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Article

Onsager's Energy Conservation of Weak Solutions for a Compressible and Inviscid Fluid

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Abstract: In this article, two classes of sufficient conditions of weak solutions are given to guarantee the energy conservation of the compressible Euler equations. Our strategy is to introduce a test function $\varphi(t)\mathbf{v}^\epsilon$ to derive the total energy. The velocity field $\mathbf{v}$ needs to be regularized both in time and space. In contrast to the noncompressible Euler equations, the compressible flows we consider here do not have a divergence-free structure. Therefore, it is necessary to make an additional estimate of the pressure p, which takes advantage of an appropriate commutator. In addition, by using the weak convergence, we show that the energy equality is conserved in a point-wise sense.

Keywords: conservation of energy; the isentropic compressible Euler equations; commutator estimate; regularity of the solutions

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1. Introduction

This paper is devoted to studying the weak solution for compressible Euler equations, which is given by

$$\begin{cases} \partial_t(\varrho\mathbf{v}) + \nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v}) + \nabla p(\varrho) = 0, \\ \partial_t\varrho + \nabla\cdot(\varrho\mathbf{v}) = 0, \end{cases} \tag{1}$$

with the initial data

$$(\varrho\mathbf{v})(0,x) = (\varrho_0\mathbf{v}_0)(x),\ \varrho(0,x) = \varrho_0(x), \tag{2}$$

here $\nabla\cdot(*) = \sum_{i=1}^{d}\partial_i(*)_i$ and $\mathbf{v}\otimes\mathbf{v}$ is noted as the component u_iu_j in the matrix. In addition, ϱ is the density of the fluid, $\mathbf{v}$ stands for velocity vector field, and $p(\varrho)$ denotes the scalar pressure. We define $\mathbf{v}_0 = 0$ on the set $\{x|\varrho_0(x) = 0\}$. For simplicity, we consider the compressible Euler equations on the periodic domain $\mathbb{T}^d$, $d = 2$ or 3, and denote the time interval $[0, T]$ by $\mathbb{I}$.

If we let $\varrho \equiv C$, then system (1) becomes the classical noncompressible Euler equations, i.e.,

$$\begin{cases} \partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{3}$$

For the domain $[0, T]\times\mathbb{T}^3$, the weak solutions considered by Onsager [1] to Equation (3) satisfy the Hölder condition

$$|\mathbf{v}(t, x+\Delta x) - \mathbf{v}(t, x)| \le C|\Delta x|^\alpha$$

for any $t \in [0, T]$, where constant C independent of $\Delta x \in \mathbb{T}^3$. In 1949, he conjectured that

(i) If $\alpha > 1/3$, the energy of every weak solution must be conserved;
(ii) If $\alpha < 1/3$, the energy of weak solutions will be dissipated.

For part (i) of the conjecture, in 1994, Constantin et al. [2] gave the first complete proof that energy is conserved as $\alpha > 1/3$ by considering the weak solutions of Equation (3) in

3D. Subsequently, the weaker assumptions of the velocity $\mathbf{v}$ in Besov spaces also lead to the conservation of energy; see [3,4]. A significant result of Conjecture (ii) came from a series of breakthrough articles by De Lellis and Székelyhidi [5,6], where they show that the energy will be dissipative for the solutions in $L_t^\infty C_x^\alpha$ if $\alpha < 1/10$. Later, De Lellis et al. [7] showed that the energy of every weak solution is dissipated if the solutions belong to $C_t C_x^\alpha$, $\alpha < 1/5$, and in 2018, this result was improved to $\alpha < 1/3$ by Isett in article [8]. Other forms of weak solutions violate the energy conservation, such as dissipative solutions to Equation (3) in 2D obtained by Choffrut [9], the uniqueness for weak solutions of the noncompressible porous media equations studied by Cordoba et al. [10], the uniqueness of weak solutions for Equation (3) due to De Lellis and Székelyhidi [11], and nonuniqueness of weak solutions for Equation (3) achieved by Isett [12].

In this work, we investigate the energy conservation of weak solutions for Equation (1). Unlike the way the homogeneous Euler equations were dealt with in [13,14], where the temporal derivative of $\mathbf{v}$ can be completely transferred to a test function, the nonhomogeneous flows contain a nonlinear term $\partial_t(\varrho\mathbf{v})$ that needs to be estimated by the time commutator. To avoid the time commutator estimate, Leslie and Shvydkoy [15] chose the test function $(\varrho\mathbf{v})^\epsilon/\varrho^\epsilon$ instead of $\mathbf{v}^\epsilon$ to multiply the momentum equation to obtain energy conservation, where convolution only works in space. However, the disadvantage is that the vacuum needs to be excluded. Recently, Feireisl et al. [16] took a direct method, Besov regularity both in space and time, which allows the authors to handle a vacuum state. If the solution satisfies

$$\begin{aligned} &\mathbf{v} \in B_3^{\alpha,\infty}(\mathbb{I} \times \mathbb{T}^d),\ \varrho, \varrho\mathbf{v} \in B_3^{\beta,\infty}(\mathbb{I} \times \mathbb{T}^d), \\ &0 \le \underline{\varrho} \le \varrho \le \overline{\varrho} \in L^\infty(\mathbb{I} \times \mathbb{T}^d),\ p \in C^2[\underline{\varrho}, \overline{\varrho}] \end{aligned} \tag{4}$$

then they showed that the energy of weak solutions is conserved in the sense of distributions. Akramov et al., in article [17], improved the assumption $p \in C^2[\underline{\varrho}, \overline{\varrho}]$ to $p \in C^{1,\gamma-1}[\underline{\varrho}, \overline{\varrho}]$, $1 < \gamma < 2$ by the inequality

$$\|p^\epsilon(\varrho) - p(\varrho^\epsilon)\|_{L^q} \le C\epsilon^{\gamma\beta}\|\varrho\|^\gamma_{B^{\beta,\infty}_{\gamma q}}.$$

However, this paper will investigate if the energy of weak solutions is conserved in a point-wise sense. In order to not add any assumptions about the pressure term p itself, we use the pressure law $p(\varrho) = \kappa\varrho^\gamma$, $\gamma > 1$. Following the ideas in [18], two types of results that ensure energy conservation are given by "trading" the regularity between variables ϱ and $\mathbf{v}$, which is the spirit of the article. The first type of result is that the density ϱ has strong regularity and assumes that the velocity belongs to the Besov space. It is concluded that the energy can be conserved for system (1) if the Hölder exponent of $\mathbf{v}$ is greater than $1/3$. The second result is that the velocity field $\mathbf{v}$ admits more regularity, which allows the existence of a less regular density ϱ. The density ϱ or the velocity $\mathbf{v}$ is given more regularity conditions to ensure energy conservation in a point-wise sense on $\mathbb{I}$, whereas the results of [16,17] hold only in a distributional sense. Similar to the idea of the treatment of the nonlinear term $\partial_t(\varrho\mathbf{v})$ in [16], we will smooth system (1) in both time and space, which allows the existence of a vacuum in the system.

The rest of the paper is organized as follows. In Section 2, we give the definition of weak solutions, some important inequalities and the energy equality of a smooth solution. Lemmas 1 and 2 are two key commutator estimates that are used to vanish the error terms. The definition of weak continuity is presented by Lemma 3, which will be used to show the energy conservation of weak solutions held in a point-wise sense. In Section 3, we state the main results of our article, and two classes of sufficient conditions are given to guarantee the energy conservation of weak solutions to Equation (1). Section 4 is devoted to elaborating on the conclusion of our paper.

2. Preliminaries

For $0 < \alpha < 1$ and $p \geq 1$, we define the Besov space as the set of all functions with the following norm

$$\|u\|_{B_p^{\alpha,\infty}(\Omega)} =: \|u\|_{L^p(\Omega)} + \sup_{\xi \in \Omega} |\xi|^{-\alpha} \|u(\cdot + \xi) - u\|_{L^p(\Omega \cap (\Omega - \xi))},$$

where the domain Ω is $[0, T] \times \mathbb{T}^d$ and $\Omega - \xi = \{y - \xi | y \in \Omega\}$. The same definition of Besov space can be found in [16].

Let η be a standard mollifier from $\mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}$, i.e., $\eta(t, x) = 0$ for $|(t, x)| \geq 1$, and $\int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \eta(t, x) dx dt = 1$. We define $\eta_\epsilon(t, x) = \frac{1}{\epsilon^{d+1}} \eta(\frac{t}{\epsilon}, \frac{x}{\epsilon})$ for any $\epsilon > 0$. If a function $f \in L^p(\Omega)$, the convolution of f is given by

$$f^\epsilon(t, x) = (f * \eta_\epsilon)(t, x).$$

For $0 < \alpha < 1$ and $p \geq 1$, by the Besov norm, we can obtain the following inequality:

$$\|\nabla u^\epsilon\|_{L^p(\Omega)} \lesssim \epsilon^{\alpha - 1} \|u\|_{B_p^{\alpha,\infty}(\Omega)}, \tag{5}$$

$$\|u^\epsilon - u\|_{L^p(\Omega)} \lesssim \epsilon^{\alpha} \|u\|_{B_p^{\alpha,\infty}(\Omega)}, \tag{6}$$

$$\|u(\cdot + \xi) - u(\cdot)\|_{L^p(\Omega)} \lesssim |\xi|^{\alpha} \|u\|_{B_p^{\alpha,\infty}(\Omega)}, \tag{7}$$

where $\|g\| \lesssim \|h\|$ denotes $\|g\| \leq C\|h\|$ for some harmless constant $C > 0$.

To state the results, we need to give the definition of weak solutions for Equation (1).

Definition 1. *A couple $(\varrho, \boldsymbol{v}, p)$ is called a weak solution of Equation (1) with initial data (2) if*

(i)

$$\int_{\mathbb{I}} \int_{\mathbb{T}^d} (\varrho \boldsymbol{v} \cdot \partial_t \widetilde{\phi} + \varrho \boldsymbol{v} \otimes \boldsymbol{v} : \nabla \widetilde{\phi} + p \nabla \cdot \widetilde{\phi}) dx dt = 0$$

for every test vector field $\widetilde{\phi} \in C_0^\infty(\mathbb{I} \times \mathbb{T}^d)$.

(ii)

$$\int_{\mathbb{I}} \int_{\mathbb{T}^d} (\varrho \partial_t \phi + \varrho \boldsymbol{v} \cdot \nabla \phi) dx dt = 0$$

for every test function $\phi \in C_0^\infty(\mathbb{I} \times \mathbb{T}^d)$.

(iii) *$(\varrho \boldsymbol{v})(t, \cdot) \rightharpoonup (\varrho_0 \boldsymbol{v}_0)(x)$ in $\mathcal{D}'(\mathbb{T}^d)$ as $t \to 0$, i.e.,*

$$\lim_{t \to 0} \int_{\mathbb{T}^d} (\varrho \boldsymbol{v})(t, x) \widetilde{\phi}(x) dx = \int_{\mathbb{T}^d} (\varrho_0 \boldsymbol{v}_0)(x) \widetilde{\phi}(x) dx$$

for every test vector field $\widetilde{\phi} \in C_0^\infty(\mathbb{T}^d)$.

(iv) *$\varrho(t, \cdot) \rightharpoonup \varrho_0(x)$ in $\mathcal{D}'(\mathbb{T}^d)$ as $t \to 0$, i.e.,*

$$\lim_{t \to 0} \int_{\mathbb{T}^d} \varrho(t, x) \phi(x) dx = \int_{\mathbb{T}^d} \varrho_0(x) \phi(x) dx$$

for every test function $\phi \in C_0^\infty(\mathbb{T}^d)$.

The following lemma is crucial for the commutator estimate. Here we rely on Lions' proof in [19].

Lemma 1 ([18,19])**.** *Let ∂ be the partial derivative in time or space. Assume that $\partial_t g$, $\nabla g \in L^p(\Omega)$, $h \in L^q(\Omega)$, $1 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} \le 1$. Then,*

$$\|\partial(gh^\epsilon) - \partial(gh)^\epsilon\|_{L^r(\Omega)} \le C(\|\partial_t g\|_{L^p(\Omega)} + \|\nabla g\|_{L^p(\Omega)})\|h\|_{L^q(\Omega)},$$

where the constant $C > 0$ does not depend on ϵ, g and h, and r is determined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Moreover, $\partial(gh^\epsilon) - \partial(gh)^\epsilon$ converges to zero in $L^r(\Omega)$ as ϵ tends to zero if $r < \infty$.

Lemma 2 ([18])**.** *Let $g \in B_p^{\alpha,\infty}(\Omega)$, $h \in L^q(\Omega)$ and $1 \le p, q \le \infty$. Then,*

$$\|gh^\epsilon - (gh)^\epsilon\|_{L^r(\Omega)} \le C\epsilon^\alpha \|h\|_{L^q(\Omega)}\|g\|_{B_p^{\alpha,\infty}(\Omega)}$$

where the constant $C > 0$ does not depend on g and h, and with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\|gh^\epsilon - (gh)^\epsilon\|_{L^r(\Omega)} \le C\epsilon^\alpha \to 0$$

as ϵ tends to zero.

Lemma 3 ([20])**.** *Let $\mathbb{Y}$ be a separable Banach space and $\overline{\mathbb{K}} \subset \mathbb{R}^m$ be compact. Assume that $f_n : \overline{\mathbb{K}} \to \mathbb{Y}^*, n = 1, 2, \cdots$ is a sequence of measurable functions such that*

$$\operatorname*{ess\,sup}_{x\in\overline{\mathbb{K}}} \|f_n(x)\|_{\mathbb{Y}^*} \le M, \ n = 1, 2, \cdots .$$

In addition, assume that the family of (real) functions

$$\langle f_n, \Psi\rangle : x \mapsto \langle f_n(x), \Psi\rangle, \ x \in \overline{\mathbb{K}}, \ n = 1, 2, \cdots$$

be equi-continuous for any fixed Ψ belonging to a dense subset in the space $\mathbb{Y}$.

Then $f_n \in C(\overline{\mathbb{K}}; \mathbb{Y}^*_{weak})$, $n = 1, 2, \cdots$, and there exists $f \in C(\overline{\mathbb{K}}; \mathbb{Y}^*_{weak})$ such that

$$f_n \to f$$

in $C(\overline{\mathbb{K}}; \mathbb{Y}^*_{weak})$ as $n \to \infty$.

Next, we will give the energy conservation of a smooth solution of system (1).

Lemma 4. *If (ϱ, v) is the smooth solution of system (1), then the following energy equality holds*

$$\begin{aligned} E(t) =: &\int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho|v|^2 + P(\varrho)\Big)(t,x)dx \\ = &\int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho_0|v_0|^2 + P(\varrho_0)\Big)(x)dx =: E(0), \end{aligned}$$

where the function $P(\varrho) = \frac{\kappa}{\gamma-1}(\varrho^\gamma - \varrho_*^{\gamma-1}\varrho)$, $\varrho_* = \min_{(t,x)\in\mathbb{R}^+\times\mathbb{T}^d}\{\varrho(t,x)\}$.

Proof. To derive the energy equality of the compressible Euler equations, we assume that $(\varrho, \mathbf{v})$ is a smooth solution to system (1). Multiplying the mass and momentum equations of system (1) by $\mathbf{v}^2$ and $2\mathbf{v}$, respectively, we obtain

$$\partial_t\varrho\mathbf{v}^2 + \mathbf{v}\nabla\varrho\mathbf{v}^2 + (\varrho\nabla\cdot\mathbf{v})\mathbf{v}^2 = 0, \tag{8}$$

$$\varrho\partial_t\mathbf{v}^2 + 2\varrho(\mathbf{v}\cdot\nabla\mathbf{v})\mathbf{v} + 2\mathbf{v}\nabla p = 0. \tag{9}$$

Combining (8) and (9) and utilizing integration by parts, it yields that

$$\frac{\partial}{\partial t}\int_{\mathbb{T}^d}\varrho|\mathbf{v}|^2dx+\int_{\mathbb{T}^d}2\nabla p\mathbf{v}dx=0. \tag{10}$$

Define $p(\varrho)=\varrho P'(\varrho)-P(\varrho)$, from the pressure law $p(\varrho)=\kappa\varrho^\gamma$, $\gamma>1$, $P(\varrho)$ can be calulated as

$$P(\varrho)=\varrho\int_{\varrho_*}^{\varrho}\frac{p(z)}{z^2}dz=\varrho\Big(\frac{\kappa}{\gamma-1}z^{\gamma-1}\Big|_{z=\varrho_*}^{z=\varrho}\Big)=\frac{\kappa}{\gamma-1}(\varrho^\gamma-\varrho_*^{\gamma-1}\varrho). \tag{11}$$

Consequently,

$$\begin{aligned}\int_{\mathbb{T}^d}\nabla p\mathbf{v}dx&=-\int_{\mathbb{T}^d}p\nabla\cdot\mathbf{v}dx=-\int_{\mathbb{T}^d}(\varrho P'(\varrho)-P(\varrho))\nabla\cdot\mathbf{v}dx\\&=-\int_{\mathbb{T}^d}(P'(\varrho)\varrho\nabla\cdot\mathbf{v}+P'(\varrho)\nabla\varrho\mathbf{v})dx=-\int_{\mathbb{T}^d}\nabla\cdot(\varrho\mathbf{v})P'(\varrho)dx\\&=\int_{\mathbb{T}^d}\partial_t\varrho P'(\varrho)dx=\frac{\partial}{\partial t}\int_{\mathbb{T}^d}P(\varrho)dx,\end{aligned} \tag{12}$$

where we used the equality $\partial_t\varrho+\nabla\cdot(\varrho\mathbf{v})=0$. From (10) and (12), we obtain the energy equality

$$\frac{d}{dt}E(t)=\frac{d}{dt}\int_{\mathbb{T}^d}\Big(\frac{1}{2}\varrho|\mathbf{v}|^2+P(\varrho)\Big)dx=0.$$

This proves that $E(t)=E(0)$. $\square$

3. Main Results

In this section, we provide two results that ensure the energy conservation of system (1) by "trading" the regularity between the velocity and the density. The first type of result gives the density ϱ strong regularity and assumes that the velocity belongs to the Besov space. It is concluded that the energy conservation of system (1) if the Hölder exponent of $\mathbf{v}$ is greater than $1/3$. The second result is to give the velocity field $\mathbf{v}$ more regularity, which allows the existence of a less regular density ϱ. The detailed results are presented as follows.

Theorem 1. *Let $(\varrho,\boldsymbol{v})$ be a solution of (1) in the distributional sense. Assume $(\varrho,\boldsymbol{v})$ satisfy*

$$\begin{aligned}&\varrho\in L^\infty(\mathbb{I}\times\mathbb{T}^d),\ \boldsymbol{v}\in B_3^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d),\\&\partial_t\varrho\in L^q(\mathbb{I}\times\mathbb{T}^d),\ \nabla\varrho\in L^r(\mathbb{I}\times\mathbb{T}^d),\\&\nabla\sqrt{\varrho}\in L^\infty(\mathbb{I};L^{\frac{3}{2}}(\mathbb{T}^d)),\ \boldsymbol{v}\in L^s(\mathbb{I}\times\mathbb{T}^d),\end{aligned} \tag{13}$$

where $\alpha>\frac{1}{3}$, $\frac{1}{q}+\frac{3}{s}\le 1$, $\frac{1}{r}+\frac{3}{s}\le 1$. Then the conservation of energy holds in the point-wise sense, i.e., for all $t\in\mathbb{I}$, we have $E(t)=E(0)$, where

$$E(t)=\int_{\mathbb{T}^d}\Big(\frac{1}{2}\varrho|\boldsymbol{v}|^2+\frac{p(\varrho)}{\gamma-1}\Big)(t,x)dx.$$

Remark 1. *Applying the isentropic pressure law $p(\varrho)=\kappa\varrho^\gamma$ instead of the pressure p, we allow the existence of a vacuum state if $\gamma>1$.*

Remark 2. *The condition $\nabla\sqrt{\varrho}\in L^\infty(\mathbb{I};L^{\frac{3}{2}}(\mathbb{T}^d))$ is to ensure $\varrho^\gamma\in C(\mathbb{I};L^k_{weak}(\mathbb{T}^d))$ and $\sqrt{\varrho}\in C(\mathbb{I};L^k_{weak}(\mathbb{T}^d))$, $k>1$, which is crucial to derive energy conservation in a point-wise sense (this can be checked in the proof of Theorem 1). We can omit $\nabla\sqrt{\varrho}\in L^\infty(\mathbb{I};L^{\frac{3}{2}}(\mathbb{T}^d))$ in assumption (13)* *if energy conservation holds only in the distributional sense, which is different from the condition* (4) *in article [16]. In fact, $\varrho \boldsymbol{v} \in B_3^{\beta,\infty}(\mathbb{I} \times \mathbb{T}^d)$ is not included in*

$$\partial_t \varrho \in L^q(\mathbb{I} \times \mathbb{T}^d), \ \nabla \varrho \in L^r(\mathbb{I} \times \mathbb{T}^d).$$

Remark 3. *Constantin et al. [2] mollified the system* (3) *only in space, thus the velocity field $\boldsymbol{v}$ only needs Besov regularity in space (that is, $\boldsymbol{v} \in L^3(\mathbb{I}; B_3^{\alpha,\infty}(\mathbb{T}^d))$) to ensure that energy is conserved. However, here we convolve the system* (1) *in both time and space, then the condition $\boldsymbol{v} \in B_3^{\alpha,\infty}(\mathbb{I} \times \mathbb{T}^d)$ is natural.*

Remark 4. *Thanks to Besov embedding theorem, we observe that $B_3^{\alpha,\infty}(\mathbb{I} \times \mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{I}; L^3(\mathbb{T}^d))$, $\alpha > \frac{1}{3}$. Thus, the assumption $\boldsymbol{v} \in L^\infty(\mathbb{I}; L^3(\mathbb{T}^d))$, which has been used in the inequality* (28) *can be removed.*

Remark 5. *The significant difference between our result and those in [15–17] is that we can establish the conservation of energy in a point-wise sense on $\mathbb{I}$, whereas it is in the sense of distribution in [16,17], and we admit the existence of a vacuum state (if $\gamma > 1$), which is excluded in [15]. In addition, we can also remove the condition $p \in C^{1,(\gamma-1)}([\underline{\varrho}, \overline{\varrho}])$ in [17]. The price to pay is that the density ϱ is given more regularity conditions to ensure energy conservation. Thus, there is no direct correlation between our result and theirs in [16,17].*

Theorem 2. *Let $(\varrho, \boldsymbol{v})$ be a solution of* (1) *in the distributional sense. Assume $(\varrho, \boldsymbol{v})$ satisfy*

$$\begin{aligned} &\varrho \in L^\infty(\mathbb{I} \times \mathbb{T}^d), \\ &\boldsymbol{v} \in B_3^{\alpha,\infty}(\mathbb{I} \times \mathbb{T}^d), \ \nabla \cdot \boldsymbol{v} \in L^\infty(\mathbb{I}; L^1(\mathbb{T}^d)), \end{aligned} \tag{14}$$

where $\alpha > \frac{1}{2}$. Then the energy conservation holds in the point-wise sense, i.e., $E(t) = E(0)$ for all $t \in \mathbb{I}$.

Remark 6. *Compared with Theorem 1, we do not need to add any regularity condition on the density besides the assumption $\varrho \in L^\infty(\mathbb{I} \times \mathbb{T}^d)$, and the vacuum state of the system can also be presented if $\gamma > 1$.*

Remark 7. *Since this theorem requires more regularity assumptions for the velocity $\boldsymbol{v}$ to compensate for the roughness of the density ϱ, we need to add the condition $\nabla \cdot \boldsymbol{v} \in L^\infty(\mathbb{I}; L^1(\mathbb{T}^d))$ to guarantee $\varrho^\gamma \in C(\mathbb{I}; L^k_{weak}(\mathbb{T}^d))$ and $\sqrt{\varrho} \in C(\mathbb{I}; L^k_{weak}(\mathbb{T}^d))$, $k > 1$. The main difference between our result and [16,17] is that, similar to the previous result, we have the ability to establish the conservation of energy in a point-wise sense up to the initial time. If energy is conserved only in the distributional sense, the assumption $\nabla \cdot \boldsymbol{v} \in L^\infty(\mathbb{I}; L^1(\mathbb{T}^d))$ can be replaced by the weaker assumption $\nabla \cdot \boldsymbol{v} \in L^1(\mathbb{I} \times \mathbb{T}^d)$.*

Remark 8. *System* (1) *can become nonhomogeneous noncompressible Euler equations by adding $\nabla \cdot \boldsymbol{v} = 0$. The energy conservation for the noncompressible Euler equations was investigated in [15,16,18]. Moreover, Chen and Yu [18] tell us that if*

$$\begin{aligned} &\varrho \in L^\infty(\mathbb{I} \times \mathbb{T}^d), \ \boldsymbol{v} \in B_p^{\beta,\infty}(\mathbb{I}; B_q^{\alpha,\infty}(\mathbb{T}^d)), \\ &\sqrt{\varrho}\boldsymbol{v} \in L^\infty(\mathbb{I}; L^2(\mathbb{T}^d)), \ \boldsymbol{v}_0 \in L^2(\mathbb{T}^d), \end{aligned}$$

where $\alpha, \beta > \frac{1}{2}$, then the energy equality conserves in a point-wise sense on $\mathbb{I}$.

Proof of Theorem 1. To prove Theorem 1, by mollifying the system (1) both in space and time, we obtain

$$\partial_t (\varrho \mathbf{v})^\epsilon + \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v})^\epsilon + \nabla p^\epsilon = 0, \tag{15}$$

$$\partial_t \varrho^\epsilon + \nabla \cdot (\varrho \mathbf{v})^\epsilon = 0. \tag{16}$$

Let $\varphi(t) \in \mathcal{D}(0, +\infty)$ be a test function, where $\mathcal{D}(0, +\infty)$ denotes the set of functions that are smooth and compactly supported on $(0, +\infty)$. To obtain the energy equality, Equation (15) is multiplied by the test function $\varphi(t)\mathbf{v}^\epsilon$ and integrated in time-space, and we have

$$\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \mathbf{v}^\epsilon [\partial_t (\varrho \mathbf{v})^\epsilon + \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v})^\epsilon + \nabla p^\epsilon] dxdt = 0. \tag{17}$$

Next, we will deal with each term in Equation (17) by Equation (16) and the appropriate commutators. The first term in (17) can be written by

$$\begin{aligned}
&\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \mathbf{v}^\epsilon \partial_t (\varrho \mathbf{v})^\epsilon dxdt \\
&= \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \mathbf{v}^\epsilon [\partial_t (\varrho \mathbf{v})^\epsilon - \partial_t (\varrho \mathbf{v}^\epsilon)] dxdt + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \mathbf{v}^\epsilon \partial_t (\varrho \mathbf{v}^\epsilon) dxdt \\
&=: \mathcal{I}_{1\epsilon} + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \partial_t \varrho |\mathbf{v}^\epsilon|^2 dxdt + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho \partial_t \frac{|\mathbf{v}^\epsilon|^2}{2} dxdt.
\end{aligned} \tag{18}$$

The second term of Equation (17) can be calculated as

$$\begin{aligned}
&\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \mathbf{v}^\epsilon \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v})^\epsilon dxdt \\
&= \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \mathbf{v}^\epsilon [\nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v})^\epsilon - \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v}^\epsilon)] dxdt + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \mathbf{v}^\epsilon \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v}^\epsilon) dxdt \\
&= \mathcal{I}_{2\epsilon} + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \nabla \cdot (\varrho \mathbf{v}) |\mathbf{v}^\epsilon|^2 dxdt + \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho \mathbf{v} \cdot \nabla \frac{|\mathbf{v}^\epsilon|^2}{2} dxdt \\
&=: \mathcal{I}_{2\epsilon} - \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \partial_t \varrho \frac{|\mathbf{v}^\epsilon|^2}{2} dxdt.
\end{aligned} \tag{19}$$

From Lemma 4, without loss of generality, we can deduce $P(\varrho) = \frac{\kappa}{\gamma-1} \varrho^\gamma$ by setting $\varrho_* = 0$ in Equation (11). By the isentropic pressure law $p(\varrho) = \kappa \varrho^\gamma$, the pressure term in (17) can be treated as

$$\begin{aligned}
&\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \mathbf{v}^\epsilon (\nabla p)^\epsilon dxdt \\
&= \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi [(\varrho^{\gamma-1} \nabla \varrho)^\epsilon - (\varrho^{\gamma-1})^\epsilon \nabla \varrho] \mathbf{v}^\epsilon dxdt + \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi (\varrho^{\gamma-1})^\epsilon \nabla \varrho \mathbf{v}^\epsilon dxdt \\
&= \mathcal{I}_{3\epsilon} + \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi [(\varrho^{\gamma-1})^\epsilon \mathbf{v}^\epsilon - \varrho^{\gamma-1} \mathbf{v}] \nabla \varrho dxdt + \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} \mathbf{v} \nabla \varrho dxdt \\
&=: \mathcal{I}_{3\epsilon} + \mathcal{I}_{4\epsilon} + \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} \nabla \varrho \mathbf{v} dxdt.
\end{aligned} \tag{20}$$

Using the mass equation and the periodicity of the domain $\mathbb{T}^d$, we deduce that

$$\begin{aligned}
&\kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \varrho^{\gamma-1} \nabla \varrho \mathbf{v} dxdt \\
&= \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} (\nabla \cdot (\varrho \mathbf{v}) - \varrho \nabla \cdot \mathbf{v}) dxdt \\
&= -\kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} \partial_t \varrho dxdt - \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^\gamma \nabla \cdot \mathbf{v} dxdt \\
&= -\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \partial_t p(\varrho) dxdt + \kappa\gamma^2 \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} \nabla \varrho \mathbf{v} dxdt.
\end{aligned}$$

This equality means

$$\kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \varrho^{\gamma-1} \nabla \varrho \mathbf{v} dxdt = \frac{1}{\gamma - 1} \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \partial_t p(\varrho) dxdt.$$

Thus, combining (18)–(20), Equation (17) can be reduced as follows

$$-\int_{\mathbb{I}}\int_{\mathbb{T}^d}\partial_t\varphi\Big(\frac{1}{2}\varrho|\mathbf{v}^\epsilon|^2+\frac{p(\varrho)}{\gamma-1}\Big)dxdt+\sum_{i=1}^{4}\mathcal{I}_{i\epsilon}=0. \tag{21}$$

To ensure that the energy equality is conserved in the distributional sense, our following work will show that $\sum_{i=1}^4\mathcal{I}_{i\epsilon}\to 0$ of (21) as ϵ tends to zero.

Utilizing Lemma 1 and the Hölder inequality, $\mathcal{I}_{1\epsilon}$ can be estimated as follows

$$\begin{aligned}|\mathcal{I}_{1\epsilon}| &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)\mathbf{v}^\epsilon(\partial_t(\varrho\mathbf{v})^\epsilon-\partial_t(\varrho\mathbf{v}^\epsilon))dxdt| \\ &\le \|\varphi\|_{L^\infty(\mathbb{I})}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|(\partial_t(\varrho\mathbf{v})^\epsilon-\partial_t(\varrho\mathbf{v}^\epsilon))\mathbf{v}^\epsilon|dxdt \\ &\lesssim (\|\partial_t\varrho\|_{L^q(\mathbb{I}\times\mathbb{T}^d)}+\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)})\|\mathbf{v}\|^2_{L^s(\mathbb{I}\times\mathbb{T}^d)},\end{aligned}$$

where $\frac{1}{q}+\frac{2}{s}\le 1$ and $\frac{1}{r}+\frac{2}{s}\le 1$. Moreover, $\mathcal{I}_{1\epsilon}\to 0$ as $\epsilon\to 0$.

In order to estimate $\mathcal{I}_{2\epsilon}$, we will divide $\mathcal{I}_{2\epsilon}$ into two parts and utilize the following commutator

$$(gh)^\epsilon-gh^\epsilon=[(gh)^\epsilon-g^\epsilon h^\epsilon]+[g^\epsilon h^\epsilon-gh^\epsilon]. \tag{22}$$

where g and h are real functions. Similar to the method used by Constantin et al. in [2], we define

$$r_\epsilon(g,h)=\int\eta_\epsilon(\mu,\tau)(\delta_{\mu,\tau}g(t,x)\delta_{\mu,\tau}h(t,x))d\mu d\tau,$$

where

$$\delta_{\mu,\tau}g(t,x)=g(t-\mu,x-\tau)-g(t,x),\ \delta_{\mu,\tau}h(t,x)=h(t-\mu,x-\tau)-h(t,x)$$

Then, one can easily check the following equality holds

$$(gh)^\epsilon-g^\epsilon h^\epsilon=r_\epsilon(g,h)-(g-g^\epsilon)(h-h^\epsilon). \tag{23}$$

We observe that $\mathcal{I}_{2\epsilon}$ can be handled as

$$\begin{aligned}\mathcal{I}_{2\epsilon} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)[\nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon-\nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v}^\epsilon)]\mathbf{v}^\epsilon dxdt \\ &= \int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon-\varrho\mathbf{v}\otimes\mathbf{v}^\epsilon]:\nabla\mathbf{v}^\epsilon dxdt \\ &= -\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\varrho[(\mathbf{v}\otimes\mathbf{v})^\epsilon-\mathbf{v}\otimes\mathbf{v}^\epsilon]:\nabla\mathbf{v}^\epsilon dxdt \\ &\quad+\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[\nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon-\nabla\cdot(\varrho(\mathbf{v}\otimes\mathbf{v})^\epsilon)]\mathbf{v}^\epsilon dxdt \\ &=: \mathcal{I}^1_{2\epsilon}+\mathcal{I}^2_{2\epsilon}.\end{aligned}$$

In view of (22), we will divide $\mathcal{I}^1_{2\epsilon}$ into two parts and estimate them separately, that is,

$$\begin{aligned}\mathcal{I}^1_{2\epsilon} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)\varrho[(\mathbf{v}\otimes\mathbf{v})^\epsilon-\mathbf{v}\otimes\mathbf{v}^\epsilon]:\nabla\mathbf{v}^\epsilon dxdt \\ &\le \int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\varrho((\mathbf{v}\otimes\mathbf{v})^\epsilon-\mathbf{v}^\epsilon\otimes\mathbf{v}^\epsilon):\nabla\mathbf{v}^\epsilon dxdt \\ &\quad+\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\varrho(\mathbf{v}^\epsilon\otimes\mathbf{v}^\epsilon-\mathbf{v}\otimes\mathbf{v}^\epsilon):\nabla\mathbf{v}^\epsilon dxdt \\ &=: \mathcal{I}^{11}_{2\epsilon}+\mathcal{I}^{12}_{2\epsilon}.\end{aligned}$$

Applying equality (23) to $\mathcal{I}_{2\epsilon}^{11}$, it follows that

$$
\begin{aligned}
|\mathcal{I}_{2\epsilon}^{11}| &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)\varrho(\mathbf{v}^\epsilon\otimes\mathbf{v}^\epsilon-(\mathbf{v}\otimes\mathbf{v})^\epsilon):\nabla\mathbf{v}^\epsilon dxdt| \\
&\lesssim |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varrho\nabla\mathbf{v}^\epsilon:r_\epsilon(\mathbf{v},\mathbf{v})dxdt|+|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varrho\nabla\mathbf{v}^\epsilon:((\mathbf{v}-\mathbf{v}^\epsilon)\otimes(\mathbf{v}-\mathbf{v}^\epsilon))dxdt| \\
&\lesssim \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\nabla\mathbf{v}^\epsilon\|_{L^3(\mathbb{I}\times\mathbb{T}^d)}(\|r_\epsilon(\mathbf{v},\mathbf{v})\|_{L^{\frac{3}{2}}(\mathbb{I}\times\mathbb{T}^d)}+\|\mathbf{v}-\mathbf{v}^\epsilon\|^2_{L^3(\mathbb{I}\times\mathbb{T}^d)}) \\
&\lesssim \epsilon^{3\alpha-1}\|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|^3_{B_3^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)}\to 0
\end{aligned}
\tag{24}
$$

as $\epsilon\to 0$ for any $\alpha>\frac{1}{3}$, where we have used $\|r_\epsilon\|_{L^{\frac{3}{2}}}\leq\epsilon^{2\alpha}\|\mathbf{v}\|^2_{B_3^{\alpha,\infty}}$ which is guaranteed by article [2]. $\mathcal{I}_{2\epsilon}^{12}$ can be estimated by Lemma 1 as follows

$$
\begin{aligned}
\mathcal{I}_{2\epsilon}^{12} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)\varrho(\mathbf{v}^\epsilon\otimes\mathbf{v}^\epsilon-\mathbf{v}\otimes\mathbf{v}^\epsilon):\nabla\mathbf{v}^\epsilon dxdt \\
&= \frac{1}{2}\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\varrho(\mathbf{v}^\epsilon-\mathbf{v})\nabla|\mathbf{v}^\epsilon|^2dxdt \\
&= -\frac{1}{2}\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\nabla\cdot(\varrho\mathbf{v}^\epsilon-\varrho\mathbf{v})|\mathbf{v}^\epsilon|^2dxdt \\
&= -\frac{1}{2}\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\nabla\cdot[\varrho\mathbf{v}^\epsilon-(\varrho\mathbf{v})^\epsilon+(\varrho\mathbf{v})^\epsilon-\varrho\mathbf{v}]|\mathbf{v}^\epsilon|^2dxdt \\
&= \frac{1}{2}\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\nabla\cdot((\varrho\mathbf{v})^\epsilon-\varrho\mathbf{v}^\epsilon)|\mathbf{v}^\epsilon|^2dxdt+\frac{1}{2}\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(\partial_t\varrho^\epsilon-\partial_t\varrho)|\mathbf{v}^\epsilon|^2dxdt \\
&\lesssim (\|\partial_t\varrho\|_{L^q(\mathbb{I}\times\mathbb{T}^d)}+\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)})\|\mathbf{v}\|^3_{L^s(\mathbb{I}\times\mathbb{T}^d)}+\|\partial_t\varrho\|_{L^q(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|^2_{L^s(\mathbb{I}\times\mathbb{T}^d)},
\end{aligned}
\tag{25}
$$

and $\mathcal{I}_{2\epsilon}^{12}\to 0$ as $\epsilon\to 0$ for any $\frac{1}{q}+\frac{3}{s}\leq 1$, $\frac{1}{r}+\frac{3}{s}\leq 1$. Similarly,

$$
\begin{aligned}
|\mathcal{I}_{2\epsilon}^{2}| &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)[\nabla\cdot(\varrho(\mathbf{v}\otimes\mathbf{v})^\epsilon)-\nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon]\mathbf{v}^\epsilon dxdt| \\
&\lesssim (\|\partial_t\varrho\|_{L^q(\mathbb{I}\times\mathbb{T}^d)}+\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)})\|\mathbf{v}\|^3_{L^s(\mathbb{I}\times\mathbb{T}^d)},
\end{aligned}
\tag{26}
$$

and $\mathcal{I}_{2\epsilon}^{2}\to 0$ as $\epsilon\to 0$ for any $\frac{1}{q}+\frac{3}{s}\leq 1$, $\frac{1}{r}+\frac{3}{s}\leq 1$.

Therefore, combining (24), (25) and (26), as ϵ tends to zero for any $\alpha>\frac{1}{3}$, one shows that $\mathcal{I}_{2\epsilon}\to 0$.

The term $\mathcal{I}_{3\epsilon}$ can be computed as

$$
\begin{aligned}
\mathcal{I}_{3\epsilon} &= \kappa\gamma\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi(t)[(\varrho^{\gamma-1}\nabla\varrho)^\epsilon-(\varrho^{\gamma-1})^\epsilon\nabla\varrho]\mathbf{v}^\epsilon dxdt \\
&\lesssim |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\int_{\mathcal{B}_\epsilon(\mu,\tau)}\eta_\epsilon(\mu,\tau)\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x-\tau)-\nabla\varrho(t,x))d\mu d\tau\mathbf{v}^\epsilon dxdt| \\
&\lesssim |\int_{\mathbb{I}}\int_{\mathbb{T}^d}\int_{\mathcal{B}_\epsilon(\mu,\tau)}\eta_\epsilon(\mu,\tau)\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x-\tau)-\nabla\varrho(t-\mu,x))d\mu d\tau\mathbf{v}^\epsilon dxdt| \\
&\quad +|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\int_{\mathcal{B}_\epsilon(\mu,\tau)}\eta_\epsilon(\mu,\tau)\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x)-\nabla\varrho(t,x))d\mu d\tau\mathbf{v}^\epsilon dxdt| \\
&\lesssim \sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x-\tau)-\nabla\varrho(t-\mu,x))\mathbf{v}^\epsilon|dxdt \\
&\quad +\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x)-\nabla\varrho(t,x))\mathbf{v}^\epsilon|dxdt \\
&\lesssim \|\varrho\|^{\gamma-1}_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|_{L^s(\mathbb{I}\times\mathbb{T}^d)},
\end{aligned}
$$

where $\mathcal{B}_\epsilon(\mu,\tau)$ is a open ball with radius ϵ. Since $C_0^\infty(\mathbb{I}\times\mathbb{T}^d)$ is dense in $L^r(\mathbb{I}\times\mathbb{T}^d)$ for any $r<\infty$, we have

$$\lim_{\epsilon\to 0}\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\|\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x-\tau)-\nabla\varrho(t-\mu,x))\|_{L^r(\mathbb{I}\times\mathbb{T}^d)}=0$$

and

$$\lim_{\epsilon\to 0}\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\|\varrho^{\gamma-1}(t-\mu,x-\tau)(\nabla\varrho(t-\mu,x)-\nabla\varrho(t,x))\|_{L^r(\mathbb{I}\times\mathbb{T}^d)}=0,$$

which means that $\mathcal{I}_{3\epsilon}\to 0$ as $\epsilon\to 0$, provided that $\frac{1}{r}+\frac{1}{s}\le 1$.

The term $\mathcal{I}_{4\epsilon}$ can be treated as

$$\begin{aligned}\mathcal{I}_{4\epsilon}&=\kappa\gamma\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1})^\epsilon\mathbf{v}^\epsilon-\varrho^{\gamma-1}\mathbf{v}]\nabla\varrho dxdt\\&=\kappa\gamma\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1})^\epsilon\mathbf{v}^\epsilon-(\varrho^{\gamma-1}\mathbf{v})^\epsilon]\nabla\varrho dxdt\\&\quad+\kappa\gamma\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1}\mathbf{v})^\epsilon-\varrho^{\gamma-1}\mathbf{v}]\nabla\varrho=dxd\\&=:\mathcal{I}_{4\epsilon}^1+\mathcal{I}_{4\epsilon}^2.\end{aligned}$$

Note that the point-wise identity (23), replacing g and h with $\varrho^{\gamma-1}$ and $\mathbf{v}$, we have

$$\begin{aligned}\mathcal{I}_{4\epsilon}^1&=\kappa\gamma\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1})^\epsilon\mathbf{v}^\epsilon-(\varrho^{\gamma-1}\mathbf{v})^\epsilon]\nabla\varrho dxdt\\&\le\kappa\gamma|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1})^\epsilon-\varrho^{\gamma-1}](\mathbf{v}^\epsilon-\mathbf{v})\nabla\varrho dxdt|\\&\quad+\kappa\gamma|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\int_{\mathcal{B}_\epsilon(\mu,\tau)}\eta_\epsilon(\mu,\tau)[\varrho^{\gamma-1}(t-\mu,x-\tau)\\&\quad-\varrho^{\gamma-1}(t,x)](\mathbf{v}(t-\mu,x-\tau)-\mathbf{v}(t,x))d\mu d\tau\nabla\varrho dxdt|\\&=:\mathcal{I}_{4\epsilon}^{11}+\mathcal{I}_{4\epsilon}^{12}.\end{aligned}$$

Utlizing the property of convolution, we know that

$$\begin{aligned}\mathcal{I}_{4\epsilon}^{11}&=\kappa\gamma|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^{\gamma-1})^\epsilon-\varrho^{\gamma-1}](\mathbf{v}^\epsilon-\mathbf{v})\nabla\varrho dxdt|\\&\lesssim\|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}^{\gamma-1}\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|_{L^s(\mathbb{I}\times\mathbb{T}^d)},\end{aligned}$$

and $\mathcal{I}_{4\epsilon}^{11}$ tends to zero as $\epsilon\to 0$, provided that $\frac{1}{r}+\frac{1}{s}\le 1$. Moreover, owning to the density of $C_0^\infty(\mathbb{I}\times\mathbb{T}^d)$ in $L^r(\mathbb{I}\times\mathbb{T}^d)$ for any $r<\infty$, $\mathcal{I}_{4\epsilon}^{12}$ can be estimated as

$$\begin{aligned}&|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi\int_{\mathcal{B}_\epsilon(\mu,\tau)}\eta_\epsilon[\varrho^{\gamma-1}(t-\mu,x-\tau)-\varrho^{\gamma-1}(t,x)](\mathbf{v}(t-\mu,x-\tau)-\mathbf{v}(t,x))d\mu d\tau\nabla\varrho dxdt|\\&\lesssim\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|[\varrho^{\gamma-1}(t-\mu,x-\tau)-\varrho^{\gamma-1}(t-\mu,x)](\mathbf{v}(t-\mu,x-\tau)-\mathbf{v}(t-\mu,x))\nabla\varrho|dxdt\\&\quad+\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|[\varrho^{\gamma-1}(t-\mu,x-\tau)-\varrho^{\gamma-1}(t-\mu,x)](\mathbf{v}(t-\mu,x)-\mathbf{v}(t,x))\nabla\varrho|dxdt\\&\quad+\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|[\varrho^{\gamma-1}(t-\mu,x)-\varrho^{\gamma-1}(t,x)](\mathbf{v}(t-\mu,x-\tau)-\mathbf{v}(t-\mu,x))\nabla\varrho|dxdt\\&\quad+\sup_{\mathcal{B}_\epsilon(\mu,\tau)}\int_{\mathbb{I}}\int_{\mathbb{T}^d}|[\varrho^{\gamma-1}(t-\mu,x)-\varrho^{\gamma-1}(t,x)](\mathbf{v}(t-\mu,x)-\mathbf{v}(t,x))\nabla\varrho|dxdt\\&\lesssim\|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}^{\gamma-1}\|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|_{L^s(\mathbb{I}\times\mathbb{T}^d)},\end{aligned}$$

and $\mathcal{I}_{4\epsilon}^{12} \to 0$ as $\epsilon \to 0$ for any $\frac{1}{r} + \frac{1}{s} \leq 1$.

On the other hand,

$$\begin{aligned}\mathcal{I}_{4\epsilon}^{2} &= \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi[(\varrho^{\gamma-1}\mathbf{v})^{\epsilon} - \varrho^{\gamma-1}\mathbf{v}]\nabla\varrho dxdt \\ &\lesssim \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}^{\gamma-1} \|\nabla\varrho\|_{L^r(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}\|_{L^s(\mathbb{I}\times\mathbb{T}^d)},\end{aligned}$$

and $\mathcal{I}_{4\epsilon}^{2} \to 0$ as ϵ tends to zero.

Therefore, letting $\epsilon \to 0$, from (13) and (21) we can obtain

$$-\int_{\mathbb{I}} \int_{\mathbb{T}^d} \partial_t\varphi\Big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{p(\varrho)}{\gamma-1}\Big)dxdt = 0, \tag{27}$$

here $\varphi(t) \in \mathcal{D}(0, +\infty)$. From the previous assumptions, we can know that

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{p(\varrho)}{\gamma-1}\Big)dx = 0$$

is established in a distributional sense.

Next, we will prove the energy is conserved in a point-wise sense up to the initial time. For this, the test function $\varphi(t)$ needs to be extended to $\varphi(t) \in \mathcal{D}(\theta, +\infty)$, where θ is fixed and $\theta < -1$. Using $p(\varrho) = \kappa\varrho^{\gamma}$, the energy equality can be written as

$$E(t) = \int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{\kappa}{\gamma-1}\varrho^{\gamma}\Big)dx.$$

Thus, we only need to show the continuity of ϱ^{γ} and $\sqrt{\varrho}\mathbf{v}$ in the strong topology as t tends to 0^+. For any fixed $\psi(x) \in C_0^{\infty}(\mathbb{T}^d)$, one obtains that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d} \psi(x)\varrho^{\gamma}(t,x)dx = -\gamma\int_{\mathbb{T}^d} \psi(x)\varrho^{\gamma-1}\nabla\cdot(\varrho\mathbf{v})dx \\ &\lesssim |\int_{\mathbb{T}^d} \psi(x)\varrho^{\gamma-1}\nabla\varrho\cdot\mathbf{v} + \nabla\psi(x)\varrho^{\gamma}\mathbf{v}dx| \\ &\lesssim (\|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}^{\gamma-1}\|\nabla\varrho\|_{L^\infty(\mathbb{I};L^{\frac{3}{2}}(\mathbb{T}^d))} + \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}^{\gamma})\|\mathbf{v}\|_{L^\infty(\mathbb{I};L^3(\mathbb{T}^d))}\end{aligned} \tag{28}$$

is bounded due to (13). By Lemma 3, thus, we have

$$\varrho^{\gamma} \in C(\mathbb{I}; L^k_{weak}(\mathbb{T}^d)),\ k > 1. \tag{29}$$

In addition, utilize the convexty of $\varrho \mapsto \varrho^{\gamma}$, one has

$$\int_{\mathbb{T}^d} \varrho^{\gamma}(t_0)dx \leq \liminf_{t\to t_0^+}\int_{\mathbb{T}^d} \varrho^{\gamma}(t)dx \ \text{ for all } t_0 \geq 0. \tag{30}$$

On the other hand, we see

$$\begin{aligned}&\limsup_{t\to 0^+}\int_{\mathbb{T}^d} |\sqrt{\varrho}\mathbf{v} - \sqrt{\varrho_0}\mathbf{v}_0|^2 dx \\ &\leq \limsup_{t\to 0^+}\int_{\mathbb{T}^d} 2\sqrt{\varrho_0}\mathbf{v}_0(\sqrt{\varrho_0}\mathbf{v}_0 - \sqrt{\varrho}\mathbf{v})dx + \limsup_{t\to 0^+}\int_{\mathbb{T}^d}\frac{2\kappa}{\gamma-1}(\varrho_0^{\gamma} - \varrho^{\gamma})dx \\ &\quad + \limsup_{t\to 0^+}\Big[\int_{\mathbb{T}^d}\Big(\varrho|\mathbf{v}|^2 + \frac{2\kappa}{\gamma-1}\varrho^{\gamma}\Big)dx - \int_{\mathbb{T}^d}\Big(\varrho_0|\mathbf{v}_0|^2 + \frac{2\kappa}{\gamma-1}\varrho_0^{\gamma}\Big)dx\Big].\end{aligned}$$

From inequality (30) and $E(0) \geq E(t)$ for all $t \in \mathbb{I}$, it yields that

$$\begin{aligned}
&\limsup_{t\to 0^+} \int_{\mathbb{T}^d} |\sqrt{\varrho}\mathbf{v} - \sqrt{\varrho_0}\mathbf{v}_0|^2 dx \\
&\leq 2\limsup_{t\to 0^+} \int_{\mathbb{T}^d} \sqrt{\varrho_0}\mathbf{v}_0(\sqrt{\varrho_0}\mathbf{v}_0 - \sqrt{\varrho}\mathbf{v}) dx \\
&\leq 2\limsup_{t\to 0^+} \int_{\mathbb{T}^d} \mathbf{v}_0(\varrho_0\mathbf{v}_0 - \varrho\mathbf{v}) dx + 2\limsup_{t\to 0^+} \int_{\mathbb{T}^d} \sqrt{\varrho}\mathbf{v}\mathbf{v}_0(\sqrt{\varrho} - \sqrt{\varrho_0}) dx \\
&= R_1 + R_2.
\end{aligned}$$

In order to show the continuity of $\sqrt{\varrho}\mathbf{v}$ in the strong topology as t tends to 0^+, we will consider the continuity of $\varrho\mathbf{v}$ and $\sqrt{\varrho}$ as t goes to 0^+. Applying the momentum equation of (1), one obtains

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} \widetilde{\psi}(x)(\varrho\mathbf{v})(t,x) dx \\
&= \int_{\mathbb{T}^d} \nabla\widetilde{\psi}(x) : (\varrho\mathbf{v}\otimes\mathbf{v}) dx + \int_{\mathbb{T}^d} \frac{\kappa}{\gamma-1}\nabla\cdot\widetilde{\psi}(x)\varrho^\gamma dx \\
&\lesssim \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|^2_{L^\infty(\mathbb{I};L^2(\mathbb{T}^d))} + \|\varrho\|^\gamma_{L^\infty(\mathbb{I}\times\mathbb{T}^d)},
\end{aligned}$$

which is bounded due to (13). Thus, we have

$$\varrho\mathbf{v} \in C(\mathbb{I}; L^2_{weak}(\mathbb{T}^d)). \tag{31}$$

Moreover, by virtue of (13) and (31), it is given by $R_1 = 0$. Similarly,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{T}^d} \psi(x)\sqrt{\varrho}(t,x) dx = -\int_{\mathbb{T}^d} \psi(x)\Big(\mathbf{v}\nabla\sqrt{\varrho} + \frac{1}{2}\sqrt{\varrho}\nabla\cdot\mathbf{v}\Big) dx \\
&= -\int_{\mathbb{T}^d} \Big(\psi(x)\nabla\sqrt{\varrho} - \frac{1}{2}\nabla(\psi(x)\sqrt{\varrho})\Big)\mathbf{v} dx = \frac{1}{2}\int_{\mathbb{T}^d} [\nabla\cdot\psi(x)\sqrt{\varrho} - \psi(x)\nabla\sqrt{\varrho}]\mathbf{v} dx \\
&\lesssim (\|\nabla\sqrt{\varrho}\|_{L^\infty(\mathbb{I};L^{\frac{3}{2}}(\mathbb{T}^d))} + \|\varrho\|^{\frac{1}{2}}_{L^\infty(\mathbb{I}\times\mathbb{T}^d)})\|\mathbf{v}\|_{L^\infty(\mathbb{I};L^3(\mathbb{T}^d))}
\end{aligned}$$

is bounded due to (13). Thus,

$$\sqrt{\varrho} \in C(\mathbb{I}; L^k_{weak}(\mathbb{T}^d)),\ k > 1. \tag{32}$$

From (13) and (32), we know that $R_2 = 0$. Thus, we deduce

$$\lim_{t\to 0^+} \int_{\mathbb{T}^d} |\sqrt{\varrho}\mathbf{v} - \sqrt{\varrho_0}\mathbf{v}_0|^2 dx = 0.$$

Similarly, we have the right temporal continuity of $\sqrt{\varrho}\mathbf{v}$ in $L^2(\mathbb{T}^d)$, that is,

$$\lim_{t\to t_0^+} \int_{\mathbb{T}^d} |(\sqrt{\varrho}\mathbf{v})(t) - (\sqrt{\varrho}\mathbf{v})(t_0)|^2 dx = 0 \ \text{ for all } t_0 \geq 0. \tag{33}$$

Moreover, by virtue of (33) and inequality $E(0) \geq E(t)$ for all $t \in \mathbb{I}$, we obtain

$$\limsup_{t\to t_0^+} \int_{\mathbb{T}^d} \varrho^\gamma(t) dx \leq \int_{\mathbb{T}^d} \varrho^\gamma(t_0) dx \ \text{ for all } t_0 \geq 0. \tag{34}$$

Therefore, combining (29), (30) and (34), we obtain that

$$\lim_{t\to t_0^+} \int_{\mathbb{T}^d} |\varrho^\gamma(t) - \varrho^\gamma(t_0)| dx = 0 \ \text{ for all } t_0 \geq 0. \tag{35}$$

Finally, we choose positive ξ and ϑ for any $t_0 > 0$ such that $\xi + \vartheta < t_0$ and define a time cut-off function

$$\Gamma_{\xi}(t) = \begin{cases} 0 & 0 \le t \le \xi, \\ \frac{t-\xi}{\vartheta} & \xi \le t \le \xi + \vartheta, \\ 1 & \xi + \vartheta \le t \le t_0, \\ \frac{\vartheta + t_0 - t}{\vartheta} & t_0 \le t \le t_0 + \vartheta, \\ 0 & t \ge t_0 + \vartheta. \end{cases}$$

Utilizing $\Gamma_{\xi}(t)$ instead of $\partial_t \varphi(t)$ in Equation (27), we can obtain

$$\frac{1}{\vartheta} \int_{\xi}^{\xi+\vartheta} \int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{\kappa}{\gamma-1}\varrho^{\gamma}\Big) dx d\mu - \frac{1}{\vartheta} \int_{t_0}^{t_0+\vartheta} \int_{\mathbb{T}^d} \Big(\frac{1}{2}\varrho|\mathbf{v}|^2 + \frac{\kappa}{\gamma-1}\varrho^{\gamma}\Big) dx d\mu = 0.$$

Letting $\vartheta \to 0$, according to (33) and (35), it follows that $E(\xi) - E(t_0) = 0$. Furthermore, sending $\xi \to 0$, it can be deduced that

$$E(t_0) - E(0) = 0$$

for all $t_0 \in \mathbb{I}$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Following the method of the previous section, one has

$$-\int_{\mathbb{I}} \int_{\mathbb{T}^d} \partial_t \varphi \Big(\frac{1}{2}\varrho^{\epsilon}|\mathbf{v}^{\epsilon}|^2 + \frac{p(\varrho)}{\gamma-1}\Big) dx dt + \sum_{i}^{3} \mathcal{J}_{i\epsilon} = 0,$$

where the pressure term is calculated as follows

$$\begin{aligned} &\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t)(\nabla p)^{\epsilon} \mathbf{v}^{\epsilon} dx dt \\ &= -\kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(\varrho^{\gamma})^{\epsilon} \nabla \cdot \mathbf{v}^{\epsilon} dx dt \\ &= -\kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi[(\varrho^{\gamma})^{\epsilon} \nabla \cdot \mathbf{v}^{\epsilon} - \varrho^{\gamma} \nabla \cdot \mathbf{v}] dx dt - \kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma} \nabla \cdot \mathbf{v} dx dt \\ &=: \mathcal{J}_{3\epsilon} - \kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma} \nabla \cdot \mathbf{v} dx dt. \end{aligned}$$

Applying the mass equation and the periodicity of $\mathbb{T}^d$, we can obtain

$$\begin{aligned} -\kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma} \nabla \cdot \mathbf{v} dx dt &= \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} \nabla \varrho \mathbf{v} dx dt \\ &= \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma-1} (\nabla \cdot (\varrho \mathbf{v}) - \varrho \nabla \cdot \mathbf{v}) dx dt \\ &= -\int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \partial_t p(\varrho) dx dt - \kappa\gamma \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi \varrho^{\gamma} \nabla \cdot \mathbf{v} dx dt. \end{aligned}$$

This equality implies that

$$-\kappa \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \varrho^{\gamma} \nabla \cdot \mathbf{v} dx dt = \frac{1}{\gamma-1} \int_{\mathbb{I}} \int_{\mathbb{T}^d} \varphi(t) \partial_t p(\varrho) dx dt.$$

In the same method as the previous Theorem 1, we need to show $\sum_i^3 \mathcal{J}_{i\epsilon} \to 0$ as ϵ goes to zero.

We handle the term $\mathcal{J}_{1\epsilon}$ as

$$\begin{aligned}
\mathcal{J}_{1\epsilon} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d} \varphi(t)[\partial_t(\varrho\mathbf{v})^\epsilon - \partial_t(\varrho^\epsilon\mathbf{v}^\epsilon)]\mathbf{v}^\epsilon dxdt \\
&= -\int_{\mathbb{I}}\int_{\mathbb{T}^d} \partial_t\varphi[(\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}^\epsilon]\mathbf{v}^\epsilon dxdt - \int_{\mathbb{I}}\int_{\mathbb{T}^d} \varphi[(\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}^\epsilon]\partial_t\mathbf{v}^\epsilon dxdt \\
&=: \mathcal{J}_{1\epsilon}^1 + \mathcal{J}_{1\epsilon}^2.
\end{aligned}$$

The first term of the above equality can be calculated as

$$\begin{aligned}
|\mathcal{J}_{1\epsilon}^1| &\lesssim \int_{\mathbb{I}}\int_{\mathbb{T}^d} |(\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}||\mathbf{v}^\epsilon|dxdt + \int_{\mathbb{I}}\int_{\mathbb{T}^d} |\varrho^\epsilon\mathbf{v} - \varrho^\epsilon\mathbf{v}^\epsilon||\mathbf{v}^\epsilon|dxdt \\
&=: \mathcal{J}_{1\epsilon}^{11} + \mathcal{J}_{1\epsilon}^{12}.
\end{aligned}$$

Thanks to Lemma 2, $\mathcal{J}_{1\epsilon}^{11}$ can be estimated as

$$\begin{aligned}
\mathcal{J}_{1\epsilon}^{11} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d} |(\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}||\mathbf{v}^\epsilon|dxdt \\
&\lesssim \epsilon^\alpha \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}\|^2_{B_2^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)} \to 0
\end{aligned}$$

as $\epsilon \to 0$ for any $\alpha > 0$. We estimate $\mathcal{J}_{1\epsilon}^{12}$ by (7) and Hölder's inequality, then

$$\begin{aligned}
\mathcal{J}_{1\epsilon}^{12} &= \int_{\mathbb{I}}\int_{\mathbb{T}^d} |\varrho^\epsilon\mathbf{v} - \varrho^\epsilon\mathbf{v}^\epsilon||\mathbf{v}^\epsilon|dxdt \\
&\lesssim \|\varrho^\epsilon\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}^\epsilon - \mathbf{v}\|_{L^2(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}^\epsilon\|_{L^2(\mathbb{I}\times\mathbb{T}^d)} \\
&\lesssim \epsilon^\alpha \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}\|^2_{B_2^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)} \to 0
\end{aligned}$$

as $\epsilon \to 0$. On the other hand, according to (5), (6) and Lemma 2, we can obtain

$$\begin{aligned}
|\mathcal{J}_{1\epsilon}^2| &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d} \varphi(t)((\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}^\epsilon)\partial_t\mathbf{v}^\epsilon dxdt| \\
&\lesssim \int_{\mathbb{I}}\int_{\mathbb{T}^d} |(\varrho\mathbf{v})^\epsilon - \varrho^\epsilon\mathbf{v}||\partial_t\mathbf{v}^\epsilon|dxdt + \int_{\mathbb{I}}\int_{\mathbb{T}^d} |\varrho^\epsilon\mathbf{v} - \varrho^\epsilon\mathbf{v}^\epsilon||\partial_t\mathbf{v}^\epsilon|dxdt \\
&\lesssim \epsilon^{2\alpha-1} \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}\|^2_{B_2^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)} \to 0
\end{aligned}$$

as ϵ tends to zero for any $\alpha > \frac{1}{2}$. Thus, $\mathcal{J}_{1\epsilon} \to 0$ as $\epsilon \to 0$ for any $\alpha > \frac{1}{2}$.

The calculation of $\mathcal{J}_{2\epsilon}$ is as follows

$$\begin{aligned}
|\mathcal{J}_{2\epsilon}| &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d} \varphi(t)[\nabla\cdot((\varrho\mathbf{v})^\epsilon \otimes \mathbf{v}^\epsilon) - \nabla\cdot(\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon]\mathbf{v}^\epsilon dxdt| \\
&\lesssim |\int_{\mathbb{I}}\int_{\mathbb{T}^d} [(\varrho\mathbf{v})^\epsilon \otimes \mathbf{v}^\epsilon - (\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon] : \nabla\mathbf{v}^\epsilon dxdt| \\
&\lesssim |\int_{\mathbb{I}}\int_{\mathbb{T}^d} [(\varrho\mathbf{v})^\epsilon \otimes \mathbf{v} - (\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon] : \nabla\mathbf{v}^\epsilon dxdt| \\
&\quad + |\int_{\mathbb{I}}\int_{\mathbb{T}^d} [(\varrho\mathbf{v})^\epsilon \otimes \mathbf{v}^\epsilon - (\varrho\mathbf{v})^\epsilon \otimes \mathbf{v}] : \nabla\mathbf{v}^\epsilon dxdt| \\
&=: \mathcal{J}_{2\epsilon}^1 + \mathcal{J}_{2\epsilon}^2.
\end{aligned}$$

From the Assumption (14) and Lemma 2, we have

$$\begin{aligned}
\mathcal{J}_{2\epsilon}^1 &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d} [(\varrho\mathbf{v})^\epsilon \otimes \mathbf{v} - (\varrho\mathbf{v}\otimes\mathbf{v})^\epsilon] : \nabla\mathbf{v}^\epsilon dxdt| \\
&\lesssim \epsilon^{2\alpha-1} \|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)} \|\mathbf{v}\|^3_{B_3^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)} \to 0
\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_{2\epsilon}^2 &= |\int_{\mathbb{I}}\int_{\mathbb{T}^d}[(\varrho\mathbf{v})^\epsilon\otimes\mathbf{v}^\epsilon-(\varrho\mathbf{v})^\epsilon\otimes\mathbf{v}]:\nabla\mathbf{v}^\epsilon dxdt|\\ &\lesssim \epsilon^{2\alpha-1}\|\varrho\|_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\mathbf{v}\|^3_{B_3^{\alpha,\infty}(\mathbb{I}\times\mathbb{T}^d)}\to 0\end{aligned}$$

as ϵ tends to zero for any $\alpha>\frac{1}{2}$.

To estimate $\mathcal{J}_{3\epsilon}$, we divide it into two parts

$$\begin{aligned}\mathcal{J}_{3\epsilon} &= -\kappa\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^\gamma)^\epsilon\nabla\cdot\mathbf{v}^\epsilon-\varrho^\gamma\nabla\cdot\mathbf{v}]dxdt\\ &\le \kappa|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^\gamma)^\epsilon\nabla\cdot\mathbf{v}^\epsilon-(\varrho^\gamma\nabla\cdot\mathbf{v})^\epsilon]dxdt|\\ &\quad+\kappa|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^\gamma\nabla\cdot\mathbf{v})^\epsilon-\varrho^\gamma\nabla\cdot\mathbf{v}]dxdt|\\ &=: \mathcal{J}_{3\epsilon}^1+\mathcal{J}_{3\epsilon}^2.\end{aligned}$$

Using the same method estimating $\mathcal{I}_{4\epsilon}^1$, we obtain

$$\begin{aligned}\mathcal{J}_{3\epsilon}^1 &= \kappa|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^\gamma)^\epsilon\nabla\cdot\mathbf{v}^\epsilon-(\varrho^\gamma\nabla\cdot\mathbf{v})^\epsilon]dxdt|\\ &\lesssim \|\varrho\|^\gamma_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\nabla\cdot\mathbf{v}\|_{L^1(\mathbb{I}\times\mathbb{T}^d)},\\ \mathcal{J}_{3\epsilon}^2 &= \kappa|\int_{\mathbb{I}}\int_{\mathbb{T}^d}\varphi[(\varrho^\gamma\nabla\cdot\mathbf{v})^\epsilon-\varrho^\gamma\nabla\cdot\mathbf{v}]dxdt|\\ &\lesssim \|\varrho\|^\gamma_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}\|\nabla\cdot\mathbf{v}\|_{L^1(\mathbb{I}\times\mathbb{T}^d)}.\end{aligned}$$

Thus, $\mathcal{J}_{3\epsilon}^1$, $\mathcal{J}_{3\epsilon}^2\to 0$ as ϵ tends to zero.

Finally, similar to the proof of Theorem 1, we show that energy is conserved in the point-wise sense on $\mathbb{I}$. The main difference is that Theorem 2 reduces the regularity of the density ϱ by enhancing the regularity of the velocity profile. For this, the following terms need to be estimated again.

From the mass equation of (1), we know that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d}\psi(x)\varrho^\gamma(t,x)dx = -\gamma\int_{\mathbb{T}^d}\psi(x)\varrho^{\gamma-1}\nabla\cdot(\varrho\mathbf{v})dx\\ &= -\int_{\mathbb{T}^d}\psi(x)\nabla\varrho^\gamma\mathbf{v}dx-\gamma\int_{\mathbb{T}^d}\psi(x)\varrho^\gamma\nabla\cdot\mathbf{v}dx\\ &= \int_{\mathbb{T}^d}\varrho^\gamma[(1-\gamma)\psi(x)\nabla\cdot\mathbf{v}+\nabla\psi(x)\mathbf{v}]dx\\ &\lesssim \|\varrho\|^\gamma_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}(\|\nabla\cdot\mathbf{v}\|_{L^\infty(\mathbb{I};L^1(\mathbb{T}^d))}+\|\mathbf{v}\|_{L^\infty(\mathbb{I};L^1(\mathbb{T}^d))})\end{aligned}$$

is bounded due to (14). Thus, one obtains

$$\varrho^\gamma\in C(\mathbb{I};L^k_{weak}(\mathbb{T}^d)),\ k>1.$$

Similarly,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}^d}\psi(x)\sqrt{\varrho}(t,x)dx = \int_{\mathbb{T}^d}\psi(x)\Big(-\mathbf{v}\nabla\sqrt{\varrho}-\frac{1}{2}\sqrt{\varrho}\nabla\cdot\mathbf{v}\Big)dx\\ &= \int_{\mathbb{T}^d}\sqrt{\varrho}\Big(\nabla\cdot(\psi(x)\mathbf{v})-\frac{1}{2}\psi(x)\nabla\cdot\mathbf{v}\Big)dx = \int_{\mathbb{T}^d}\sqrt{\varrho}\Big(\frac{1}{2}\psi(x)\nabla\cdot\mathbf{v}+\nabla\psi(x)\mathbf{v}\Big)dx\\ &\lesssim \|\varrho\|^{\frac{1}{2}}_{L^\infty(\mathbb{I}\times\mathbb{T}^d)}(\|\nabla\cdot\mathbf{v}\|_{L^\infty(\mathbb{I};L^1(\mathbb{T}^d))}+\|\mathbf{v}\|_{L^\infty(\mathbb{I};L^1(\mathbb{T}^d))}),\end{aligned}$$

thanks to (14), it is bounded. Therefore,

$$\sqrt{\varrho} \in C(\mathbb{I}; L^k_{weak}(\mathbb{T}^d)), \ k > 1.$$

By Assumption (14), one can easily check that $\varrho \mathbf{v} \in C(\mathbb{I}; L^2_{weak}(\mathbb{T}^d))$. This completes the proof of Theorem 2. $\square$

4. Conclusions

In this paper, we investigate the relationship between the regularity of weak solutions and the energy conservation for the isentropic compressible Euler equations. By "trading" regularity between the density and velocity profile, we provide two types of sufficient conditions on the regularity of weak solutions to ensure energy conservation in the point-wise sense. The innovations of this paper include: (i) the energy conservation of weak solutions can be established in a point-wise sense on $\mathbb{I}$; (ii) our method can deal with the vacuum case with adiabatic coefficient $\gamma > 1$. This work is of great significance for the study of fluid structure changes, such as the velocity and the density.

For future research direction, one can try to consider the compressible Euler equations of weak solutions that exhibit uniqueness and satisfy energy conservation.

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