,n-1$, and then applying the triangular inequality. □

Special means

Let $\kappa,\tau\in\mathbb{R}$, then:

The generalized arithmetic mean is given by $A(\kappa,\tau)=\frac{\kappa^\gamma+\tau^\gamma}{2^\gamma}$.

The generalized p-Logarithmic mean is given by:

$$
L_p(\kappa,\tau)=\left[\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\frac{\tau^{(p+1)\gamma}-\kappa^{(p+1)\gamma}}{(\tau-\kappa)^\gamma}\right)\right]^{\frac{1}{p}},\quad p\in\mathbb{Z}\backslash\{0,-1\},\text{ and }\kappa\neq\tau.
$$

Proposition 2. *Let $\kappa,\tau\in\mathbb{R}$ with $0<\kappa<\tau$; then, we have for $n\geq 2$*

$$
\begin{aligned}
&|4^\gamma A(\kappa^n,\tau^n)-A^n(\kappa,\tau)-3^\gamma\Gamma(\gamma+1)L_n^n(\kappa,\tau)| \\
\leq&\frac{3^\gamma(\tau-\kappa)^\gamma}{4^\gamma}\left(\frac{\Gamma(1+p\gamma)}{\Gamma(1+(p+1)\gamma)}\left(\frac{4^{p+1}-1}{3^{p+1}}\right)^\gamma\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\right)^{\frac{1}{q}} \\
&\times\left(\frac{\Gamma(1+n\gamma)}{\Gamma(1+(n-1)\gamma)}\right)^{\frac{1}{q}} \\
&\times\left(\left(\kappa^{(n-1)\gamma q}+\left(\frac{\kappa+\tau}{2}\right)^{(n-1)\gamma q}\right)^{\frac{1}{q}}+\left(\left(\frac{\kappa+\tau}{2}\right)^{(n-1)\gamma q}+\tau^{(n-1)\gamma q}\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Proof. The assertion arises from the application of Theorem 7 to the function $\Phi:(0,+\infty)\to\mathbb{R}^\gamma$ defined by $\Phi(\chi)=\chi^{n\gamma}$. □

5. Conclusions

For sufficiently nice functions, convexity is determined by looking at its second derivative. However, the case in the fractional form quite different and determined by using the inequalities. Thus, in this work, we discussed the fractal Milne-type quadrature formula. We began by introducing a novel generalized identity, which was used to prove some new fractal Milne-type inequalities via generalized convexity. In addition, our findings were demonstrated that relevant to the quadrature formula's error estimates and to special means. The findings could lead to additional research on this intriguing topic and generalizations for other types of generalized convexity and for weighted formulas, and for upper dimensions.

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References

1. Alomari, M.; Liu, Z. New error estimations for the Milne's quadrature formula in terms of at most first derivatives. *Konuralp J. Math.* **2013**, *1*, 17–23.
2. Kashuri, A.; Mohammed, P.O.; Abdeljawad, T.; Hamasalh, F.; Chu, Y. New Simpson-type integral inequalities for s-convex functions and their applications. *Math. Probl. Eng.* **2020**, *2020*, 8871988. [CrossRef]
3. Lakhdari, A.; Meftah, B. Some fractional weighted trapezoid type inequalities for preinvex functions. *Int. J. Nonlinear Anal.* **2022** *1*, 3567–3587.
4. Pečarić, J.; Varošanec, S. A note on Simpson's inequality for functions of bounded variation. *Tamkang J. Math.* **2000**, 3, 239–242. [CrossRef]
5. Sarikaya, M.Z.; Set, E.; Ozdemir, M.E. On new inequalities of Simpson's type for s-convex functions. *Comput. Math. Appl.* **2010**, *8*, 2191–2199. [CrossRef]
6. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Fractals and Fractional Calculus in Continuum Mechanics (Udine, 1996), CISM Courses and Lect., 378; Springer: Vienna, Austria, 1997; pp. 223–276.
7. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and applications of fractional differential equations. In *North-Holland Mathematics Studies*; Elsevier Science B.V.: Amsterdam, The Netherlands, 2006; Volume 204.
8. Mandelbrot, B.B. *The Fractal Geometry of Nature*; Macmillan: New York, NY, USA, 1983.
9. Yang, X.-J. *Advanced Local Fractional Calculus and Its Applications*; World Science Publisher: New York, NY, USA, 2012.
10. Yang, X.J. *Local Fractional Functional Analysis and Its Applications*; Asian Academic Publisher Limited: Hong Kong, China, 2011.
11. Yang, X.J.; Baleanu, D.; Srivastava, H.M. *Local Fractional Integral Transforms and Their Applications*; Academic Press: Cambridge, MA, USA, 2015.
12. Ahmad, J.; Mohyud-Din, S.T. Solving wave and diffusion equations on Cantor sets. *Proc. Pakistan Acad. Sci.* **2015**, *1*, 81–87.
13. Yang, X.-J.; Srivastava, H.M.; He, J.-H.; Baleanu, D. Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives. *Phys. Lett. A* **2013**, *377*, 1696–1700. [CrossRef]
14. Yang, X.-J.; Srivastava, H.M. An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *29*, 499–504. [CrossRef]
15. Yang, X.-J.; Machado, J. A.T.; Nieto, J.J. A new family of the local fractional PDEs. *Fund. Inform.* **2017**, *151*, 63–75. [CrossRef]
16. Abdeljawad, T.; Rashid, S.; Hammouch, Z.; Chu, Y.-M. Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications. *Adv. Difference Equ.* **2020**, *406*, 27. [CrossRef]
17. Akkurt, A.; Sarikaya, M.Z.; Budak, H.; Yildirim, H. Generalized Ostrowski type integral inequalities involving generalized moments via local fractional integrals. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2017**, *111*, 797–807. [CrossRef]
18. Al Quarashi, M.; Rashid, S.; Khalid, A.; Karaca, Y.; Chu, Y.-M. New computations of Ostrowski type inequality pertaining to fractal style with applications, *Fractals* **2021**, *29*, 2140026. [CrossRef]
19. Chen, G.-S. Generalizations of Hölder's and some related integral inequalities on fractal space. *J. Funct. Spaces Appl.* **2013**, *9*, 198405. [CrossRef]
20. Erden, S.; Sarikaya, M.Z. Generalized Pompeiu type inequalities for local fractional integrals and its applications. *Appl. Math. Comput.* **2016**, *274*, 282–291. [CrossRef]
21. Iftikhar, S.; Kumam, P.; Erden, S. Newton's-type integral inequalities via local fractional integrals. *Fractals* **2020**, *3*, 2050037. [CrossRef]
22. Kiliçman, A.; Saleh, W. Notions of generalized s-convex functions on fractal sets. *J. Inequal. Appl.* **2015**, *312*. s13660-015-0826-x. [CrossRef]
23. Lakhdari, A.; Saleh, W.; Meftah, B.; Iqbal, A. Corrected Dual–Simpson–Type Inequalities for Differentiable Generalized Convex Functions on Fractal Set. *Fractal. Fract.* **2022**, *6*, 710. [CrossRef]
24. Liu, Q.; Sun, W. A Hilbert-type fractal integral inequality and its applications. *J. Inequal. Appl.* **2017**, *83*, 8. [CrossRef]
25. Meftah, B.; Souahi, A.; Merad, M. Some local fractional Maclaurin-type inequalities for generalized convex functions and their applications. *Chaos Solitons Fractals* **2022**, *162*, 112504. [CrossRef]
26. Mo, H.; Sui, X. Hermite-Hadamard-type inequalities for generalized s-convex functions on real linear fractal set $\mathbb{R}^{\gamma}$ $(0 < \gamma < 1)$. *Math. Sci.* **2017**, *3*, 241–246.
27. Sarikaya, M.Z.; Tunc, T.; Budak, H. On generalized some integral inequalities for local fractional integrals. *Appl. Math. Comput.* **2016**, *276*, 316–323. [CrossRef]
28. Sarikaya, M.Z.; Budak, H.; Erden, S. On new inequalities of Simpson's type for generalized convex functions. *Korean J. Math.* **2019**, *27*, 279–295.

29. Set, E.; Sarikaya, M.Z.; Uygun, N. On new inequalities of Simpson's type for generalized quasi-convex functions. *Adv. Inequal. Appl.* **2017**, *3*, 1–11.
30. Yang, Y.-J.; Baleanu, D.; Yang, X.-J. Analysis of fractal wave equations by local fractional Fourier series method. *Adv. Math. Phys.* **2013**, *2013*, 632309. [CrossRef]