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Jackson Differential Operator Associated with Generalized Mittag–Leffler Function

Adel A. Attiya [1,2,*], Mansour F. Yassen [3,4] and Abdelhamid Albaid [5]

1 Department of Mathematics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Department of Mathematics, College of Science and Humanities in Al-Aflaj, Prince Sattam Bin Abdulaziz University, Al-Aflaj 11912, Saudi Arabia; mf.ali@psau.edu.sa
4 Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
5 Department of Physics, College of Science, University of Ha'il, Ha'il 81451, Saudi Arabia; albaid1979@gmail.com
* Correspondence: aattiy@mans.edu.eg

Abstract: Quantum calculus plays a significant role in many different branches such as quantum physics, hypergeometric series theory, and other physical phenomena. In our paper and using quantitative calculus, we introduce a new family of normalized analytic functions in the open unit disk, which relates to both the generalized Mittag–Leffler function and the Jackson differential operator. By using a differential subordination virtue, we obtain some important properties such as coefficient bounds and the Fekete–Szegő problem. Some results that represent special cases of this family that have been studied before are also highlighted.

Keywords: Mittag–Leffler function; Jackson differential operator; quantum calculus; analytic functions; univalent functions; subordination relation; differential subordination; operators in geometric function theory; Fekete–Szegő function; Gaussian hypergeometric function

MSC: 30C45; 30C80; 30C50; 05A30; 33E12; 33C05

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1. Introduction

First, let us assume that Δ represents a family of analytic functions as the form below

$$u(\eta) = \eta + \sum_{k=2}^{\infty} a_k \eta^k, \; \eta \in \mathbb{D}, \tag{1}$$

where $\mathbb{D}$ represents the set of all values η in the open unit disk $\eta \in \mathbb{C}$ satisfying $|\eta| < 1$.

The *Hadamard product* of both functions $u_1(\eta) = \eta + \sum_{k=2}^{\infty} a_k \eta^k$ and $u_2 = \eta + \sum_{k=2}^{\infty} b_k \eta^k$ in Δ is defined by (see [1])

$$(u_1 * u_2)(\eta) = \eta + \sum_{k=2}^{\infty} a_k b_k \eta^k, \; \eta \in \mathbb{D}.$$

For the analytic functions u_1 and u_2 $(u_1, u_2 \in \mathbb{D})$, the function u_1 is called *subordinate* to the function u_2, and it is written $u_1(\eta) \prec u_2(\eta)$; if we have a function ω (*Schwarz function*), which is analytic in the open disk $\mathbb{D}$ with the conditions $\omega(0) = 0$ and $|\omega(\eta)| < 1, \eta \in \mathbb{D}$ satisfies $u_1(\eta) = u_2(\omega(\eta))$ for all $\eta \in \mathbb{D}$. If the function $u_2 \in S$ (the class of univalent functions in $\mathbb{D}$), then (cf., e.g., [2,3])

$$u_1(\eta) \prec u_2(\eta) \Leftrightarrow u_1(0) = u_2(0) \quad \text{and} \quad u_1(\mathbb{D}) \subset u_2(\mathbb{D}).$$

The Geometric Function Theory (GFT) is an important branch of the field of complex analysis, which is concerned with the study of many important geometric properties in complex analysis, which are related to analytic functions and have numerous applications in different fields, such as analytic number theory, dynamical systems, and fractal calculus. It also has many applications in special functions and probability distributions as well as in fuzzy algebra and other fields.

There are many classes of normalized functions which appear simultaneously with studying GFT, e.g.,

A function $u(\eta) \in \Delta$ belongs to the class of starlike functions $\mathcal{S}^*$ if it satisfies

$$\mathrm{Re}\left(\frac{\eta u'(\eta)}{u(\eta)}\right) > 0 \quad (\eta \in \mathbb{D}),$$

and a function $u(\eta) \in \Delta$ belongs to the class of starlike functions of order v denoted by $\mathcal{S}^*(v)$ if it satisfies

$$\mathrm{Re}\left(\frac{\eta u'(\eta)}{u(\eta)}\right) > v \quad (\eta \in \mathbb{D}),$$

for $v\ (0 \leq v < 1)$.

In addition, a function $u(\eta) \in \Delta$ belongs to the class of convex functions $\mathcal{C}^*$ if it satisfies

$$\mathrm{Re}\left(1 + \frac{\eta u''(\eta)}{u'(\eta)}\right) > 0 \quad (\eta \in \mathbb{D}),$$

and a function $u(\eta) \in \Delta$ belongs to the class of convex functions of order v denoted by $\mathcal{C}^*(v)$ if it satisfies

$$\mathrm{Re}\left(1 + \frac{\eta u''(\eta)}{u'(\eta)}\right) > v \quad (\eta \in \mathbb{D}),$$

for some $v\ (0 \leq v < 1)$.

For more details of the classes $\mathcal{S}^*$, $\mathcal{S}^*(v)$, $\mathcal{C}^*$ and $\mathcal{C}^*(v)$, see, e.g., Robertson [4], Bulboaca [2], Miller and Mocanu [5] and Duren [3].

Moreover, a function $u(\eta) \in \Delta$ is said to be in the class C if it satisfies

$$\mathrm{Re}\left(\frac{u(\eta)}{\eta}\right) > 0 \quad (\eta \in \mathbb{D}),$$

and $u(\eta) \in \Delta$ belongs to the class $C(v)$ if it satisfies

$$\mathrm{Re}\left(\frac{u(\eta)}{\eta}\right) > v \quad (\eta \in \mathbb{D}),$$

for $v\ (0 \leq v < 1)$, the classes C and $C(v)$ were studied by MacGregor [6] and Èzrohi [7], respectively.

Furthermore, a function $u(\eta) \in \Delta$ belongs to the class $\mathcal{B}$ if it satisfies

$$\mathrm{Re}\left(u'(\eta)\right) > 0 \quad (\eta \in \mathbb{D}),$$

and the function $u(\eta) \in \Delta$ belongs to the class $\mathcal{B}(v)$ if it satisfies

$$\mathrm{Re}\left(u'(\eta)\right) > v \quad (\eta \in \mathbb{D}),$$

for some $v\ (0 \leq v < 1)$. The class $\mathcal{B}$ was studied by Goel [8] and Yamaguchi [9]. In addition, the class $\mathcal{B}(v)$ was studied by Chen [10,11] and Goel [12]; see also [13].

Furthermore, let

$$\mathfrak{R} := \left\{ \mathfrak{P} : \mathfrak{P}(\eta) = \sum_{k=0}^{\infty} b_k \eta^k,\ b_0 = 1,\ \mathrm{Re}\,\mathfrak{P}(\eta) > 0\ (b_0 = 1, \eta \in \mathbb{D}) \right\},$$

denote all the Carathéodory functions (see [14,15]).

Quantum calculus plays a significant role in the quantum physics and hypergeometric series theory as well as other physical phenomena. The applications of q-differentiation and also q-integration were defined and introduced by Jackson [16,17]; see also [18–24].

The Mittag–Leffler function $E_\alpha(\eta)$ $(\eta \in \mathbb{C})$ was obtained by Mittag–Leffler [25,26] which in the form

$$E_\alpha(\eta) = \sum_{k=0}^{\infty} \frac{\eta^k}{\Gamma(\alpha k + 1)},$$
$$(\alpha \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0).$$

Wiman [27] introduced Wim's function $E_{\alpha,\beta}(\eta)$ $(\eta \in \mathbb{C})$ in the form

$$E_{\alpha,\beta}(\eta) = \sum_{k=0}^{\infty} \frac{\eta^k}{\Gamma(\alpha k + \beta)},$$

where α and β are complex values in $\mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$.

Prabhakar [28] introduced the function $E^{\delta}_{\alpha,\beta}(\eta)$ $(\eta \in \mathbb{C})$ in the form

$$E^{\delta}_{\alpha,\beta}(\eta) = \sum_{k=0}^{\infty} \frac{(\delta)_k\, \eta^k}{\Gamma(\alpha k + \beta)\, k!},$$
$$(\alpha, \beta, \delta \in \mathbb{C};\ \mathrm{Re}(\alpha) > 0;\ \mathrm{Re}(\beta) > 0;\ \mathrm{Re}(\delta) > 0),$$

where $(\delta)_n$ is the *Pochhammer symbol*:

$$(\delta)_n = \frac{\Gamma(\delta + n)}{\Gamma(\delta)} = \begin{cases} 1, & n = 0 \\ \delta(\delta+1)\ldots(\delta+n-1) \end{cases}. \tag{2}$$

For the Mittag–Leffler function and related articles, see for example [29–34].

Raina's function ([35]; see also [18]) is defined by

$$_{\mathcal{N}}\mathcal{H}_{a,b}(\eta) = \sum_{k=0}^{\infty} \frac{\mathcal{N}(k)}{\Gamma(ak + b)} \eta^k,\ \eta \in \mathbb{D},$$

where a and b are complex values in $\mathbb{C}$, $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$ and the sequence $\{\mathcal{N}(k)\}_{k \in \mathbb{N}_0}$ is bounded $(\mathcal{N}(k) \in \mathbb{C})$.

Remark 1.

1. *If* $\mathcal{N}(k) = 1$ $(k \geq 0)$, *then* Raina's function *gives the* Mittag–Leffler function.
2. *If* $(n)_k$ *is the well-known Pochhammer symbol,* $\mathcal{N}(k) = \dfrac{(a)_k (b)_k}{(c)_k}$, $a = 1$ *and* $b = 1$, *then Raina's function reduces to the following Gaussian hypergeometric function:*

$$_2F_1(a, b; c; \eta) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{\eta^k}{\Gamma(k+1)},\ \eta \in \mathbb{D}.$$

Definition 1 ([16])**.** *The Jackson derivative of* $u(\eta)$ *is defined as follows*

$$(\mathfrak{d}_l\,)u(\eta) := \frac{u(\eta) - u(l\,\eta)}{\eta(1 - l)},\ (0 < l < 1).$$

Therefore,

$$\mathfrak{d}_q\left(\eta^k\right) = \frac{1-l^k}{1-l}\,\eta^{k-1},\ k \in \mathbb{N}\cup\{0\},$$

when $u(\eta)$ has the form (1), then we have

$$\left(\mathfrak{d}_q\right)u(\eta) = 1 + \sum_{k=2}^{\infty} a_k\,[k]_q\,\eta^{k-1},$$

where

$$[k]_l := \frac{1-l^k}{1-l}.$$

In addition, note that

$$\mathfrak{d}_l\,\kappa = 0 \ \text{ and } \ \lim_{l\to 1^-}\left(\mathfrak{d}_l\right)u(\eta) = u'(\eta),$$

where κ is a constant in $\mathbb{C}$.

When $s \in \mathbb{C}$, then the *q-shifted factorial* denoted by $(s;q)_\tau$ is defined as follows (see [16])

$$(s;q)_\tau := \prod_{j=0}^{\tau-1}\left(1-q^j s\right),\ \tau \in \mathbb{N} := \{1,2,\ldots\},\quad (s;q)_0 = 1. \tag{3}$$

By using (3), we can formulate *q-shifted gamma function* as follows:

$$\left(q^t;q\right)_\tau = \frac{\Gamma(s+\tau)(1-q)^\tau}{\Gamma_q(s)}\ \ (t\in\mathbb{N})$$

where

$$\Gamma_q(s) = \frac{(q;q)_\infty(1-q)^{1-s}}{(q^s;q)_\infty}\ \ (0<q<1)$$

and

$$(s;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j s\right).$$

In geometric function theory, there are many famous operators dealing with normalized functions, e.g.,

Let D^α be a differential operator $D^\alpha : \Delta \to \Delta$ defined as follows

$$D^\alpha u(\eta) = \frac{\eta}{(1-\eta)^{\alpha+1}} * u(\eta) \qquad (\alpha > -1),$$

D^α represents Ruscheweyh derivatives as defined by Ruscheweyh [36], which can be in the form

$$D^\alpha u(\eta) = \eta + \sum_{k=2}^{\infty}\frac{(\alpha-1)_{k-1}}{(k-1)!}a_k\,\eta^k,\ \eta\in\mathbb{D},\quad (\alpha>-1),$$

where $(\alpha)_k$ denotes the Pochhammer symbol defined by (2).

In addition, for $u(\eta) \in \Delta$ and $\eta \in \mathbb{D}$, the following integral operators $A(u)$, $L(u)$ and $L_\gamma(u)$ are defined as

$$A(u)\,(\eta) = \int_0^\eta \frac{u(t)}{t}\,dt\,,$$

$$L(u)\,(\eta) = \frac{2}{\eta}\int_0^\eta u(t)\ dt$$

and

$$L_{\gamma}(u)\,(\eta) = \frac{1+\gamma}{\eta^{\gamma}} \int\limits_{0}^{\eta} u(t)\; t^{\gamma-1}\; dt \qquad (\gamma > -1)\,.$$

The operators $A(u)$ and $L(u)$ are the Alexander operator and Libera operator, which were introduced by Alexander [37] and Libera [38], respectively. $L_{\gamma}(u)$ represents a generalized Bernardi operator; the operator $L_{\gamma}(u)$ when $\gamma \in \mathbb{N} = \{1, 2, \ldots\}$ was introduced by Bernardi [39].

Moreover, Jung et al. [40] introduced the following integral operator:

$$I^{\sigma}(u)\,(\eta) = \frac{2^{\sigma}}{\eta\,\Gamma(\sigma)} \int\limits_{0}^{\eta} \left(\log\left(\frac{\eta}{t}\right)\right)^{\sigma-1} u(t)\; dt \quad (\sigma > 0,\; u(\eta)\, \in \Delta),$$

they showed that

$$I^{\sigma}(u)\,(\eta) = \eta\, + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^{\sigma} a_k\, \eta^k.$$

The operator $I^{\sigma}(u)$ is closely related to multiplier transformations studied earlier by Flett [41].

Furthermore, denote by $J_{s,b}\,(u) \,:\, \Delta \to \Delta$ the Srivastava–Attiya operator, which is introduced by Srivastava and Attiya [42]; see also ([43]) defined by

$$J_{s,b}\,(u)\,(\eta) = G_{s,b}(\eta) \;=\; (1+b)^s \left[\varphi(\eta, s,\, b) - b^{-s}\right] * u(\eta) \quad (\eta \in \mathbb{D}\,),$$

where $\varphi(\eta, s,\, b)$ is the general Hurwitz–Lerch–Zeta function defined by (cf., e.g., ([44], p. 121 *et seq.*)) and

$$(b \in \mathbb{C} \setminus \mathbb{Z}_0^-\,,\; \mathbb{Z}_0^- = \mathbb{Z}^- \cup \{0\} = \{0,\, -1,\, -2, \ldots\},\; s \in \mathbb{C}\;,\; \eta \in \mathbb{D})$$

For $\mathcal{N}(0) \neq 0$, the normalized function ${}_{\mathcal{N}}\mathcal{L}_{d,b}$ (see [18]) is defined by

$$ {}_{\mathcal{N}}\mathcal{L}_{d,b}(\eta) := \eta + \sum_{k=2}^{\infty} \frac{\mathcal{N}(k-1)\Gamma(b)}{\mathcal{N}(0)\Gamma(d(k-1)+b)}\, \eta^k,\; \eta \in \mathbb{D}. \tag{4}$$

If $\mathcal{N}(k) = (k+1)^{-r}$, $r \in \mathbb{R}$ is a non-negative value, $a = 0$ and $b = 1$; then, the operator (4) is the integral operator defined by Sălăgean (Sălăgean integral operator of order r) (see [45]).

Now, by using the q-gamma function, the class of normalized functions ${}_{q,\mathcal{N}}\mathcal{L}_{d,b}(\eta)$ is defined as follows:

$$ {}_{q,\mathcal{N}}\mathcal{L}_{d,b}(\eta) := \eta + \sum_{k=2}^{\infty} \Phi_k(d, b, \mathcal{N}, q)\eta^k,\; \eta \in \mathbb{D},$$

where

$$\Phi_k(d, b, \mathcal{N}, q) := \frac{\mathcal{N}(k-1)\Gamma_q(b)}{\mathcal{N}(0)\Gamma_q(d(k-1)+b)}, \tag{5}$$

with $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ and $\mathcal{N}(0) \neq 0$.

Considering the quantum operator $\mathfrak{d}_q$, Attiya et al. [18] introduced the q-Raina differential operator ${}_{\mathcal{N}}\mathcal{L}_q^n : \Delta \to \Delta$ by

$$\begin{aligned}
{}_{\mathcal{N}}\mathcal{L}_q^0(a,b)u(\eta) &= u(\eta) *_{q,\mathcal{N}} \mathcal{L}_{d,b}(\eta),\\
{}_{\mathcal{N}}\mathcal{L}_q^1(a,b)u(\eta) &= \eta\mathfrak{d}_q\Big({}_{\mathcal{N}}\mathcal{L}_q^0(a,b)u(\eta)\Big),\\
{}_{\mathcal{N}}\mathcal{L}_q^2(a,b)u(\eta) &= {}_{\mathcal{N}}\mathcal{L}_q^1(a,b)\Big({}_{\mathcal{N}}\mathcal{L}_q^1(a,b)u(\eta)\Big),\\
&\dots\\
{}_{\mathcal{N}}\mathcal{L}_q^k(a,b)u(\eta) &= {}_{\mathcal{N}}\mathcal{L}_q^1(a,b)\Big({}_{\mathcal{N}}\mathcal{L}_q^{k-1}(a,b)u(\eta)\Big),\ u \in \Delta,\ k \in \mathbb{N},\ k \geq 2.
\end{aligned} \tag{6}$$

Employing the above definition and if $u \in \Delta$, which is in the form (1), then we have

$${}_{\mathcal{N}}\mathcal{L}_q^n(a,b)u(\eta) = \eta + \sum_{k=2}^{\infty}[k]_q^n \frac{\mathcal{N}(k-1)\Gamma_q(b)}{\mathcal{N}(0)\Gamma_q(d(k-1)+b)} a_k\eta^k,\ \alpha \neq 0.$$

Analogously to ${}_{\mathcal{N}}\mathcal{L}_q^n$, we add the significant parameter α ($\alpha \neq 0$) for a new operator $\mathcal{L}_q^n(\mathcal{N}, a, b, \alpha)$ as follows:

$$\begin{aligned}
\mathcal{L}_q^n(\mathcal{N}, a, b, \alpha)u(\eta) &= \eta + \sum_{k=2}^{\infty}\left(\frac{[k]_q + (\alpha-1)}{\alpha}\right)^n \frac{\mathcal{N}(k-1)\Gamma_q(b)}{\mathcal{N}(0)\Gamma_q(d(k-1)+b)} a_k\eta^k,\ \alpha \neq 0 \\
&= \eta + \sum_{k=2}^{\infty}\left(\frac{[k]_q + (\alpha-1)}{\alpha}\right)^n \Phi_k(d,b,\mathcal{N},q)a_k\eta^k,\ \eta \in \mathbb{D},
\end{aligned} \tag{7}$$

where $\Phi_k(d,b,\mathcal{N},q)$ is given by (5)

Remark 2.

(i) *Putting $\alpha = 1$, in (7), we obtain the q-Raina differential operator defined in [18].*
(ii) *Putting $\alpha = 1$ and $\mathcal{N}(k-1) = 1$ ($k \geq 1$), (7), we obtain the q-differential operator of [22].*
(iii) *Putting $\alpha = 1$, $d = 0$ and $\mathcal{N}(k-1) = 1$ ($k \geq 1$) in (7), we obtain the Sălăgean q-differential operator defined in [46].*
(iv) *Putting $\alpha = 1$, $\mathcal{N}(k-1) = 1$ and $q = 1$ in (7), we obtain the class studied by Bansal and Prajapat [47] (see also [48]).*

Definition 2. *Let us define the convex analytic function $\Omega_{j,\Im}$ in $\mathbb{D}$ as follows:*

$$\Omega_{j,\Im}(\eta) := \begin{cases} \dfrac{1+\eta}{1-\eta}, & \text{if } j = 0,\\ F_1(j,\Im), & \text{if } j = 1,\\ F_2(j,\Im), & \text{if } 0 < j < 1,\\ F_3(j,\Im), & \text{if } j > 1,\end{cases}$$

where $\Im \in \mathbb{C} \setminus \{0\}$, and the following functions are defined by (see [49])

$$\begin{aligned}
F_1(j,\Im)(\eta) &= 1 + \frac{2\Im}{\pi^2}\left(\log\left(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\right)\right)^2,\\
F_2(j,\Im)(\eta) &= 1 + \frac{2\Im}{1-j^2}\sinh^2\left(\frac{2}{\pi}\arccos(j)\,\mathrm{arctanh}(\sqrt{\eta})\right),\\
F_3(j,\Im)(\eta) &= 1 + \frac{\Im}{1-j^2} + \frac{\Im}{j^2-1}\sin\left(\frac{\pi}{2Y(t)}\int_0^{\ell(\eta)/\sqrt{t}} \frac{d\zeta}{\sqrt{1-\zeta^2}\sqrt{1-(\zeta t)^2}}\right),
\end{aligned}$$

where $\ell(\eta) = \dfrac{\eta - \sqrt{t}}{1 - \sqrt{t\eta}}$, $t \in (0,1)$, *taken with* $t = \cosh\left(\dfrac{\pi Y'(t)}{4Y(t)}\right)$, $Y(t)$ *is the well-known Legendre's complete elliptic integral from the first kind and* $Y'(t)$ *is the complementary integral of Legendre's function* $Y(t)$, *which satisfies* $(Y'(t))^2 = 1 - (Y(t))^2$.

Now, we define and introduce the class $\mathcal{S}_{q,\Im}^{n,j}(\alpha, d, b)$ of analytic functions as follows:

Definition 3. *The function* $u \in \Delta$ *is called to be in the class* $\mathcal{S}_{q,\Im}^{n,j}(\alpha, d, b)$ *if we have the following subordination relation*

$$\frac{\alpha\left(\mathcal{L}_q^{n+1}(\mathcal{N}, d, b, \alpha)u(\eta)\right)}{\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha)u(\eta)} + 1 - \alpha \prec \Omega_{j,\Im}(\eta), \ \ \alpha \neq 0$$

where $\Omega_{j,\Im}$ *in the form (see also [18,49,50])*

$$\Omega_{j,\Im}(\eta) = 1 + \gamma_1\eta + \gamma_2\eta^2 + \ldots, \ \eta \in \mathbb{D}, \tag{8}$$

is given by Definition 2.

Definition 4. *When* $q \to 1^-$, *then the function* $u \in \mathcal{S}_{q,\Im}^{n,j}(\alpha, d, b)$ *is said to be in the class* $\mathcal{S}_{\Im}^{n,j}(\alpha, d, b)$.

Lemma 1 ([51])**.** *Let* $G(\eta) = \sum_{k=0}^{\infty} g_k\eta^k$ *be a univalent convex function in* $\mathbb{D}$ *satisfying the inequality*

$$H(\eta) = \sum_{k=0}^{\infty} h_k\eta^k \prec G(\eta).$$

Then, $|h_k| \leq |g_1|$ *for all* $k \geq 1$.

Lemma 2 ([52])**.** *Let* $P(\eta) = 1 + \sum_{k=1}^{\infty} p_k\eta^k$ *be analytic in* $\mathbb{D}$ *that satisfies* $\operatorname{Re} P(\eta) > 0 \ (\eta \in \mathbb{D})$. *Then,*

$$\left|p_2 - \aleph p_1^2\right| \leq 2\max\{1; |2\aleph - 1|\}, \ \aleph \in \mathbb{C}.$$

2. Estimation Coefficient for the Class $\mathcal{S}_{q,\Im}^{n,j}(\alpha, d, b)$

The following theorem is related to functions in the class $\mathcal{S}_{\Im}^{n,j}(\alpha, d, b)$

Theorem 1. *If* u *is in the class* $\mathcal{S}_{\Im}^{n,j}(\alpha, d, b)$, *then*

$$\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha)u(\eta) \prec \eta \exp\left(\int_0^{\eta} \frac{\Omega_{j,\Im}(\omega(\chi)) - 1}{\chi} d\chi\right),$$

where ω *is a Schwarz function where* $\omega(0) = 0$ *and also,* $\omega(\eta)| < 1$, $\eta \in \mathbb{D}$. *Furthermore, for* $|\eta| := \varrho < 1$, *we obtain*

$$\exp\left(\int_0^1 \frac{\Omega_{j,\Im}(-\varrho) - 1}{\varrho} d\varrho\right) \leq \left|\frac{\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha)u(\eta)}{\eta}\right| \leq \exp\left(\int_0^1 \frac{\Omega_{j,\Im}(\varrho) - 1}{\varrho} d\varrho\right).$$

Proof. Since $u \in \mathcal{S}_{\Im}^{n,j}(\alpha, d, b)$, then

$$\frac{\left(\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha)u(\eta)\right)'}{\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha)u(\eta)} - \frac{1}{\eta} = \frac{\Omega_{j,\Im}(\omega(\eta)) - 1}{\eta}, \ \eta \in \mathbb{D}. \tag{9}$$

Integrating both sides of (9), it follows that

$$\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta) \prec \eta \exp\left(\int_0^\eta \frac{\Omega_{j,\Im}(\chi)-1}{\chi}d\chi\right),$$

which is equivalent to

$$\frac{\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)}{\eta} \prec \exp\left(\int_0^\eta \frac{\Omega_{j,\Im}(\chi)-1}{\chi}d\chi\right).$$

Since

$$\Omega_{j,\Im}(-\varrho|\eta|) \leq \operatorname{Re}\left(\Omega_{j,\Im}(\omega(\eta\varrho))\right) \leq \Omega_{j,\Im}(\varrho|\eta|),$$

this yields

$$\int_0^1 \frac{\Omega_{j,\Im}(-\varrho|\eta|)-1}{\varrho}d\varrho \leq \int_0^1 \frac{\operatorname{Re}(\Omega_{j,\Im}(\omega(\eta\varrho)))-1}{\varrho}d\varrho \leq \int_0^1 \frac{\Omega_{j,\Im}(\varrho|\eta|)-1}{\varrho}d\varrho.$$

Therefore, we obtain

$$\int_0^1 \frac{\Omega_{j,\Im}(-\varrho|\eta|)-1}{\varrho}d\varrho \leq \log\left|\frac{\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)}{\eta}\right| \leq \int_0^1 \frac{\Omega_{j,\Im}(\varrho|\eta|)-1}{\varrho}d\varrho,$$

then

$$\exp\left(\int_0^1 \frac{\Omega_{j,\Im}(-\varrho)-1}{\varrho}d\varrho\right) \leq \left|\frac{\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)}{\eta}\right| \leq \exp\left(\int_0^1 \frac{\Omega_{j,\Im}(\varrho)-1}{\varrho}d\varrho\right).$$

□

Remark 3. *Theorem 1 represents a generalization of results of some authors, e.g.,*

1. *Putting $\alpha = 1$, in Theorem 1, we have the result due to Attiya et al. ([18], Theorem 6).*
2. *Putting $\alpha = 1$ and $\mathcal{N}(k) = 1$ for all $k \geq 1$, in Theorem 1, we have the result due to Noor and Razzaque ([22], Theorem 6).*
3. *Putting $\alpha = 1, \mathcal{N}(k) = 1$ $(k \geq 1)$, $d = 0$ and $b = 1$, then in Theorem 1, we have the result due to Hussain et al. ([53], Theorem 3.1).*

Theorem 2. *If u belongs to the class $\mathcal{S}_{q,\Im}^{n,j}(\alpha,d,b)$, then*

$$|a_2| \leq \frac{|\gamma_1|}{q\left|\frac{q}{\alpha}+1\right|^n \Phi_2(d,b,\mathcal{N},q)}, \text{ and}$$

$$|a_k| \leq \frac{|\gamma_1|}{q[k-1]_q\left|1+\frac{q\,[k-1]_q}{\alpha}\right|^n \Phi_k(d,b,\mathcal{N},q)} \prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{q[j]_q\left|1+\frac{q\,[j]_q}{\alpha}\right|^n}\right),\ k \geq 3$$

where $\alpha \neq 0$ and γ_1 is defined by (8).

Proof. Letting

$$P(\eta) = \frac{\alpha\left(\mathcal{L}_q^{n+1}(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)} + 1 - \alpha$$

therefore,

$$P(\eta) = \frac{\eta\mathfrak{d}_q\left(\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}_q^n(\mathcal{N},d,b,\alpha)u(\eta)} \quad (\eta \in \mathbb{D}),$$

letting $P(\eta) = 1 + \sum\limits_{k=1}^{\infty} p_k \eta^k$, which gives

$$\eta \mathfrak{d}_q \Big(\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha) u(\eta) \Big) = \Big(\mathcal{L}_q^n(\mathcal{N}, d, b, \alpha) u(\eta) \Big) \; P(\eta), \; \eta \in \mathbb{D},$$

therefore, we have

$$\begin{aligned} &\eta + \sum_{k=2}^{\infty} \left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n [k]_q \, \Phi_k(d, b, \mathcal{N}, q) a_k \eta^k \\ &= \left(\eta + \sum_{k=2}^{\infty} \left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_k(d, b, \mathcal{N}, q) a_k \eta^k \right) \left(1 + \sum_{k=1}^{\infty} p_k \eta^k \right) \\ &= \sum_{k=0}^{\infty} p_k \eta^{k+1} + \sum_{k=0}^{\infty} p_k \eta^k \cdot \sum_{k=2}^{\infty} \left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_k(d, b, \mathcal{N}, q) a_k \eta^k. \quad (p_0 = 1) \\ &= \eta + \sum_{k=2}^{\infty} \left(p_{k-1} + \sum_{j=1}^{k-1} \left(\frac{[j+1]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_{j+1}(d, b, \mathcal{N}, q) \; a_{j+1} \; p_{k-j-1} \right) \eta^k. \end{aligned}$$

By matching the coefficients of η^k of the equality mentioned above, we obtain

$$\begin{aligned} \left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n [k]_q \, \Phi_k(d, b, \mathcal{N}, q) a_k = p_{k-1} + \left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_k(d, b, \mathcal{N}, q) a_k \\ + \sum_{j=1}^{k-2} \left(\frac{[j+1]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_{j+1}(d, b, \mathcal{N}, q) \; a_{j+1} \; p_{k-j-1}, \end{aligned}$$

which gives

$$\left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n ([k]_q \; - 1) \Phi_k(d, b, \mathcal{N}, q) a_k = p_{k-1} + \sum_{j=1}^{k-2} \left(\frac{[j+1]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_{j+1}(d, b, \mathcal{N}, q) \; a_{j+1} \; p_{k-j-1}.$$

Accordingly, we obtain

$$a_k = \frac{1}{\left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n ([k]_q \; - 1) \Phi_k(d, b, \mathcal{N}, q)} \left(\sum_{j=1}^{k-1} \left(\frac{[j]_q + (\alpha - 1)}{\alpha} \right)^n \Phi_j(d, b, \mathcal{N}, q) \; a_j \; p_{k-j} \right),$$

for some calculation implies that

$$a_k = \frac{1}{\left(\frac{[k]_q + (\alpha - 1)}{\alpha} \right)^n ([k]_q \; - 1) \Phi_k(d, b, \mathcal{N}, q)} \sum_{j=1}^{k-1} \left(\frac{[j]_q + (\alpha - 1)}{\alpha} \right)^n \frac{\mathcal{N}(j-1) \Gamma_q(b)}{\mathcal{N}(0) \Gamma_q(d(j-1) + b)} a_j \; p_{k-j}.$$

In view of Lemma 1, since $|p_k| \leq |\gamma_1|$, we obtain

$$|a_k| \leq \frac{|\gamma_1|}{\left| \frac{[k]_q + (\alpha - 1)}{\alpha} \right|^n ([k]_q \; - 1) \Phi_k(d, b, \mathcal{N}, q)} \sum_{j=1}^{k-1} \left| \frac{[j]_q + (\alpha - 1)}{\alpha} \right|^n \frac{\mathcal{N}(j-1) \Gamma_q(b)}{\mathcal{N}(0) \Gamma_q(d(j-1) + b)} |a_j|.$$

For $k = 2$, we have

$$\begin{aligned} |a_2| &\leq \frac{|\gamma_1|}{q |1 + \frac{q}{\alpha}|^n \; \Phi_2(d, b, \mathcal{N}, q)} \sum_{j=1}^{1} \left| \frac{[j]_q + (\alpha - 1)}{\alpha} \right|^n \frac{\mathcal{N}(j-1) \Gamma_q(b)}{\mathcal{N}(0) \Gamma_q(d(j-1) + b)} |a_j| \\ &= \frac{|\gamma_1|}{q |1 + \frac{q}{\alpha}|^n \; \Phi_2(d, b, \mathcal{N}, q)}, \end{aligned}$$

while if $k=3$, and using the above inequality, then

$$|a_3| \leq \frac{|\gamma_1|}{q[2]_q\left|1+\frac{q\,[2]_q}{\alpha}\right|^n \Phi_3(d,b,\mathcal{N},q)}\left(1+\frac{|\gamma_1|}{q\left|1+\frac{q}{\alpha}\right|^n}\right).$$

For $k \geq 3$, the following inequality is valid by mathematical induction

$$|a_k| \leq \frac{|\gamma_1|}{q[k-1]_q\left|1+\frac{q\,[k-1]_q}{\alpha}\right|^n \Phi_k(d,b,\mathcal{N},q)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{q[j]_q\left|1+\frac{q\,[j]_q}{\alpha}\right|^n}\right),\ k \geq 3.$$

which completes our proof. □

For special cases of Theorem 2, we obtain the following corollaries

Corollary 1 ([18], Theorem 2)**.** *If $u \in \mathcal{S}^{n,j}_{q,\Im}(1,d,b)$, then we have*

$$|a_2| \leq \frac{|\gamma_1|}{[2]_q^n\ ([2]_q-1)\Phi_2(d,b,\mathcal{N},q)},\ \textit{and}$$

$$|a_k| \leq \frac{|\gamma_1|}{[k]_q^n\ ([k]_q-1)\Phi_k(d,b,\mathcal{N},q)}\prod_{j=1}^{k-2}\left(1+\frac{|\rho_1|}{[j+1]_q-1}\right),\ k \geq 3$$

with γ_1 given by (8).

Corollary 2 ([22], Theorem 8)**.** *If $u \in \mathcal{S}^{n,j}_{q,\Im}(1,d,b)$ and $\mathcal{N}(k)=1$ for all $k \geq 1$, then*

$$|a_2| \leq \frac{|\gamma_1|}{[2]_q^n\Phi_2(d,b,1,q)([2]_q-1)},\ \textit{and}$$

$$|a_k| \leq \frac{|\gamma_1|}{[k]_q^n\Phi_k(d,b,1,q)([k]_q-1)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{[j+1]_q-1}\right),\ k \geq 3,$$

with γ_1 given by (8).

Corollary 3 ([53], Theorem 3.2)**.** *If $u \in \mathcal{S}^{n,j}_{q,\Im}(1,0,1)$ and $\mathcal{N}(k)=1$ for all $k \geq 1$, then*

$$|a_2| \leq \frac{|\gamma_1|}{[2]_q^n\Phi_2(0,1,1,q)([2]_q-1)},\textit{and}$$

$$|a_k| \leq \frac{|\gamma_1|}{[k]_q^n\Phi_k(0,1,1,q)([k]_q-1)}\prod_{j=1}^{k-2}\left(1+\frac{|\gamma_1|}{[j+1]_q-1}\right),\ k \geq 3,$$

where γ_1 is given by (8).

3. Fekete–Szegő Problem Associated with Class $\mathcal{S}^{n,j}_{q,\Im}(\alpha,d,b)$

In the following theorem, we will give estimation for the Fekete–Szegő problem for the class $\mathcal{S}^{n,j}_{q,\Im}(\alpha,d,b)$.

Theorem 3. *If $u \in \mathcal{S}^{n,j}_{q,\Im}(\alpha,d,b)$, then*

$$|a_3-\psi a_2^2| \leq \frac{|\gamma_1|}{2T\ \Phi_3(d,b,\mathcal{N},q)}\max\{1;|2\Psi-1|\},$$

where $\psi \in \mathbb{C}$, *and*

$$\Psi := \Psi(d,b,\mathcal{N},q) = \frac{2T\Phi_3(d,b,\mathcal{N},q)}{\gamma_1}\left(\frac{U}{T\Phi_3(d,b,\mathcal{N},q)} - \frac{\psi\gamma_1^2}{q^2\left(1+\frac{q}{\alpha}\right)^{2n}\Phi_2^2(d,b,\mathcal{N},q)}\right), \tag{10}$$

with

$$T = \left(1-[3]_q\right)\left(\frac{[3]_q-(1-\alpha)}{\alpha}\right)^n$$

and

$$U = \frac{1}{4}\left(\gamma_1\left(\frac{q-\gamma_1}{4q}\right) - \gamma_2^2\right),$$

where γ_1 *and* γ_2 *defined by* (8).

Proof. From the condition $u \in \mathcal{S}_{q,\Im}^{n,j}(\alpha,d,b)$, we have

$$\frac{\alpha\left(\mathcal{L}_q^{n+1}(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}_q^{n}(\mathcal{N},d,b,\alpha)u(\eta)} + 1 - \alpha = \Omega_{j,\Im}(\omega(\eta)),$$

where ω is a Schwarz function ($\omega(0)=0$, $\omega(\eta)| < 1$).

Let $p \in \mathcal{P}$ be defined as follows

$$p(\eta) = \frac{1+\omega(\eta)}{1-\omega(\eta)} = 1 + p_1\eta + p_2\eta^2 + \ldots,\ \eta \in \mathbb{D},$$

which implies

$$\omega(\eta) = \frac{p_1}{2}\eta + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)\eta^2 + \ldots,\ \eta \in \mathbb{D}$$

and

$$\Omega_{j,\Im}(\omega(\eta)) = 1 + \frac{\gamma_1 p_1}{2}\eta + \left(\frac{\gamma_2 p_1^2}{4} + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)\gamma_1\right)\eta^2 + \ldots, \eta \in \mathbb{D}.$$

Therefore, we obtain

$$\begin{aligned}\frac{\alpha\left(\mathcal{L}_q^{n+1}(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}_q^{n}(\mathcal{N},d,b,\alpha)u(\eta)} + 1 - \alpha &= 1 + \left(qa_2\left(1+\frac{q}{\alpha}\right)^n \Phi_2(d,b,\mathcal{N},q)\right)\eta \\ &+ \left(\left([3]_q - 1\right)\left(\tfrac{[3]_q-1}{\alpha}+1\right)^n\Phi_3(d,b,\mathcal{N},q)a_3 - \left(q\left(1+\tfrac{q}{\alpha}\right)^{2n}\Phi_2^2(d,b,\mathcal{N},q)\right)a_2^2\right)\eta^2 + \ldots,\ \eta \in \mathbb{D},\end{aligned}$$

thus, the following coefficients can be determined as follows:

$$\begin{aligned}a_2 &= \frac{\gamma_1 p_1}{q[\left(1+\frac{q}{\alpha}\right)^n\Phi_2(d,b,\mathcal{N},q)},\\ a_3 &= \frac{1}{T\Phi_3(d,b,\mathcal{N},q)\left([3]_q-1\right)}\left(-\frac{\gamma_1 p_2}{2} + p_1^2\left(-\frac{\gamma_2^2}{4} + \frac{\gamma_1}{4} - \frac{\gamma_1^2}{q}\right)\right)\\ a_3 - \psi a_2^2 &= \frac{1}{T\Phi_3(d,b,\mathcal{N},q)}\left(-\frac{\gamma_1 p_2}{2} + U\,p_1^2\right)\\ &\quad - \psi\left(\frac{\gamma_1 p_1}{q[\left(1+\frac{q}{\alpha}\right)\Phi_2(d,b,\mathcal{N},q)}\right)^2.\end{aligned}$$

A simple computation yields

$$a_3 - \psi a_2^2 = \frac{-\gamma_1}{2T\,\Phi_3(d,b,\mathcal{N},q)}(p_2 - \Psi p_1^2),$$

where $\psi \in \mathbb{C}$ and Ψ is defined by (10). Therefore, by using Lemma 2, we obtain the desired result. □

Theorem 3 generalizes some of the previous results, e.g.,

Corollary 4 ([18], Theorem 3)**.** *If $u \in \mathcal{S}_{q,\Im}^{n,j}(1,d,b)$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\rho_1|}{2[3]_q^n \Phi_3(a,b,\mathcal{M},q)([3]_q - 1)} \max\{1; |2\Psi - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\Psi := \Psi(a,b,\mathcal{M},q) = \frac{1}{2}\left(1 - \frac{\rho_2}{\rho_1} - \rho_1\left(\frac{1}{[2]_q - 1} - \psi\frac{[3]_q^n([3]_q - 1)}{2\Phi_2(a,b,\mathcal{M},q)\Big([2]_q^n([2]_q - 1)\Big)^2}\right)\right), \tag{11}$$

with ρ_1 and ρ_2 defined by (8).

Corollary 5 ([22], Theorem 10)**.** *If $u \in \mathcal{S}_{q,\Im}^{n,j}(1,d,b)$ with $\mathcal{N}(k) = 1$ for all $k \geq 1$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\gamma_1|}{2[3]_q^n \Phi_3(d,b,1,q)([3]_q - 1)} \max\{1; |2\widehat{\Psi} - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\widehat{\Psi} := \Psi(d,b,1,q) = \frac{1}{2}\left(1 - \frac{\gamma_2}{\gamma_1} - \gamma_1\left(\frac{1}{[2]_q - 1} - \psi\frac{[3]_q^n([3]_q - 1)}{2\Phi_2(d,b,1,q)\Big([2]_q^n([2]_q - 1)\Big)^2}\right)\right),$$

with γ_1 and γ_2 given by (8).

Corollary 6 ([53], Theorem 3.3)**.** *If $u \in \mathcal{S}_{q,\Im}^{n,j}(1,0,1)$ with $\mathcal{N}(k) = 1$ for all $k \geq 1$, then*

$$|a_3 - \psi a_2^2| \leq \frac{|\gamma_1|}{2[3]_q^n \Phi_3(0,1,1,q)([3]_q - 1)} \max\{1; |2\widetilde{\Psi} - 1|\},$$

where $\psi \in \mathbb{C}$, and

$$\widetilde{\Psi} := \Psi(0,1,1,q) = \frac{1}{2}\left(1 - \frac{\gamma_2}{\gamma_1} - \gamma_1\left(\frac{1}{[2]_q - 1} - \psi\frac{[3]_q^n([3]_q - 1)}{2\Phi_2(0,1,1,q)\Big([2]_q^n([2]_q - 1)\Big)^2}\right)\right),$$

with γ_1 and γ_2 defined by (8).

The following result is related to the sufficient condition of the functions belonging to the class $\mathcal{S}_{q,\Im}^{n,j}(\alpha,d,b)$.

Theorem 4. *Let $u \in \Delta$ be of the form (1). If*

$$\sum_{k=2}^{\infty}\left(\left([k]_q-1\right)(j+1)+|\Im|\right)\,|\Phi_k(d,b,\mathcal{N},q)|\left(\frac{[k]_q-1}{\alpha}+1\right)^n|a_k|\le|\Im|,$$

then $u \in \mathcal{S}^{n,j}_{q,\Im}(\alpha,d,b)$.

Proof. Obviously, we have

$$\begin{aligned}
&\left|\frac{\eta\eth_q\left({}_m\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)\right)}{{}_m\mathcal{L}^n_q(d,b)u(\eta)}-1\right| \\
&=\left|\frac{\eta\eth_q\left({}_m\mathcal{L}^n_q(d,b)u(\eta)\right)-\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}{{}_m\mathcal{L}^n_q(d,b)u(\eta)}\right| \\
&=\left|\frac{\sum_{k=2}^{\infty}([k]_q-1)\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)a_k\eta^k}{\eta+\sum_{k=2}^{\infty}\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)a_k\eta^k}\right| \\
&\le\frac{\sum_{k=2}^{\infty}\left|([k]_q-1)\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)\right||a_k|}{1-\sum_{k=2}^{\infty}\left|\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)\right||a_k|},\ \eta\in\mathbb{D},
\end{aligned}$$

and from the assumption of the theorem

$$1-\sum_{k=2}^{\infty}\left|\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)\right||a_k|>0.$$

Since

$$\begin{aligned}
&\left|\frac{j}{\Im}\left(\frac{\eta\eth_q\left(\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}-1\right)\right|-\mathrm{Re}\left(\frac{1}{\Im}\left(\frac{\eta\eth_q\left(\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)\right)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}-1\right)\right) \\
&\le\frac{j}{|\Im|}\left|\left(\frac{\eta\eth_q\left({}_m\mathcal{L}^n_q(d,b)u(\eta)\right)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}-1\right)\right|+\left|\frac{1}{\Im}\right|\left|\frac{\eta\eth_q\left({}_m\mathcal{L}^n_q(d,b)u(\eta)\right)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}-1\right| \\
&=\frac{j+1}{|\Im|}\left|\left(\frac{\eta\eth_q\left({}_m\mathcal{L}^n_q(d,b)u(\eta)\right)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}-1\right)\right|=\frac{j+1}{|\Im|}\left|\frac{\eta\eth_q\left(\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)\right)-\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}{\mathcal{L}^n_q(\mathcal{N},d,b,\alpha)u(\eta)}\right| \\
&\le\frac{j+1}{|\Im|}\left(\frac{\sum_{k=2}^{\infty}\left|([k]_q-1)\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)\right||a_k|}{1-\sum_{k=2}^{\infty}\left|\left(\frac{[k]_q-1}{\alpha}+1\right)^n\Phi_k(d,b,\mathcal{N},q)\right||a_k|}\right)\le 1,\ \eta\in\mathbb{D},
\end{aligned}$$

we obtain $u \in \mathcal{S}^{n,j}_{q,\Im}(\alpha,d,b)$. □

We can see Theorem 4 generalizes other previously obtained results, e.g.,

Corollary 7 ([18], Theorem 4)**.** *Let $u \in \Lambda$ be in the form (1). If*

$$\sum_{k=2}^{\infty}\left(\left([k]_q-1\right)(j+1)+|\wp|\right)|\Phi_k(a,b,\mathcal{M},q)|[n]_q^k|a_k|\le|\wp|,$$

then $h \in \mathcal{S}^{n,j}_{q,\Im}(1, d, b)$.

Corollary 8 ([22], Theorem 12)**.** *Let* $u \in \Delta$ *be of the form* (1)*. If*

$$\sum_{k=2}^{\infty} \left(\left([k]_q - 1 \right) (j+1) + |\Im| \right) |\Phi_k(d, b, 1, q)| [k]_q^n |a_k| \leq |\Im|,$$

then

$$\frac{\eta \eth_q \left(\mathcal{L}_q^n(d,b) u(\eta) \right)}{\mathcal{L}_q^n(d,b) u(\eta)} \prec \Omega_{j,\Im}(\eta).$$

That is, $u \in \mathcal{S}^{n,j}_{q,\Im}(\alpha, d, b)$*, when* $\mathcal{N}(k) = 1$ *for all* $k \geq 1$.

Corollary 9 ([53], Theorem 3.4)**.** *Let* $u \in \Delta$ *be of the form* (1)*. If*

$$\sum_{k=2}^{\infty} \left(\left([k]_q - 1 \right) (j+1) + |\Im| \right) |\Phi_k(0, 1, 1, q)| [k]_q^n |a_k| \leq |\Im|,$$

then

$$\frac{\eta \eth_q \left(\mathcal{L}_q^n(0,1) u(\eta) \right)}{\mathcal{L}_q^n(0,1) u(\eta)} \prec \Omega_{j,\Im}(\eta).$$

That is, $u \in \mathcal{S}^{n,j}_{q,\Im}(0, 1)$*, when* $\mathcal{N}(k) = 1$ *for all* $k \geq 1$.

4. Conclusions

By using a Jackson differential operator and generalized Mittag–Leffler function, we introduced the class $\mathcal{S}^{n,j}_{q,\Im}(\alpha, d, b)$ of analytic functions in the unit disk. The coefficient bounds and the Fekete–Szegő problem have been obtained by using differential subordination in the geometric function theorem. A sufficient condition for the coefficients of the functions belonging to $\mathcal{S}^{n,j}_{q,\Im}(\alpha, d, b)$ has been considered. Furthermore, we highlighted some special results of cases for the class $\mathcal{S}^{n,j}_{q,\Im}(\alpha, d, b)$, which have been studied before. In the future work, using other classes of analytic functions which are associated with operators in Geometric Function Theory (GFT), the authors may study various new geometric properties and their applications in GFT.

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