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Third-Order Differential Subordination for Meromorphic Functions Associated with Generalized Mittag-Leffler Function

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Abstract: Using the results of third-order differential subordination, we introduce certain families of admissible functions and discuss some applications of third-order differential subordination for meromorphic functions associated with a linear operator containing a generalized Mittag-Leffler function.

Keywords: analytic functions; third-order differential subordination; Hadamard product; admissible functions; meromorphic functions; Mittag-Leffler function; linear operator

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1. Introduction and Preliminaries

Let $\mathcal{H}(\Delta)$ denote the class of all analytic functions in $\Delta = \{\omega \in \mathbb{C} : |\omega| < 1\}$. For $a \in \mathbb{C}$ and $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$, let $\mathcal{H}[a, j]$ be the subclass of analytic functions defined by:

$$\mathcal{H}[a,j] = \left\{ f \in \mathcal{H}(\Delta) : f(\omega) = a + \sum_{n=j}^{\infty} a_n \omega^n \ + \ldots \right\}.$$

Furthermore, suppose that $\mathcal{H}[1, j] = \mathcal{H}_j$.

For two functions $f, g \in \mathcal{H}(\Delta)$, the function $f(\omega)$ is called subordinate to $g(\omega)$, denoted by $f(\omega) \prec g(\omega)$, if there exists a Schwarz function $\vartheta(\omega)$, which is analytic in unit disk Δ with $\vartheta(0) = 0$ and $|\vartheta(\omega)| < 1 (\omega \in \Delta)$, satisfies $f(\omega) = g(\vartheta(\omega))$ for all $\omega \in \Delta$. Moreover, if the function g is a univalent function in Δ, then $f(\omega) \prec g(\omega)$ if and only if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$ (see [1–3]).

Let $\phi(r, s, t, u; \omega) : \mathbb{C}^4 \times \Delta \to \mathbb{C}$ and $h(\omega)$ be univalent in unit disk Δ. Furthermore, if $g(\omega)$ is analytic in Δ satisfies:

$$\phi\Big(g(\omega), \omega g'(\omega), \omega^2 g''(\omega), \omega^3 g'''(\omega); \omega\Big) \prec h(\omega), \tag{1}$$

then $g(\omega)$ is a solution of the above differential subordination (1). The univalent function $\varrho(\omega)$ is said to be a dominant of the solutions of (1) if $g(\omega)$ is subordinate to $\varrho(\omega)$ for all $g(\omega)$ satisfying (1). A univalent dominant $\tilde{q}$ such that satisfies $\tilde{q} \prec \varrho$ for all dominants of (1) is called the best dominant (see [4]).

Furthermore, let $\sum(p, j)$ be the family of functions $f(\omega)$ of the form:

$$f(\omega) = \omega^{-p} + \sum_{n=j}^{\infty} a_n \omega^{n-p} \quad (p, j \in \mathbb{N} = \{1, 2, \ldots\}), \tag{2}$$

which are analytic and also $p-$valent in the punctured disk $\Delta^* = \Delta \backslash \{0\}$ and set $\sum(1,1) = \sum$.

For functions f given by (2) and $g \in \sum(p,j)$ given by

$$g(\omega) = \omega^{-p} + \sum_{n=j}^{\infty} b_n \omega^{n-p} \quad (\omega \in \Delta^*), \tag{3}$$

the Hadamard product (or convolution) of two functions f and g is defined by

$$(f * g)(\omega) = \omega^{-p} + \sum_{n=j}^{\infty} a_n b_n \omega^{n-p} = (g * f)(\omega).$$

The Mittag-Leffler function $E_\alpha(\omega)(\omega \in \mathbb{C})$ is defined as (see [5,6]):

$$E_\alpha(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + 1)} (\alpha \in \mathbb{C}, \Re\{\alpha\} > 0). \tag{4}$$

Srivastava and Tomovski [7] introduced the generalized Mittag-Leffler function $E_{\alpha,\beta}^{\gamma,k}(\omega)$ (with $j = 1$) in the form (see also [8]):

$$E_{\alpha,\beta}^{\gamma,k}(\omega) = \frac{1}{\Gamma(\beta)} + \sum_{n=j}^{\infty} \frac{(\gamma)_{nk}\, \omega^n}{\Gamma(\alpha n + \beta)\, n!}, \tag{5}$$

where $\beta, \gamma \in \mathbb{C}$, $\Re\{\alpha\} > \max\{0; \Re\{k\} - 1\}; \Re\{k\} > 0$, $\Re\{\alpha\} = 0$ at $\Re\{k\} = 1$ with $\beta \neq 0$ and $(\gamma)_m$ is the Pochhammer symbol defined as:

$$(\gamma)_m = \frac{\Gamma(\gamma + m)}{\Gamma(\gamma)} = \begin{cases} 1, & \text{if} \quad m = 0, \\ \gamma(\gamma+1)\ldots(\gamma+m-1), & \text{if} \quad m \in \mathbb{N}. \end{cases}$$

We now define the function $\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(\omega)$ by

$$\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(\omega) = \omega^{-p} \Gamma(\beta) E_{\alpha,\beta}^{\gamma,k}(\omega). \tag{6}$$

Corresponding to the function $\mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(\omega)$ defined by (6), Aouf and Seoudy [9] introduced a linear operator $\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} : \sum(p,j) \to \sum(p,j)$ by

$$\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(\omega) = \mathcal{B}_{p,\alpha,\beta}^{\gamma,k}(\omega) * f(\omega) = \omega^{-p} + \sum_{n=j}^{\infty} \frac{\Gamma(\beta)\, (\gamma)_{nk}}{\Gamma(\beta + \alpha n)\, n!} a_n \omega^{n-p}. \tag{7}$$

$$(\Re\{\alpha\} > \max\{0; \Re\{k\} - 1\}; \Re\{k\} > 0, \Re\{\alpha\} = 0 \; at \; \Re\{k\} = 1 \; with \; \beta \neq 0).$$

We note that

$$\mathcal{T}_{p,0,\beta}^{1,1} f(\omega) = f(\omega) \quad \text{and} \quad \mathcal{T}_{p,0,\beta}^{2,1} f(\omega) = \omega f'(\omega) + (p+1) f(\omega).$$

Furthermore, it is easily verified from (7) that

$$k\omega \left(\mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(\omega) \right)' = \gamma \mathcal{T}_{p,\alpha,\beta}^{\gamma+1,k} f(\omega) - (\gamma + pk) \mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(\omega) \tag{8}$$

and

$$\alpha\omega \left(\mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k} f(\omega) \right)' = \beta \mathcal{T}_{p,\alpha,\beta}^{\gamma,k} f(\omega) - (\beta + p\alpha)\, \mathcal{T}_{p,\alpha,\beta+1}^{\gamma,k} f(\omega). \tag{9}$$

To obtain our results, we will use the following definitions and lemmas.

Definition 1 ([4], p. 441). *Let Q be the set of all functions ϱ that are analytic and univalent on $\bar{\Delta}\backslash E(\varrho)$ where*

$$E(\varrho) = \left\{\zeta \in \partial\Delta : \lim_{\omega\to\zeta} \varrho(\omega) = \infty\right\},$$

and are such that $\min|\varrho'(\zeta)| = \rho > 0$ for $\zeta \in \partial\Delta\backslash E(\varrho)$. Further, let $Q(a)$ denote the subclass of Q consisting of functions ϱ for which $\varrho(0) = a$ and $Q(1) \equiv Q_1$.

Definition 2 ([4], Theorem 1, p. 449). *If $\Omega \subseteq \mathbb{C}$, $\varrho \in Q$ and $j \geq 2$. Let $\Psi_j[\Omega, \varrho]$ be the family of admissible functions consisting of functions $\psi : \mathbb{C}^4 \times \Delta \to \mathbb{C}$, which satisfy the condition of admissibility as:*

$$\psi(r, s, t, u; \omega) \notin \Omega$$

whenever

$$r = \varrho(\zeta), s = m\zeta\varrho'(\zeta), \Re\left\{\frac{t}{s} + 1\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{u}{s}\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $\omega \in \Delta$, $\zeta \in \partial\Delta\backslash E(\varrho)$ and $m \geq j$.

Lemma 1 ([4], Theorem 1, p. 449). *Let $g \in \mathcal{H}[a, j]$ with $j \geq 2$. Furthermore, let $\varrho \in Q(a)$ and satisfy the following conditions:*

$$\Re\left\{\frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0 \quad \textit{and} \quad \left|\frac{\omega g'(\omega)}{\varrho'(\zeta)}\right| \leq m,$$

where $\omega \in \Delta$, $\zeta \in \partial\Delta\backslash E(\varrho)$ and $m \geq j$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_j[\Omega, \varrho]$ and

$$\psi(g(\omega), \omega g'(\omega), \omega^2 g''(\omega), \omega^3 g'''(\omega); \omega) \in \Omega$$

then $g(\omega) \prec \varrho(\omega)$.

Several authors have obtained many important results involving various operators related by differential subordination and differential superordination (for example, see [10–19]).

In the present paper, by making use of the third-order differential subordination theorems of Antonino and Miller [4] (see also the recent works by Tang et al. [20–22]), we determine the sufficient conditions for certain appropriate classes of admissible functions so that

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega)$$

and

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega),$$

where $\varrho(\omega)$ is given univalent functions in Δ with $\varrho \in Q_1 \cap H_j$. In addition, we obtain some special cases of these classes of admissible functions. Our results derived in the present paper and, together with other papers that appeared in recent years, will pave the way for further study in the direction of the third-order subordination theory.

2. Third-Order Differential Subordination Results with $\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}$

Unless otherwise mentioned, we assume throughout this paper that $f \in \sum(p, j)$, $k \neq 0$, $\gamma \neq 0, -1, -2$, $\zeta \in \partial\Delta\backslash E(\varrho)$, $\theta \in [0, 2\pi)$ and $\omega \in \Delta$.

Definition 3. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in Q_1 \cap \mathcal{H}_j$. Let $\Phi_1[\Omega, \varrho]$ be the family of admissible functions consists of functions $\phi : \mathbb{C}^4 \times \Delta \to \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a, b, c, d; \omega) \notin \Omega,$$

whenever

$$a = \varrho(\zeta), b = \varrho(\zeta) + \frac{km\zeta\varrho'(\zeta)}{\gamma},$$

$$\Re\left\{\frac{c(\gamma+1) - b(2\gamma+1)b + \gamma a}{k(b-a)}\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{d(\gamma+1)[(\gamma+2)-3(\gamma+k+1)c] + \left[3\gamma^2+3(2k+1)\gamma+2k^2+3k+1\right]b - \left(\gamma^2+3k\gamma+2k^2\right)a}{k^2(b-a)}\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\},$$

where $\omega \in \Delta$, $\zeta \in \partial\Delta \backslash E(\varrho)$, and $m \geq j \geq 2$.

Theorem 1. *If $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_1[\Omega, \varrho]$. If $f \in \sum(p, j)$ and $\varrho \in Q_1$ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0 \quad \text{and} \quad \left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)\right)'\right| \leq m\left|\varrho'(\zeta)\right|, \tag{10}$$

then

$$\left\{\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\right) : \omega \in \Delta\right\} \subset \Omega \tag{11}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega).$$

Proof. Define $g(\omega)$ in unit disk Δ by

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) = g(\omega) \quad (\omega \in \Delta). \tag{12}$$

Differentiating (12) with respect to ω and using the recurrence relation (8), we have

$$\omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega) = g(\omega) + \frac{k}{\gamma}\omega g'(\omega). \tag{13}$$

Differentiating (13) with respect to ω and also using the recurrence relation (8), we obtain

$$\omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega) = g(\omega) + \frac{k(2\gamma+k+1)}{\gamma(\gamma+1)}\omega g'(\omega) + \frac{k^2}{\gamma(\gamma+1)}\omega^2 g''(\omega). \tag{14}$$

Further computations show that

$$\omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega) = g(\omega) + \frac{k\left[3\gamma^2+3(k+2)\gamma+k^2+3k+2\right]}{\gamma(\gamma+1)(\gamma+2)}\omega g'(\omega)$$
$$+ \frac{3k^2(\gamma+k+1)}{\gamma(\gamma+1)(\gamma+2)}\omega^2 g''(\omega) + \frac{k^3}{\gamma(\gamma+1)(\gamma+2)}\omega^3 g'''(\omega). \tag{15}$$

Let

$$\left.\begin{aligned}
&a = r,\\
&b = r + \tfrac{k}{\gamma}s,\\
&c = r + \tfrac{k(2\gamma+k+1)}{\gamma(\gamma+1)}s + \tfrac{k^2}{\gamma(\gamma+1)}t,\\
&d = r + \tfrac{k\left[3\gamma^2+3\gamma(k+2)+k^2+3k+2\right]}{\gamma(\gamma+1)(\gamma+2)}s + \tfrac{3k^2(\gamma+k+1)}{\gamma(\gamma+1)(\gamma+2)}t + \tfrac{k^3}{\gamma(\gamma+1)(\gamma+2)}u.
\end{aligned}\right\} \tag{16}$$

we now define the transformation $\psi(r,s,t,u;\omega) : \mathbb{C}^4 \times \Delta \to \mathbb{C}$ by

$$\begin{aligned}
\psi(r,s,t,u;\omega) &= \phi(a,b,c,d;\omega)\\
&= \phi\bigg(r, r + \frac{k}{\gamma}s, r + \frac{k(2\gamma+k+1)}{\gamma(\gamma+1)}s + \frac{k^2}{\gamma(\gamma+1)}t,\\
&\quad r + \tfrac{k\left[3\gamma^2+3\gamma(k+2)+k^2+3k+2\right]}{\gamma(\gamma+1)(\gamma+2)}s + \tfrac{3k^2(\gamma+k+1)}{\gamma(\gamma+1)(\gamma+2)}t + \tfrac{k^3}{\gamma(\gamma+1)(\gamma+2)}u;\omega\bigg).
\end{aligned} \tag{17}$$

Then, using relations (12)–(15), we have

$$\begin{aligned}
&\psi\Big(g(\omega), \omega g'(\omega), \omega^2 g''(\omega), \omega^3 g'''(\omega);\omega\Big)\\
&= \phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega);\omega\Big).
\end{aligned} \tag{18}$$

Note that

$$\frac{t}{s} + 1 = \frac{c(\gamma+1) - b(2\gamma+1) + \gamma a}{k(b-a)}$$

and

$$\frac{u}{s} = \frac{d(\gamma+1)[(\gamma+2) - 3c(\gamma+k+1)] + \left[3\gamma^2 + 3(2k+1)\gamma + 2k^2 + 3k + 1\right]b - \left(\gamma^2 + 3k\gamma + 2k^2\right)a}{k^2(b-a)}.$$

Further note that the condition of admissibility for function $\phi \in \Phi_1[\Omega, \varrho]$ of Definition 3 is equivalent to the condition of admissibility for the function $\psi \in \Psi_n[\Omega, \varrho]$, which is given in Definition 2. Thus, the proof of Theorem 1 follows from Lemma 1. □

The following result will be an extension of Theorem 1 when the behavior of the function $\varrho(\omega)$ on $\partial\Delta$ is unknown.

Corollary 1. *If $\Omega \subseteq \mathbb{C}$ and ϱ is a univalent in Δ with $\varrho \in Q_1$. Let $\phi \in \Phi_1[\Omega, \varrho_\rho]$ for some $\rho \in (0,1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If $f \in \sum$ and ϱ_ρ, satisfy the following conditions:*

$$\Re\left\{\frac{\zeta\varrho''_\rho(\zeta)}{\varrho'_\rho(\zeta)}\right\} \geq 0 \quad \text{and} \quad \left|\omega\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)\Big)'\right| \leq m\left|\varrho'_\rho(\zeta)\right|, \tag{19}$$

then

$$\left\{\phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega);\omega\Big) : \omega \in \Delta\right\} \subset \Omega,$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega).$$

Proof. From Theorem 1 we have $\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho_\rho(\omega)$ and since $\varrho_\rho(\omega) \prec \varrho(\omega)$, we conclude that $\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega)$. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, therefore, we have a conformal mapping h from Δ into the domain Ω such that $h(\Delta)$ is equal to Ω. Then, we denote the class $\Phi_1[h(\Delta), \varrho]$ by $\Phi_1[h, \varrho]$. The next two corollaries are immediate consequences of Theorem 1 and Corollary 1.

Corollary 2. *If h is univalent function in Δ and let $\phi \in \Phi_1[h, \varrho]$, also suppose that $\varrho \in Q_1$ satisfies (10). Then*

$$\phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\Big) \prec h(\omega), \tag{20}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega).$$

Corollary 3. *If ϱ is univalent function in Δ with $\varrho \in Q_1$ and $\phi \in \Phi_1\big[h, \varrho_\rho\big]$ for some $\rho \in (0,1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If ϱ_ρ satisfies the conditions in (19), then the subordination relation (20) implies that*

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega).$$

The following corollary shows the connection between the best dominant of a third order differential subordination and the solution of the corresponding third-order differential equation.

Corollary 4. *If h is univalent function in Δ and ψ is given by (18) where $\phi \in \Phi_1[h, \varrho]$. If the differential equation*

$$\psi\Big(\varrho(\omega), \omega\varrho'(\omega), \omega^2\varrho''(\omega), \omega^3\varrho'''(\omega); \omega\Big) = h(\omega)$$

has a solution ϱ with $\varrho \in Q_1$ that satisfies the conditions (10), then subordination (20) implies that

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec \varrho(\omega),$$

and ϱ is the best dominant of (20).

Proof. Since

$$\begin{aligned}
&\phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\Big) \\
&= \psi\Big(g(\omega), \omega g'(\omega), \omega^2 g''(\omega), \omega^3 g'''(\omega); \omega\Big) \prec h(\omega),
\end{aligned} \tag{21}$$

then $g(\omega)$ is a solution of (21), and from Corollary 2 we obtain that $g(\omega) \prec \varrho(\omega)$, that is, ϱ is a dominant of (21). Furthermore,

$$\begin{aligned}
&\phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\Big) \\
&= \psi\Big(g(\omega), \omega g'(\omega), \omega^2 g''(\omega), \omega^3 g'''(\omega); \omega\Big) \prec h(\omega) \\
&= \psi\Big(\varrho(\omega), \omega\varrho'(\omega), \omega^2\varrho''(\omega), \omega^3\varrho'''(\omega); \omega\Big),
\end{aligned}$$

which means that ϱ is the best dominant of (21). □

Theorem 1 will be applied when $\varrho(\omega) = 1 + M\omega, M > 0$. By Definition 3, the family of admissible functions $\Phi_1[\Omega, \varrho]$, denoted now by $\Phi_1[\Omega, M]$ as follows:

Definition 4. *If $\Omega \subseteq \mathbb{C}$ and let $M > 0$. The family of admissible functions $\Phi_1[\Omega, M]$ consists of the functions $\phi : \mathbb{C}^4 \times \Delta \to \mathbb{C}$, which satisfy the following admissibility condition*

$$\phi\Big(1 + Me^{i\theta}, 1 + \frac{(\gamma+km)Me^{i\theta}}{\gamma}, 1 + \frac{k^2L+[km(2\gamma+k+1)+\gamma(\gamma+1)]Me^{i\theta}}{\gamma(\gamma+1)},$$

$$1 + \frac{k^3N+3k^2(\gamma+k+1)L+\big\{km\big[3\gamma^2+3\gamma(k+2)+k^2+3k+2\big]+\gamma(\gamma+1)(\gamma+2)\big\}Me^{i\theta}}{\gamma(\gamma+1)(\gamma+2)}; \omega\Big) \notin \Omega$$

whenever $\omega \in \Delta$, $\Re\{Le^{-i\theta}\} \geq m(m-1)M$ and $\Re\{Ne^{-i\theta}\} \geq 0$ for every $\theta \in [0, 2\pi]$ and $m \geq j \geq 2$.

Using the definition of the family of admissible functions, from the result in Theorem 1, we have the following result.

Corollary 5. *If $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_1[\Omega, M]$. If*

$$\left|\omega\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)\Big)'\right| \leq mM, \tag{22}$$

then

$$\phi\Big(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\Big) \in \Omega \quad (\omega \in \Delta),$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec 1 + M\omega.$$

3. Further Results Involving $\mathcal{T}^{\gamma,k}_{p,\alpha,\beta}$

In this section, using the recurrence relation (9), we obtain interesting results of differential subordination associated with $\mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)$. The proofs of our results presented in this section are similar to the previous section and will be omitted.

Definition 5. *If $\Omega \subseteq \mathbb{C}$ and $\varrho \in Q_1 \cap \mathcal{H}_j$. Let $\Phi_2[\Omega, \varrho]$ be the family of admissible functions, consisting of functions $\phi : \mathbb{C}^4 \times \Delta \to \mathbb{C}$ that satisfy the condition of admissibility:*

$$\phi(a, b, c, d; \omega) \notin \Omega,$$

whenever

$$a = \varrho(\zeta), b = \varrho(\zeta) + \frac{m\alpha\zeta\varrho'(\zeta)}{\beta+2},$$

$$\Re\left\{\frac{c(\beta+1) - b(2\beta+3) + a(\beta+2)}{\alpha(b-a)}\right\} \geq m\Re\left\{1 + \frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\}$$

and

$$\Re\left\{\frac{(\beta+1)[\beta d-3(\beta+\alpha+1)c]+\big[3\beta^2+3(2\alpha+3)\beta+2\alpha^2+9\alpha+7\big]b-\big[\beta^2+(3\alpha+4)\beta+2\alpha^2+6\alpha+4\big]a}{\alpha^2(b-a)}\right\} \geq m^2\Re\left\{\frac{\zeta^2\varrho'''(\zeta)}{\varrho'(\zeta)}\right\}$$

where $\omega \in \Delta$, $\zeta \in \partial\Delta \backslash E(\varrho)$, and $m \geq j \geq 2$.

Theorem 2. *If $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_2[\Omega, \varrho]$. If $f \in \sum(p,j)$ and $\varrho \in Q_1$ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta\varrho''(\zeta)}{\varrho'(\zeta)}\right\} \geq 0 \quad and \quad \left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega)\right)'\right| \leq m|\varrho'(\zeta)|, \tag{23}$$

then

$$\left\{\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta+2} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta+1} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\right) : \omega \in \Delta\right\} \subset \Omega\,, \tag{24}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega).$$

The following result will be an extension of Theorem 2 when the behavior of the function $\varrho(\omega)$ on $\partial\Delta$ is unknown.

Corollary 6. *If $\Omega \subseteq \mathbb{C}$ and ϱ is univalent in Δ with $\varrho \in Q_1$. Let $\phi \in \Phi_2[\Omega, \varrho_\rho]$ for some $\rho \in (0,1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If $f \in \sum(p,j)$ and ϱ_ρ satisfy the following conditions:*

$$\Re\left\{\frac{\zeta\varrho''_\rho(\zeta)}{\varrho'_\rho(\zeta)}\right\} \geq 0 \quad and \quad \left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega)\right)'\right| \leq m\left|\varrho'_\rho(\zeta)\right|, \tag{25}$$

then

$$\left\{\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+2} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+1} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega); \omega\right) : \omega \in \Delta\right\} \subset \Omega$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, therefore, we have a conformal mapping h from Δ into the domain Ω such that $h(\Delta)$ is equal to Ω. Then, we denote the class $\Phi_2[h(\Delta), \varrho]$ by $\Phi_2[h, \varrho]$. The next two corollaries are immediate consequences of Theorem 2 and Corollary 6.

Corollary 7. *If h is univalent function in Δ and let $\phi \in \Phi_2[h, \varrho]$, also suppose that $\varrho \in Q_1$ satisfies conditions (10). Then*

$$\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+2} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+1} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega); \omega\right) \prec h(\omega), \tag{26}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega).$$

Corollary 8. *If ϱ is univalent function in Δ with $\varrho \in Q_1$ and $\phi \in \Phi_2[h, \varrho_\rho]$ for some $\rho \in (0,1)$, where $\varrho_\rho(\omega) = \varrho(\rho\omega)$. If ϱ_ρ satisfies conditions (25), then the subordination (26) implies that*

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega).$$

The following corollary shows the connection between the best dominant of a third-order differential subordination and the solution of the corresponding third-order differential equation.

Corollary 9. *If h is univalent function in Δ and ψ is given by (18) where $\phi \in \Phi_2[h, \varrho]$. If the differential equation*

$$\psi\left(\varrho(\omega), \omega\varrho'(\omega), \omega^2\varrho''(\omega), \omega^3\varrho'''(\omega); \omega\right) = h(\omega)$$

has a solution ϱ with $\varrho \in Q_1$ that satisfies conditions (23), then subordination (26) implies that

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec \varrho(\omega),$$

and ϱ is the best dominant of (26).

Theorem 2 will be applied when $\varrho(\omega) = 1 + M\omega, M > 0$. By Definition 5, the family of admissible functions $\Phi_2[\Omega, \varrho]$, denoted now by $\Phi_2[\Omega, M]$ as follows:

Definition 6. *If $\Omega \subseteq \mathbb{C}$ and let $M > 0$. The family of admissible functions $\Phi_2[\Omega, M]$ consists of the functions $\phi : \mathbb{C}^4 \times \Delta \to \mathbb{C}$, which satisfy the following admissibility condition*

$$a = 1 + Me^{i\theta}, b = 1 + Me^{i\theta} + \frac{m\alpha Me^{i\theta}}{\beta+2},$$

$$\phi\Big(1 + Me^{i\theta}, 1 + \tfrac{(\beta+2+\alpha m)Me^{i\theta}}{\beta+2}, 1 + \tfrac{\alpha^2 L + [\alpha m(2\beta+\alpha+3) + (\beta+1)(\beta+2)]Me^{i\theta}}{(\beta+1)(\beta+2)},$$
$$1 + \tfrac{\alpha^3 N + 3(\beta+\alpha+1)\alpha^2 L + \left[\alpha m\left(3\beta^2 + 3(\alpha+2)\beta + \alpha^2 + 3\alpha + 2\right) + \beta(\beta+1)(\beta+2)\right]Me^{i\theta}}{\beta(\beta+1)(\beta+2)}; \omega\Big) \notin \Omega$$

whenever $\omega \in \Delta$, $\Re\{Le^{-i\theta}\} \geq m(m-1)M$ and $\Re\{Ne^{-i\theta}\} \geq 0$ for every $\theta \in [0, 2\pi]$ and $m \geq j \geq 2$.

Using the definition of the family of admissible functions, from the result in Theorem 2, we have the following result.

Corollary 10. *If $\Omega \subseteq \mathbb{C}$ and $\phi \in \Phi_2[\Omega, M]$. If we suppose that*

$$\left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega)\right)'\right| \leq mM, \tag{27}$$

then

$$\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+2} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+1} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega); \omega\right) \in \Omega \quad (\omega \in \Delta),$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec 1 + M\omega.$$

4. Some Applications

If we take $\Omega = \{\chi \in \mathbb{C} : |\chi - 1| < M\}$, and $\Phi_1[\Omega, M]$ is simply denoted by $\Phi_1[M]$ and Corollary 5 reduces to the next corollary.

Corollary 11. *Let $\phi \in \Phi_1[M]$ and suppose that*

$$\left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)\right)'\right| \leq mM, \tag{28}$$

then

$$\left|\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+2,k}_{p,\alpha,\beta} f(\omega), \omega^p \mathcal{T}^{\gamma+3,k}_{p,\alpha,\beta} f(\omega); \omega\right) - 1\right| < M \quad (\omega \in \Delta), \tag{29}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec 1 + M\omega. \tag{30}$$

Corollary 12. *Suppose that*

$$\left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega)\right)'\right| \leq mM, \tag{31}$$

then

$$\left|\omega^p \mathcal{T}^{\gamma+1,k}_{p,\alpha,\beta} f(\omega) - 1\right| < M \ \ (\omega \in \Delta), \tag{32}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega) \prec 1 + M\omega. \tag{33}$$

Proof. The result follows from Corollary 11 by putting $\phi(a,b,c,d;\omega) = b$. $\square$

Putting $\gamma = k = 1$ and $\alpha = 0$ in Corollary 12, and noting that

$$\mathcal{T}^{1,1}_{p,0,\beta} f(\omega) = f(\omega) \quad \text{and} \quad \mathcal{T}^{2,1}_{p,0,\beta} f(\omega) = \omega f'(\omega) + (p+1) f(\omega) \tag{34}$$

we obtain the following result :

Example 1. *If $f \in \sum(p,j)$ satisfies the following conditions*

$$\left|\omega(\omega^p f(\omega))'\right| \leq mM \tag{35}$$

and

$$\left|\omega^{p+1} f'(\omega) + (p+1)\omega^p f(\omega) - 1\right| < M, \tag{36}$$

then

$$|\omega^p f(\omega) - 1| < M. \tag{37}$$

Furthermore, putting $\gamma = k = 1$ and $\alpha = 0$ in Corollary 12 and $p = j = 1$, we obtain the following result:

Example 2. *If $f \in \sum$ satisfies the following inequalities*

$$\left|\omega^2 f'(\omega) + \omega f(\omega)\right| \leq mM \tag{38}$$

and

$$\left|\omega^2 f'(\omega) + 2\omega f(\omega) - 1\right| < M, \tag{39}$$

then

$$|\omega f(\omega) - 1| < M. \tag{40}$$

Putting $\Omega = \{\chi \in \mathbb{C} : |\chi - 1| < M\}$, as special case and $\Phi_2[\Omega, M]$ is simply denoted by $\Phi_2[M]$ and Corollary 10 reduces to the next corollary.

Corollary 13. *Let $\phi \in \Phi_2[M]$ and suppose that*

$$\left|\omega\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega)\right)'\right| \leq mM, \tag{41}$$

then

$$\left|\phi\left(\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+2} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+1} f(\omega), \omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta} f(\omega); \omega\right) - 1\right| < M \ \ (\omega \in \Delta), \tag{42}$$

which implies

$$\omega^p \mathcal{T}^{\gamma,k}_{p,\alpha,\beta+3} f(\omega) \prec 1 + M\omega. \tag{43}$$

5. Conclusions

In our present investigation, we have determined the sufficient conditions for classes $\Phi_1[\Omega, \varrho]$ and $\Phi_2[\Omega, \varrho]$ of admissible functions to obtain some interesting results of third-order differential subordination for meromorphically multivalent functions that include a linear operator $T^{\gamma,k}_{p,\alpha,\beta}$ associated with the generalized Mittag-Leffler function. Furthermore, some special cases of these classes of admissible functions and some important inequalities have been derived. Our results are connected with those in several earlier works, which are related to the theory of differential subordination and superordination of Geometric Function Theory.

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