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Investigation of the Second-Order Hankel Determinant for Sakaguchi-Type Functions Involving the Symmetric Cardioid-Shaped Domain

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Abstract: Determining the sharp bounds for coefficient-related problems that appear in the Taylor–Maclaurin series of univalent functions is one of the most difficult aspects of studying geometric function theory. The purpose of this article is to establish the sharp bounds for a variety of problems, such as the first three initial coefficient problems, the Zalcman inequalities, the Fekete–Szegö type results, and the second-order Hankel determinant for families of Sakaguchi-type functions related to the cardioid-shaped domain. Further, we study the logarithmic coefficients for both of these classes.

Keywords: starlike and convex functions; convolution; Zalcman functionals; Hankel determinant problems; logarithmic coefficients

MSC: 30C45; 30C50

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1. Preliminary Concepts

In order to effectively comprehend the core principle that underlies our important findings, we must analyze some essential function theory literature. For this, we use the symbols $\mathbb{U}_d$ and $\mathcal{A}$, which represent, respectively, the open unit disc and the analytic (or holomorphic) functions' family normalized by $f(0) = f'(0) - 1 = 0$. Particularly, if $f \in \mathcal{A}$, then it can be written in terms of the series expansion:

$$f(z) = z + \sum_{l=2}^{\infty} a_l z^l, \quad (z \in \mathbb{U}_d). \tag{1}$$

Moreover, remember that by notation $\mathcal{S}$, we denote the family of univalent functions with series expansion (1). This family was first taken into account by Köebe in 1907. Aleman and Constantin [1] recently reported on a stunning relationship between fluid dynamics and univalent function theory. In fact, they presented a straightforward technique that demonstrates how to employ a univalent harmonic map to find explicit solutions to incompressible two-dimensional Euler equations. It has many applications in differents fields of applied sciences such as fluid dynamics, modern mathematical physics, nonlinear integrable system theory, and the theory of partial differential equations.

The "Bieberbach conjecture", which is the most well-known outcome of function theory, was contributed by Bieberbach [2] in 1916. This hypothesis states that if f in $\mathcal{S}$, then $|a_n| \leq n$ for every $n \geq 2$. Furthermore, he demonstrated this result for $n = 2$. It is evident that a lot of well-regarded researchers have employed a range of techniques to address this issue. This hypothesis was settled for $n = 3$ by Schaeffer and Spencer [3] and also by Löwner [4] utilizing the variational method and the Löwner differential equation, respectively. Later, Jenkins [5] proved the same coefficient inequality $|a_3| \leq 3$ with the use of quadratic differentials. Garabedian and Schiffer [6] used the variational technique to determine that $|a_4| \leq 4$. Pederson and Schiffer [7] calculated that $|a_5| \leq 5$ by using the Garabedian–Schiffer inequality ([8], p. 108). Moreover, by using Ozawa [9] and the Grunsky inequality ([8], p. 60), Pederson [10] proved that $|a_6| \leq 6$. Numerous scholars have tried for a long time to prove this conjecture for $n \geq 7$, but no one has succeeded. Utilizing hypergeometric functions, de-Branges [11] finally proved this conjecture for every $n \geq 2$ in 1985. Zalcman proposed the inequality $\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2$ with $n \geq 2$ for $f \in \mathcal{S}$, in 1960, as a way of establishing the Bieberbach conjecture. Consequently, there have been several works [12–14] on the Zalcman conjecture and its generalized version $\left|\lambda a_n^2 - a_{2n-1}\right| \leq \lambda n^2 - 2n + 1$ $(\lambda \geq 0)$ for various subfamilies of the set $\mathcal{S}$, but this conjecture was open for many years. Finally, utilizing the holomorphic homotopy of univalent functions, Krushkal first proved this conjecture in [15] for $n \leq 6$ and then settled it in an unpublished article [16] for all $n \geq 2$. Later in 1999, Ma [17] presented a generalized Zalcman conjecture for $f \in \mathcal{S}$ that

$$|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1) \text{ for } n \geq 2, m \geq 2,$$

and he showed it for the family $\mathcal{S}^*$, but for the class $\mathcal{S}$, it is still an unsolved problem.

The following determinant $\mathcal{D}_{\lambda,n}(f)$ with $n, \lambda \in \mathbb{N} = \{1, 2, \ldots\}$, known as the Hankel determinant, was studied by Pommerenke [18,19] for the function $f \in \mathcal{S}$

$$\mathcal{D}_{\lambda,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\lambda-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\lambda} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+\lambda-1} & a_{n+\lambda} & \cdots & a_{n+2\lambda-2} \end{vmatrix}.$$

This determinant is important for many different studies, including those of power series with integral coefficients by Polya ([20], p. 323) and Cantor [21] and singularities by Hadamard ([20], 329) and Edrei [22], among many others.

Specifically, the following notations are used to identify the first- and second-order Hankel determinants, respectively:

$$\begin{aligned} \mathcal{D}_{2,1}(f) &= \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \\ \mathcal{D}_{2,2}(f) &= \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \end{aligned} \tag{2}$$

In the literature, there are few articles in which the bounds of the Hankel determinant were studied for the function f belonging to the family $\mathcal{S}$. The best-known sharp inequality for the function $f \in \mathcal{S}$ is given by $|\mathcal{D}_{2,n}(f)| \leq \eta$, where η is an absolute constant, which was proven by Hayman [23]. Furthermore, for the same family $\mathcal{S}$, it was found in [24] that $|\mathcal{D}_{2,2}(f)| \leq \eta$ for $1 \leq \eta \leq 11/3$. After these results, it was and still is a challenging problem for researchers to obtain the sharp bounds of the Hankel determinants for a specific class of functions. The first article [25] in which the authors successfully determined the sharp estimates of $|\mathcal{D}_{2,2}(f)|$ for the two fundemental subfamilies of the set $\mathcal{S}$ of univalent functions by using the concepts of the Caratheodory functions was published in 2007. These two determinants are well-studied in the literature [26–29] for diverse subfamilies of univalent functions; however, there are very few published papers [30,31], where the determinant's sharp bounds are determined. The interested readers may also appreciate the work of authors [32–36] in which they proved sharp bounds of the third-order Hankel determinant for some novel subfamilies of univalent functions.

Between 1916 and 1985, researchers have tried to solve this problem, and as a result, several interesting subfamilies of the class $\mathcal{S}$ have been introduced. Some of the fundamental families are defined as follows:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ f \in \mathcal{S} : \Re\mathfrak{e}\left(\frac{zf'(z)}{f(z)} \right) > 0, \ \ (z \in \mathbb{U}_d) \right\}, \\
\mathcal{K} &= \left\{ f \in \mathcal{S} : \Re\mathfrak{e}\left(\frac{(zf'(z))'}{f'(z)} \right) > 0, \ \ (z \in \mathbb{U}_d) \right\}, \\
\mathcal{C} &= \left\{ f \in \mathcal{S} : \Re\mathfrak{e}\left(\frac{zf'(z)}{h(z)} \right) > 0 \ \text{with } h \in \mathcal{S}^* \ (z \in \mathbb{U}_d) \right\}.
\end{aligned}$$

Using the familiar idea of subordination, Ma [37] contributed the unified subclass of the family $\mathcal{S}$, shown below, by assuming that the function ϕ in $\mathbb{U}_d$ is univalent with $\phi'(0) > 0$ and $\Re\mathfrak{e}\phi > 0$. Moreover, he claimed that the area $\phi(\mathbb{U}_d)$ is symmetric along the real line axis and has a star-shaped form about the point $\phi(0) = 1$.

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \phi(z), \ \ (z \in \mathbb{U}_d) \right\}.$$

He concentrated on some results, such as the theorem of covering, growth, and distortion. Several subfamilies of the collection $\mathcal{S}$ have been looked at as specific options for the class $\mathcal{S}^*(\phi)$ throughout the past few years. In the study described above, the following families are particularly noteworthy.

(i). $\mathcal{S}^*_{\mathcal{L}} \equiv \mathcal{S}^*(\sqrt{1+z})$ [38], $\mathcal{S}^*_{\exp} \equiv \mathcal{S}^*(\exp(z))$ [39], $\mathcal{S}^*_{\tanh} \equiv \mathcal{S}^*(1+\tanh(z))$ [40],

(ii). $\mathcal{S}^*_{\cos} \equiv \mathcal{S}^*(\cos(z))$ [41], $\mathcal{S}^*_{pet} \equiv \mathcal{S}^*\left(1+\sinh^{-1} z\right)$ [42], $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh(z))$ [43],

(iii). $\mathcal{S}^*_{\sin} \equiv \mathcal{S}^*(1+\sin(z))$ [44], $\mathcal{S}^*_{car} \equiv \mathcal{S}^*\left(1+z+\frac{1}{2}z^2\right)$ [45],

(iv). $\mathcal{S}^*_{(n-1)\mathcal{L}} \equiv \mathcal{S}^*(\Psi_{n-1}(z))$ [46] with $\Psi_{n-1}(z) = 1 + \frac{n}{n+1}z + \frac{1}{n+1}z^n$ for $n \geq 2$.

By proposing the family $\mathcal{S}^*_s$ of starlike functions concerning symmetric points in 1959, Sakaguchi [47] generalized the family $\mathcal{S}^*$ of starlike functions. This idea was then exploited by Das and Singh [48] to develop the family $\mathcal{K}_s$ of convex functions with symmetric points in 1977. In both of these papers, they gave the following analytic descriptions of these classes:

$$\begin{aligned}
\mathcal{S}^*_s &= \left\{ g \in \mathcal{S} : \Re\mathfrak{e} \frac{2zf'(z)}{f(z) - f(-z)} > 0, \ \ (z \in \mathbb{U}_d) \right\}, \\
\mathcal{K}_s &= \left\{ g \in \mathcal{S} : \Re\mathfrak{e} \frac{2(zf'(z))'}{(f(z) - f(-z))'} > 0, \ \ (z \in \mathbb{U}_d) \right\}.
\end{aligned}$$

In the same paper, Sakaguchi also claimed that the class $\mathcal{S}^*_s$ is a subfamily of the set $\mathcal{C}$ of close-to-convex functions and that it contains the families of convex and odd starlike functions with regard to the origin. Following that, several mathematicians introduced numerous new univalent function subfamilies with respect to symmetric points and studied coefficient-type problems; see a few of them in [49–52].

Now, with the use of the above facts, we now define the families $\mathcal{SS}^*_{car}$ and $\mathcal{SK}_{car}$ by the following representations:

$$\mathcal{SS}^*_{car} = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \psi_{car}(z) \qquad (z \in \mathbb{U}_d) \right\}, \tag{3}$$

and

$$\mathcal{SK}_{car} = \left\{ f \in \mathcal{S} : \frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \psi_{car}(z) \qquad (z \in \mathbb{U}_d) \right\}, \tag{4}$$

with

$$\psi_{car}(z) = 1 + z + \frac{1}{2}z^2.$$

The objective of this article is to calculate the sharp limits of the initial coefficients a_n with $n = 2, 3, 4$, the Zalcman, Fekete–Szegö, and Krushkal inequalities, as well as the determinant $|\mathcal{D}_{2,2}(f)|$ for the families $\mathcal{SS}^*_{car}$ and $\mathcal{SK}_{car}$ using a novel technique. The sharp bounds of the logarithmic coefficients for these newly established classes are also a subject of investigation in the current paper.

2. Set of Lemmas

Let $\mathfrak{B}_0$ be the family of Schwarz functions. Then, the function $\mathfrak{w} \in \mathfrak{B}_0$ may be expressed as a power series

$$\mathfrak{w}(z) = \sum_{n=1}^{\infty} w_n z^n. \tag{5}$$

The subsequent Schwarz function lemmas are required to prove our primary findings.

Lemma 1 ([53]). *Let $\mathfrak{w} \in \mathfrak{B}_0$ have the series expansion form (5). Then, we have*

$$\left| w_3 + \sigma w_1 w_2 + \varsigma w_1^3 \right| \le 1,$$

where $(\sigma, \varsigma) \in \mathbb{D}_1 \cup \mathbb{D}_2$, with

$$\begin{aligned} \mathbb{D}_1 &= \left\{ |\sigma| \le \frac{1}{2},\ -1 \le \varsigma \le 1 \right\}, \\ \mathbb{D}_2 &= \left\{ \frac{1}{2} \le |\sigma| \le 2,\ \frac{4}{27}(1 + |\sigma|)^3 - (1 + |\sigma|) \le \varsigma \le 1 \right\}. \end{aligned}$$

Lemma 2 ([54]). *If $\mathfrak{w} \in \mathfrak{B}_0$ is in the form (5), then*

$$|w_2| \le 1 - |w_1|^2, \tag{6}$$

$$|w_n| \le 1,\ n \ge 1. \tag{7}$$

Additionally, inequality (6) may be made better by

$$\left| w_2 + \eta w_1^2 \right| \le \max\{1, |\eta|\},\ \ \eta \in \mathbb{C}. \tag{8}$$

Lemma 3 ([55]). *If $\mathfrak{w} \in \mathfrak{B}_0$ is in the form (5), then*

$$|w_3| \le 1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}, \tag{9}$$

$$|w_4| \le 1 - |w_1|^2 - |w_2|^2. \tag{10}$$

Lemma 4 ([56]). *If $\mathfrak{w} \in \mathfrak{B}_0$ is in the form (5), then*

$$\left| w_1 w_3 - w_2^2 \right| \le 1 - |w_1|^2.$$

Lemma 5 ([57]). *If $\mathfrak{w} \in \mathfrak{B}_0$ is in the form (5) and $\gamma \in \mathbb{C}$, then*

$$\left| w_4 + 2w_1 w_3 + \eta w_2^2 + (1 + 2\eta) w_1^2 w_2 + \eta w_1^4 \right| \le \max\{1, |\eta|\}, \tag{11}$$

and

$$\left| w_4 + (1 + \eta) w_1 w_3 + w_2^2 + (1 + 2\eta) w_1^2 w_2 + \eta w_1^4 \right| \le \max\{1, |\eta|\}. \tag{12}$$

3. Coefficient Bounds for $\mathcal{SS}^*_{car}$

We start with the coefficient bounds of $f \in \mathcal{SS}^*_{car}$.

Theorem 1. *Let $f \in \mathcal{SS}^*_{car}$ have the series expansion form (1). Then,*

$$|a_2| \le \frac{1}{2},\ |a_3| \le \frac{1}{2},\ |a_4| \le \frac{1}{4}.$$

All these bounds are sharp.

Proof. Assume that $f \in \mathcal{SS}^*_{car}$. From the definition, it follows that there exists a Schwarz function $\mathfrak{w}$, such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = \psi_{car}(\mathfrak{w}(z)). \tag{13}$$

Using (1), we achieve

$$\frac{2zf'(z)}{f(z) - f(-z)} := 1 + 2a_2 z + 2a_3 z^2 + (4a_4 - 2a_2 a_3)z^3 + \left(4a_5 - 2a_3^2\right) z^4 + \cdots . \tag{14}$$

Let

$$\mathfrak{w}(z) = w_1 z + w_2 z^2 + w_3 z^3 + w_4 z^4 + \cdots . \tag{15}$$

By simple calculation and applying the series representation of (15), we obtain

$$\begin{aligned} \psi_{car}(\mathfrak{w}(z)) &= 1 + w_1 z + \left(w_2 + \frac{1}{2}w_1^2\right) z^2 + (w_3 + w_1 w_2) z^3 \\ &\quad + \left(w_1 w_3 + w_4 + \frac{1}{2}w_2^2\right) z^4 + \cdots . \end{aligned} \tag{16}$$

Now, by comparing (14) and (16), we achieve

$$a_2 = \frac{1}{2}w_1, \tag{17}$$

$$a_3 = \frac{1}{2}w_2 + \frac{1}{4}w_1^2, \tag{18}$$

$$a_4 = \frac{1}{4}w_3 + \frac{3}{8}w_1 w_2 + \frac{1}{16}w_1^3, \tag{19}$$

$$a_5 = \frac{1}{4}w_1 w_3 + \frac{1}{4}w_4 + \frac{1}{4}w_2^2 + \frac{1}{32}w_1^4 + \frac{1}{8}w_1^2 w_2. \tag{20}$$

Applying Lemma 2 to both (17) and (18), we obtain

$$|a_2| \le \frac{1}{2} \quad \text{and} \quad |a_3| \le \frac{1}{2}.$$

Rearranging (19), we have

$$|a_4| = \frac{1}{4}\left|w_3 + \frac{3}{2}w_1 w_2 + \frac{1}{4}w_1^3\right|.$$

By using the triangle inequality and Lemma 1 with $\sigma = \frac{3}{2}$ and $\varsigma = \frac{1}{4}$, we obtain

$$|a_4| \le \frac{1}{4}.$$

The bounds on the estimation of $|a_2|$, $|a_3|$, and $|a_4|$ are the best possible by using, respectively, the following extremal functions

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1+z+\frac{1}{2}z^2, \tag{21}$$

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1+z^2+\frac{1}{2}z^4, \tag{22}$$

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1+z^3+\frac{1}{2}z^6. \tag{23}$$

□

Theorem 2. *Let $f \in \mathcal{SS}^*_{car}$. Then,*

$$\left|a_3-\lambda a_2^2\right| \le \max\left\{\frac{1}{2}, \left|\frac{1-\lambda}{4}\right|\right\}.$$

The above-stated result is the best one.

Proof. From (17) and (18), we have

$$\begin{aligned}\left|a_3-\lambda a_2^2\right| &= \frac{1}{2}\left|w_2+\frac{1}{2}w_1^2-\frac{1}{2}\lambda w_1^2\right|, \\ &= \frac{1}{2}\left|w_2+\left(\frac{1-\lambda}{2}\right)w_1^2\right|.\end{aligned}$$

Applying the triangle inequality and Lemma 2, we obtain

$$\left|a_3-\lambda a_2^2\right| \le \max\left\{\frac{1}{2}, \left|\frac{1-\lambda}{4}\right|\right\}.$$

The obtained bound is sharp by considering $\mathfrak{w}(z)=z^2$. □

After putting $\lambda = 1$ into Theorem 2, we obtain the following corollary.

Corollary 1. *Let $f \in \mathcal{SS}^*_{car}$. Then,*

$$\left|a_3-a_2^2\right| \le \frac{1}{2}.$$

The equality is obtained by the extremal function provided by (22).

Next, we provide the bounds of the Zalcman inequalities for $f \in \mathcal{SS}^*_{car}$.

Theorem 3. *Let $f \in \mathcal{SS}^*_{car}$ have the series expansion form (1). Then,*

$$|a_4-a_2a_3| \le \frac{1}{4}, \tag{24}$$

and

$$\left|a_5-a_3^2\right| \le \frac{1}{4}. \tag{25}$$

Both the inequalities (24) and (25) are sharp.

Proof. Utilizing (17)–(19), we obtain

$$|a_4-a_2a_3| = \frac{1}{4}\left|w_3+\frac{1}{2}w_1w_2-\frac{1}{4}w_1^3\right|;$$

so, by taking $\sigma = \frac{1}{2}$ and $\varsigma = -\frac{1}{4}$ in Lemma 1, it yields

$$|a_4 - a_2 a_3| \leq \frac{1}{4}.$$

To estimate $d_5 - d_3^2$, we write this expression as follows:

$$\begin{aligned}
\left|a_5 - a_3^2\right| &= \frac{1}{4}\left|w_4 + w_1 w_3 - \frac{1}{2}w_1^2 w_2 - \frac{1}{8}w_1^4\right| \\
&= \frac{1}{4}\left|\frac{1}{2}\left(w_4 + 2w_1 w_3 - \frac{1}{2}w_2^2 - \frac{1}{2}w_1^4\right) + \frac{1}{2}\left(w_4 + \frac{1}{4}w_1^4 - w_1^2 w_2 + \frac{1}{2}w_2^2\right)\right| \\
&\leq \frac{1}{8}\left|w_4 + 2w_1 w_3 - \frac{1}{2}w_2^2 - \frac{1}{2}w_1^4\right| + \frac{1}{8}\left|w_4 + \frac{1}{4}w_1^4 - w_1^2 w_2 + \frac{1}{2}w_2^2\right| \\
&= \frac{1}{8}U_1 + \frac{1}{8}U_2,
\end{aligned}$$

where

$$\begin{aligned}
U_1 &= \left|w_4 + 2w_1 w_3 - \frac{1}{2}w_2^2 - \frac{1}{2}w_1^4\right|, \\
U_2 &= \left|w_4 + \frac{1}{4}w_1^4 - w_1^2 w_2 + \frac{1}{2}w_2^2\right|.
\end{aligned}$$

Taking $\eta = -\frac{1}{2}$ in (11), we obtain $U_1 \leq 1$. For U_2, using Lemma 3, we have

$$\begin{aligned}
U_2 &\leq |w_4| + \frac{1}{4}|w_1|^4 + \left|w_1^2\right||w_2| + \frac{1}{2}|w_2|^2 \\
&\leq 1 - |w_1|^2 - |w_2|^2 + \frac{1}{4}|w_1|^4 + |w_1|^2|w_2| + \frac{1}{2}|w_2|^2 \\
&\leq 1 - |w_1|^2 - \frac{1}{2}|w_2|^2 + |w_1|^2|w_2| + \frac{1}{4}|w_1|^4.
\end{aligned}$$

After some simple calculations, we obtain $U_2 \leq 1$. Hence, we obtain

$$\left|a_5 - a_3^2\right| \leq \frac{1}{8}U_1 + \frac{1}{8}U_2 \leq \frac{1}{4}.$$

Thus, the proof of Theorem 3 is complete.
Equalities are obtained in both the inequalities by choosing the functions

$$\begin{aligned}
\frac{2zf'(z)}{f(z) - f(-z)} &= 1 + z^3 + \frac{1}{2}z^6 \\
\frac{2zf'(z)}{f(z) - f(-z)} &= 1 + z^4 + \frac{1}{2}z^8.
\end{aligned} \tag{26}$$

□

Now, our focus turns to studying the second-order Hankel determinant for $\mathcal{SS}^*_{car}$.

Theorem 4. *If $f \in \mathcal{SS}^*_{car}$ has the series form (1), then*

$$|\mathcal{D}_{2,2}(f)| \leq \frac{1}{4}.$$

The above-stated result is the best possible by using the extremal function provided in (22).

Proof. From (17)–(19), we easily obtain

$$\begin{aligned} |\mathcal{D}_{2,2}(f)| &= \frac{1}{4}\left|\frac{1}{2}\left(w_2^2 - w_1 w_3\right) + \frac{1}{2}\left(w_2^2 + \frac{1}{4}w_1^4 + \frac{1}{2}w_1^2 w_2\right)\right| \\ &\leq \frac{1}{8}\left|w_2^2 - w_1 w_3\right| + \frac{1}{8}\left|w_2^2 + \frac{1}{4}w_1^4 + \frac{1}{2}w_1^2 w_2\right| \\ &= \frac{1}{8}T_1 + \frac{1}{8}T_2, \end{aligned}$$

where

$$T_1 = \left|w_2^2 - w_1 w_3\right|,$$

and

$$T_2 = \left|w_2^2 + \frac{1}{4}w_1^4 + \frac{1}{2}w_1^2 w_2\right|.$$

Using Lemma 4, we obtain $T_1 \leq 1$, since

$$\begin{aligned} \left|w_2^2 + \frac{1}{4}w_1^4 + \frac{1}{2}w_1^2 w_2\right| &\leq \left(1 - |w_1|^2\right)^2 + \frac{1}{4}|w_1|^4 + \frac{1}{2}|w_1|^2\left(1 - |w_1|^2\right) \\ &= 1 - \frac{3}{2}|w_1|^2 + \frac{3}{4}|w_1|^4 = \chi\left(|w_1|^2\right), \end{aligned}$$

where

$$\chi(t) = 1 - \frac{3}{2}t + \frac{3}{4}t^2.$$

It is easy to observe that χ is a decreasing function on $t \in [0,1]$; thus, we have $\chi(t) \leq \chi(0) = 1$. As $|w_1|^2 \in [0,1]$, we obtain $T_1 \leq 1$. Hence, we obtain

$$|\mathcal{D}_{2,2}(f)| \leq \frac{1}{8}T_1 + \frac{1}{8}T_2 \leq \frac{1}{4}.$$

The assertion of Theorem 4 is thus proved. □

Theorem 5. *Let $f \in \mathcal{SS}^*_{car}$. Then,*

$$|\mathcal{D}_{2,3}(f)| = \left|a_3 a_5 - a_4^2\right| \leq \frac{1}{8}.$$

This above-stated result is the best possible with the extremal function given by (22).

Proof. From (18)–(20), we have

$$\begin{aligned} |\mathcal{D}_{2,3}(f)| &= \frac{1}{8}\left|\left(w_2 + \frac{1}{2}w_1^2\right)w_4 - \frac{1}{8}w_1^2 w_2^2 - \frac{1}{2}w_1 w_3\left(w_2 - \frac{1}{2}w_1^2\right)\right. \\ &\quad \left. - \frac{1}{2}w_3^2 + \frac{1}{32}w_1^6 + w_2^3\right|. \end{aligned}$$

Applying the triangle inequality and Lemma 3, we have

$$\begin{aligned} |\mathcal{D}_{2,3}(f)| \leq{}& \frac{1}{8}\left[\left(|w_2| + \frac{1}{2}|w_1|^2\right)\left(1 - |w_1|^2 - |w_2|^2\right) + \frac{1}{8}|w_1|^2|w_2|^2 + \frac{1}{32}|w_1|^6 + |w_2|^3\right. \\ & \left. + \frac{1}{2}|w_1|\left(1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}\right)\left(|w_2| + \frac{1}{2}|w_1|^2\right) + \frac{1}{2}\left(1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}\right)^2\right]. \end{aligned}$$

In the inequality above, let $h(|w_1|,|w_2|)$ denote the right hand side and let $x=|w_1|$, $y=|w_2|$. Since

$$\begin{aligned}\frac{\partial h}{\partial d} &= \frac{1}{4(1+x)^2}\Big[8y^3-(1+x)6xy^2-\Big(5x^3-5x^2+8\Big)(1+x)y \\ &\quad -2(x-1)(1+x)^3(x+2)\Big],\end{aligned}$$

replacing y^2 by y, we achieve

$$\frac{\partial h}{\partial y}\geq\frac{k(x,y)}{4(1+x)^2},$$

with

$$k(x,y)=8y^3-(1+x)\Big(5x^3-5x^2+6x+8\Big)y+2(1-x)(1+x)^3(x+2).$$

A simple algebraic calculation demonstrates that the critical points of f justify

$$\left\{\begin{array}{c}(14+2x+20x^3)y-2\big(5-6x-5x^2\big)(1+x)^2=0\\ 24y^2-(1+x)\big(5x^3-5x^2+6x+8\big)=0\end{array}\right..$$

So, in $\Delta=\big\{(x,y):0\leq x\leq 1,\ 0\leq y\leq 1-x^2\big\}$, there is only one critical point (x_0,y_0) where $x_0=0.2336...$ and $y_0=0.6874\cdots$. For this point,

$$k(x_0,y_0)=1.2305\cdots.$$

On the boundary of Δ, we obtain

$$\begin{aligned}k(x,0) &= 2(1-x)(x+2)(1+x)^3\geq k(1,0)=0,\\ k(0,y) &= 4\Big(1-2y+2y^3\Big)\geq k\left(0,\frac{\sqrt{3}}{3}\right)=4-\frac{16\sqrt{3}}{9},\\ k\Big(x,1-x^2\Big) &= (1-x)(1+x)^2\Big(4-8x-x^2+3x^3\Big).\end{aligned}$$

It is simple task to show that $4-8x-x^2+3x^3>0$ in $[0,1]$; so,

$$k(x,0)\geq 0 \text{ for } (x,y)\in\Delta.$$

This means that

$$\frac{\partial h}{\partial y}\geq 0 \text{ for } (x,y)\in\Delta.$$

Consequently,

$$h(x,y)\leq h\Big(x,1-x^2\Big)=1+\frac{13}{32}x^6-\frac{1}{2}x^4-\frac{7}{8}x^2.$$

Therefore, $h\big(x,1-x^2\big)$ is decreasing for $x\in[0,1]$; so, $h(x,y)\leq h\big(x,1-x^2\big)\leq 1$. Hence,

$$|\mathcal{D}_{2,3}(f)|\leq\frac{1}{8}.$$

Thus, the result is proved. □

4. Coefficient Bounds for $\mathcal{SK}_{car}$

We start with the estimates of some initial coefficients of $f\in\mathcal{SK}_{car}$.

Theorem 6. *If $f \in \mathcal{SK}_{car}$ is of the form* (1), *then*

$$\begin{aligned} |a_2| &\leq \frac{1}{4}, \\ |a_3| &\leq \frac{1}{6}, \\ |a_4| &\leq \frac{1}{16}. \end{aligned}$$

All these bounds are sharp.

Proof. Using the Alexander-type relation along with the obtained coefficient estimates of the class $\mathcal{SS}_{car}$, we easily obtain the above bounds. The bounds on the estimation of $|a_2|$, $|a_3|$, and $|a_4|$ are sharp using the extremal functions provided, respectively, by

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 1+z+\frac{1}{2}z^2, \tag{27}$$

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 1+z^2+\frac{1}{2}z^4, \tag{28}$$

$$\frac{2(zf'(z))'}{(f(z)-f(-z))'} = 1+z^3+\frac{1}{2}z^6. \tag{29}$$

□

Theorem 7. *Let $f \in \mathcal{SK}_{car}$ have the series expansion* (1). *Then,*

$$\left|a_3 - \lambda a_2^2\right| \leq \max\left\{\frac{1}{6}, \left|\frac{4-3\lambda}{48}\right|\right\}.$$

This result is sharp.

Proof. Utilizing the definition of the class $\mathcal{SK}_{car}$, we easily have

$$a_2 = \frac{1}{4}w_1, \tag{30}$$

$$a_3 = \frac{1}{6}w_2 + \frac{1}{12}w_1^2, \tag{31}$$

$$a_4 = \frac{1}{16}w_3 + \frac{3}{32}w_1w_2 + \frac{1}{64}w_1^3, \tag{32}$$

$$a_5 = \frac{1}{20}w_1w_3 + \frac{1}{20}w_4 + \frac{1}{20}w_2^2 + \frac{1}{160}w_1^4 + \frac{1}{40}w_1^2w_2. \tag{33}$$

From (30) and (31), we obtain

$$\begin{aligned} \left|a_3 - \lambda a_2^2\right| &= \frac{1}{6}\left|w_2 + \frac{1}{2}w_1^2 - \frac{3}{8}\lambda w_1^2\right|, \\ &= \frac{1}{6}\left|w_2 + \left(\frac{4-3\lambda}{8}\right)w_1^2\right|. \end{aligned}$$

Applying the triangle inequality and Lemma 2, we obtain

$$\left|a_3 - \lambda a_2^2\right| \leq \max\left\{\frac{1}{6}, \left|\frac{4-3\lambda}{48}\right|\right\}.$$

□

Putting $\lambda = 1$, we achieve the following corollary.

Corollary 2. *Let $f \in \mathcal{SK}_{car}$ be the series form (1). Then,*

$$\left|a_3 - a_2^2\right| \leq \frac{1}{6}.$$

The equality is obtained by using (28).

Now, we discuss the Zalcman functionals for $f \in \mathcal{SK}_{car}$.

Theorem 8. *Let $f \in \mathcal{SK}_{car}$ be the series form (1). Then,*

$$|a_4 - a_2 a_3| \leq \frac{1}{16}. \tag{34}$$

The above-stated result is the best possible using the extremal function provided in (29).

Proof. From (30)–(32), we obtain

$$|a_4 - a_2 a_3| = \frac{1}{16}\left|w_3 + \frac{5}{6}w_1 w_2 - \frac{1}{12}w_1^3\right|;$$

so, taking $\sigma = \frac{5}{6}$ and $\varsigma = -\frac{1}{12}$ in Lemma 1 yields

$$|a_4 - a_2 a_3| \leq \frac{1}{16}.$$

The assertion of Theorem 8 is thus proved. □

The Hankel determinants for the class $\mathcal{SK}_{car}$ are now our focus.

Theorem 9. *If $f \in \mathcal{SK}_{car}$ is of the form (1), then*

$$|\mathcal{D}_{2,2}(f)| \leq \frac{1}{36}.$$

The above-stated result is the best possible using the extremal function provided by (28).

Proof. From (30)–(32), we have

$$\begin{aligned}
\left|a_2 a_4 - a_3^2\right| &= \frac{1}{36}\left|w_2^2 + \frac{7}{64}w_1^4 - \frac{9}{16}w_1 w_3 + \frac{5}{32}w_1^2 w_2\right|, \\
&= \frac{1}{36}\left|\frac{1}{2}\left(w_2^2 - w_1 w_3\right) + \frac{1}{2}\left(\frac{7}{32}w_1^4 - \frac{1}{8}w_1 w_3 + \frac{5}{16}w_1^2 w_2 + w_2^2\right)\right|, \\
&\leq \frac{1}{72}\left|w_2^2 - w_1 w_3\right| + \frac{1}{72}\left|\frac{7}{32}w_1^4 - \frac{1}{8}w_1 w_3 + \frac{5}{16}w_1^2 w_2 + w_2^2\right|, \\
&= \frac{1}{72}Q_1 + \frac{1}{72}Q_2.
\end{aligned}$$

Using Lemma 4, it is clear that $Q_1 \leq 1$. For Q_2, using Lemma 3, we have

$$\begin{aligned}
|Q_2| &\leq \frac{7}{32}|w_1|^4 + \frac{1}{8}|w_1|\left(1 - |w_1|^2 - \frac{|w_2|^2}{1 + |w_1|}\right) + \frac{5}{16}|w_1|^2|w_2| + |w_2|^2, \\
&\leq \frac{7}{32}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 - \frac{|w_1||w_2|^2}{8(1 + |w_1|)} + \frac{5}{16}|w_1|^2|w_2| + |w_2|^2, \\
&\leq \frac{7}{32}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 + |w_2|^2\left(1 - \frac{|w_1|}{8(1 + |w_1|)}\right) + \frac{5}{16}|w_1|^2|w_2|.
\end{aligned} \tag{35}$$

Since $\left(1-\frac{|w_1|}{8(1+|w_1|)}\right)>0$, we can put $|w_2|\le 1-|w_1|^2$ in (35), and we have

$$|Q_2|\le \frac{7}{32}|w_1|^4+\frac{1}{8}|w_1|-\frac{1}{8}|w_1|^3+\left(1-|w_1|^2\right)^2\left(1-\frac{|w_1|}{8(1+|w_1|)}\right)+\frac{5}{16}|w_1|^2\left(1-|w_1|^2\right).$$

Let us put $|w_1|=x$ and $x\in(0,1]$, and we obtain

$$|Q_2|\le 1-\frac{25}{16}x^2+\frac{25}{32}x^4=F(x).$$

As $F'(x)\leqq 0$, $F(x)$ is the decreasing function of x, and it gives the maximum value at $x=0$

$$|Q_2|\le 1.$$

Hence, we obtain that

$$\left|a_2a_4-a_3^2\right|\le \frac{1}{72}Q_1+\frac{1}{72}Q_2\le\frac{1}{36}.$$

The assertion of Theorem 9 is thus proved. □

Theorem 10. *If $f\in\mathcal{SK}_{car}$ is of the form (1), then*

$$|\mathcal{D}_{2,3}(f)|\le\frac{1}{120}.$$

This result is the best possible using the extremal function defined by (28).

Proof. From (31)–(33), we have

$$\begin{aligned}\left|a_3a_5-a_4^2\right| &= \frac{1}{120}\left|\left(w_2+\frac{1}{2}w_1^2\right)w_4-\frac{7}{128}w_1^2w_2^2+\frac{17}{512}w_1^6-\frac{13}{32}w_1w_3\left(w_2-\frac{17}{26}w_1^2\right)\right.\\ &\quad\left.+\frac{3}{128}w_1^4w_2+w_2^3-\frac{15}{32}w_3^2\right|.\end{aligned}$$

Using Lemma 3 and the triangle inequality, we obtain

$$\begin{aligned}\left|a_3a_5-a_4^2\right| &\le \frac{1}{120}\left[\left(|w_2|+\frac{1}{2}|w_1|^2\right)\left(1-|w_1|^2-|w_2|^2\right)+\frac{17}{512}|w_1|^6+\frac{3}{128}|w_1|^4|w_2|\right.\\ &\quad+\frac{7}{128}|w_1|^2|w_2|^2+\frac{13}{32}|w_1|\left(1-|w_1|^2-\frac{|w_2|^2}{1+|w_1|}\right)\left(|w_2|+\frac{17}{26}|w_1|^2\right)\\ &\quad\left.+|w_2|^3+\frac{15}{32}\left(1-|w_1|^2-\frac{|w_2|^2}{1+|w_1|}\right)^2\right].\end{aligned}$$

Let $h(|w_1|,|w_2|)$ represent the right hand side of the above inequality and let $x=|w_1|$, $y=|w_2|$. Since

$$\begin{aligned}\frac{\partial h}{\partial y} &= \frac{1}{128(1+x)^2}\left[240y^3-156x(1+x)y^2-2(1+x)\left(91x^3-63x^2+120\right)y\right.\\ &\quad\left.+\left(3x^4-52x^3-128x^2+52x+128\right)(1+x)^2\right],\end{aligned}$$

omitting x^4 in the last component and replacing y^2 by y, we obtain

$$\frac{\partial h}{\partial y}\ge\frac{k(x,y)}{128(1+x)^2},$$

with

$$k(x,y) = 240y^3 - 2(1+x)\left(91x^3 - 63x^2 + 78x + 120\right)y + 4(13x+32)(1-x)(1+x)^3.$$

A simple algebraic calculation demonstrates that the critical points of f justify

$$\begin{cases} (783x^3 + 168x^2 + 60x + 396)y - 4(77 - 65x^2 - 102x)(1+x)^2 = 0 \\ 720y^2 - 2(1+x)(91x^3 - 63x^2 + 78x + 120) = 0 \end{cases}.$$

So, in $\Xi = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1 - x^2\}$, there is only one critical point (x_0, y_0), where $x_0 = 0.2472...$ and $y_0 = 0.6884\cdots$. For this point,

$$k(x_0, y_0) = 49.0898\cdots.$$

On the boundary of Ξ, we have

$$\begin{aligned} k(x,0) &= (52x+128)(1-x)(1+x)^3 \ge k(1,0) = 0, \\ k(0,y) &= 240y^3 - 240y + 128 \ge k\left(0, \frac{\sqrt{3}}{3}\right) = 128 - \frac{160\sqrt{3}}{3}, \\ k\left(x, 1-x^2\right) &= 2(1-x)(1+x)^2\left(29x^3 - 31x^2 - 108x + 64\right). \end{aligned}$$

It is a simple task to show that $29x^3 - 31x^2 - 108x + 64 > 0$ in $[0,1]$; so,

$$k(x,0) \ge 0 \text{ for } (x,y) \in \Xi.$$

This means that

$$\frac{\partial h}{\partial x} \ge 0 \text{ for } (x,y) \in \Xi.$$

Consequently,

$$h(x,y) \le h\left(x, 1-x^2\right) = 1 - \frac{137}{128}x^2 - \frac{9}{128}x^4 + \frac{89}{512}x^6.$$

Therefore, $h(x, 1-x^2)$ is decreasing for $x \in [0,1]$; so, $h(x,y) \le h(x, 1-x^2) \le 1$. Therefore,

$$|\mathcal{D}_{2,3}(f)| \le \frac{1}{120}.$$

Thus, the proof is complete. □

5. Logarithmic Coefficients for $\mathcal{SS}^*_{car}$ and $\mathcal{SK}_{car}$

The logarithmic coefficients of a given function f, denoted by $\gamma_n := \gamma_n(f)$, are defined as

$$\frac{1}{2}\log\left(\frac{f(z)}{z}\right) = \sum_{n=1}^{\infty} \gamma_n z^n. \tag{36}$$

The theory of Schlicht functions is significantly impacted by these coefficients in various estimations. In 1985, de-Branges [11] determined that

$$\sum_{k=1}^{n} k(n-k+1)|\gamma_n|^2 \le \sum_{k=1}^{n} \frac{n-k+1}{k}, \text{ for } n \ge 1,$$

and for the particular function $f(z) = z/\left(1 - e^{i\theta}z\right)$ with $\theta \in \mathbb{R}$, equality is attained. Evidently, this inequality gives rise to the broadest formulation of the well-known Bieberbach–Robertson–Milin conjectures involving the Taylor coefficients of f that belong to $\mathcal{S}$. For more information on how de-Brange's claim was explained, see [58–60]. Brennan's conjecture for conformal mappings was answered by Kayumov [61] in 2005 by taking the logarithmic coefficients into account. Several works [62–64] that have significantly advanced the study of logarithmic coefficients are included in this article.

It is easy to determine from the definition given above that the logarithmic coefficients for f belonging to $\mathcal{S}$ are given by

$$\gamma_1 = \frac{1}{2}a_2 \tag{37}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) \tag{38}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{39}$$

The Hankel determinant $\mathcal{D}_{q,n}\left(F_f/2\right)$ with logarithmic coefficients was initially developed by Kowalczyk and Lecko in [65,66] and is given by

$$\mathcal{D}_{q,n}\left(F_f/2\right) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \tag{40}$$

In particular, it is noted that

$$\mathcal{D}_{2,1}\left(F_f/2\right) = \begin{vmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 \end{vmatrix} = \left|\gamma_1\gamma_3 - \gamma_2^2\right|.$$

For more details on the investigation of logarithmic coefficients, see the articles [67–71].

Theorem 11. *Let $f \in \mathcal{SS}^*_{car}$ be the series form (1). Then,*

$$|\gamma_1| \le \frac{1}{4},$$

$$|\gamma_2| \le \frac{1}{4},$$

$$|\gamma_3| \le \frac{1}{8}.$$

All these bounds are sharp.

Proof. Applying (17)–(19) in (37)–(39), we obtain

$$\gamma_1 = \frac{1}{4}w_1, \tag{41}$$

$$\gamma_2 = \frac{1}{4}w_2 + \frac{1}{16}w_1^2, \tag{42}$$

$$\gamma_3 = \frac{1}{8}w_3 + \frac{1}{16}w_1w_2 - \frac{1}{96}w_1^3. \tag{43}$$

The bounds of γ_1 and γ_2 are clear.
For γ_3 rearranging (43), we obtain

$$|\gamma_3| = \frac{1}{8}\left|w_3 + \frac{1}{2}w_1w_2 - \frac{1}{12}w_1^3\right|.$$

Using Lemma 1 with $\sigma = \frac{1}{2}$, $\varsigma = -\frac{1}{12}$, and the triangle inequality, we obtain

$$|\gamma_3| = \frac{1}{8}.$$

The equalities hold for the function given by (21)–(23) and using (37)–(39). □

Theorem 12. *Let $f \in \mathcal{SS}^*_{car}$ have the series representation (1). Then,*

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{1-\lambda}{16}\right|\right\}.$$

The above-stated result is the best possible.

Proof. From (41) and (42), we have

$$\begin{aligned} \left|\gamma_2 - \lambda\gamma_1^2\right| &= \frac{1}{4}\left|w_2 + \frac{1}{4}w_1^2 - \frac{\lambda}{4}w_1^2\right|, \\ &= \frac{1}{4}\left|w_2 + \left(\frac{1-\lambda}{4}\right)w_1^2\right|. \end{aligned}$$

Using Lemma 2 and the triangle inequality, we have

$$\left|\gamma_2 - \lambda_1^2\right| \leq \max\left\{\frac{1}{4}, \left|\frac{1-\lambda}{16}\right|\right\}.$$

□

Putting $\lambda = 1$, we obtain the following consequence.

Corollary 3. *Let $f \in \mathcal{SS}^*_{car}$ be the series form (1). Then,*

$$\left|\gamma_2 - \gamma_1^2\right| \leq \frac{1}{4}.$$

The equality is determined by using (37), (38), and (22).

Theorem 13. *If $f \in \mathcal{SS}^*_{car}$ has the expansion form (1), then*

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{8}.$$

The equality is determined by using (37)–(39) and (23).

Proof. From (41)–(43), we obtain

$$|\gamma_3 - \gamma_1\gamma_2| = \frac{1}{8}\left|w_3 - \frac{5}{24}w_1^3\right|.$$

Applying the triangle inequality and Lemma 3, we have

$$\left|w_3 - \frac{5}{24}w_1^3\right| \leq 1 - |w_1|^2 - \frac{|w_2|^2}{1+|w_1|} + \frac{5}{24}|w_1|^3.$$

After some simple calculations, we obtain

$$\left|w_3 - \frac{5}{24}w_1^3\right| \leq 1.$$

Hence, we obtain

$$|\gamma_3 - \gamma_1\gamma_2| \leq \frac{1}{8}.$$

Thus, the result is complete. □

Theorem 14. *If $f \in \mathcal{SS}^*_{car}$ is given by (1), then*

$$\left|\mathcal{D}_{2,1}\left(F_f/2\right)\right| \leq \frac{1}{16}.$$

The equality is determined by using (37)–(39) and (22).

Proof. From (41)–(43), we have

$$\begin{aligned}
\left|\gamma_1\gamma_3 - \gamma_2^2\right| &= \frac{1}{16}\left|\frac{1}{2}\left(w_2^2 - w_1w_3\right) + \frac{1}{2}\left(w_2^2 + \frac{5}{24}w_1^4 + \frac{1}{2}w_1^2w_2\right)\right| \\
&\leq \frac{1}{32}\left|w_2^2 - w_1w_3\right| + \frac{1}{32}\left|w_2^2 + \frac{5}{24}w_1^4 + \frac{1}{2}w_1^2w_2\right| \\
&= \frac{1}{32}R_1 + \frac{1}{32}R_2,
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \left|w_2^2 - w_1w_3\right| \\
R_2 &= \left|w_2^2 + \frac{5}{24}w_1^4 + \frac{1}{2}w_1^2w_2\right|.
\end{aligned}$$

Using Lemma 4, we obtain $R_1 \leq 1$, since

$$\begin{aligned}
\left|w_2^2 + \frac{1}{4}w_1^4 + \frac{1}{2}w_1^2w_2\right| &\leq \left(1 - |w_1|^2\right)^2 + \frac{5}{24}|w_1|^4 + \frac{1}{2}|w_1|^2\left(1 - |w_1|^2\right) \\
&= 1 - \frac{3}{2}|w_1|^2 + \frac{17}{24}|w_1|^4 = \xi\left(|w_1|^2\right),
\end{aligned}$$

where

$$\xi(t) = 1 - \frac{3}{2}t + \frac{17}{24}t^2.$$

It is easy to observe that ξ is a decreasing function on $t \in [0,1]$; thus, we have $\xi(t) \leq \xi(0) = 1$. As $|w_1|^2 \in [0,1]$, we obtain $R_2 \leq 1$. Hence, we obtain

$$\left|\mathcal{D}_{2,1}\left(F_f/2\right)\right| \leq \frac{1}{32}R_1 + \frac{1}{32}R_2 \leq \frac{1}{16}.$$

Thus, the assertion of Theorem 14 is proved. □

Now, we study the logarithmic coefficients for the family $f \in \mathcal{SK}_{car}$.

Theorem 15. *If $f \in \mathcal{SK}_{car}$ is given by (1), then*

$$\begin{aligned}
|\gamma_1| &\leq \frac{1}{8}, \\
|\gamma_2| &\leq \frac{1}{12}, \\
|\gamma_3| &\leq \frac{1}{32}.
\end{aligned}$$

The bounds are sharp.

Proof. Applying (30)–(32) in (37)–(39), we obtain

$$\gamma_1 = \frac{1}{8}w_1, \tag{44}$$

$$\gamma_2 = \frac{1}{12}w_2 + \frac{5}{192}w_1^2, \tag{45}$$

$$\gamma_3 = \frac{1}{32}w_3 + \frac{5}{192}w_1 w_2. \tag{46}$$

The bounds of γ_1 and γ_2 are clear.
Rearranging γ_3, we obtain

$$|\gamma_3| = \frac{1}{32}\left|w_3 + \frac{5}{6}w_1 w_2\right|.$$

Using the triangle inequality and Lemma 3, we have

$$\left|w_3 + \frac{5}{6}w_1 w_2\right| \leq 1 - |w_1|^2 - \frac{|w_2|^2}{1+|w_1|} + \frac{5}{6}|w_1||w_2|.$$

After some simple calculations, we obtain

$$\left|w_3 + \frac{5}{6}w_1 w_2\right| \leq 1.$$

Hence,

$$|\gamma_3| \leq \frac{1}{32}.$$

The equalities holds for the function given (27)–(29) and using (37)–(39). □

Theorem 16. *If $f \in \mathcal{SK}_{car}$ is of the form (1), then*

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \max\left\{\frac{1}{12}, \left|\frac{5-3\lambda}{192}\right|\right\}.$$

The above-stated result is the best possible.

Proof. From (44) and (45), we obtain

$$\begin{aligned}\left|\gamma_2 - \lambda\gamma_1^2\right| &= \frac{1}{12}\left|w_2 - \frac{3\lambda}{16}w_1^2 + \frac{5}{16}w_1^2\right| \\ &= \frac{1}{12}\left|w_2 + \left(\frac{5-3\lambda}{16}\right)w_1^2\right|.\end{aligned}$$

Using (8) and the triangle inequality, we obtain

$$\left|\gamma_2 - \lambda\gamma_1^2\right| \leq \max\left\{\frac{1}{12}, \left|\frac{5-3\lambda}{192}\right|\right\}.$$

□

Putting $\lambda = 1$, we achieve the following corollary.

Corollary 4. *If $f \in \mathcal{SK}_{car}$ is given by (1), then*

$$\left|\gamma_2 - \gamma_1^2\right| \leq \frac{1}{12}.$$

The equality is determined by using (37), (38) and (28).

Theorem 17. *If $f \in \mathcal{SK}_{car}$ is of the form (1), then*

$$|\gamma_3 - \gamma_1\gamma_2| \le \frac{1}{32}.$$

The equality is determined by using (37)–(39), and (29).

Proof. From (44)–(46), we obtain

$$|\gamma_3 - \gamma_1\gamma_2| = \frac{1}{32}\left|w_3 + \frac{1}{2}w_1w_2 - \frac{5}{48}w_1^3\right|;$$

so, taking $\sigma = \frac{1}{2}$ and $\varsigma = -\frac{5}{48}$ in Lemma 1 yields

$$|\gamma_3 - \gamma_1\gamma_2| \le \frac{1}{32}.$$

Thus, the proof is complete. □

Theorem 18. *If $f \in \mathcal{SK}_{car}$ is of the form (1), then*

$$\left|\mathcal{D}_{2,1}\left(F_f/2\right)\right| \le \frac{1}{144}.$$

The equality is determined by using (37)–(39) and (28).

Proof. From (44)–(46), we have

$$\begin{aligned}
\left|\gamma_1\gamma_3 - \gamma_2^2\right| &= \frac{1}{144}\left|w_2^2 + \frac{25}{256}w_1^4 - \frac{9}{16}w_1w_3 + \frac{5}{32}w_1^2w_2\right| \\
&= \frac{1}{144}\left|\frac{1}{2}\left(w_2^2 - w_1w_3\right) + \frac{1}{2}\left(\frac{25}{128}w_1^4 - \frac{1}{8}w_1w_3 + \frac{5}{16}w_1^2w_2 + w_2^2\right)\right| \\
&\le \frac{1}{288}\left|w_2^2 - w_1w_3\right| + \frac{1}{288}\left|\frac{25}{128}w_1^4 - \frac{1}{8}w_1w_3 + \frac{5}{16}w_1^2w_2 + w_2^2\right| \\
&= \frac{1}{288}M_1 + \frac{1}{288}M_2,
\end{aligned}$$

where

$$M_1 = \left|w_2^2 - w_1w_3\right|,$$

and

$$M_2 = \left|\frac{25}{128}w_1^4 - \frac{1}{8}w_1w_3 + \frac{5}{16}w_1^2w_2 + w_2^2\right|.$$

Using Lemma 4, it is clear that $M_1 \le 1$. For M_2, using Lemma 3, we have

$$\begin{aligned}
|M_2| &\le \frac{25}{128}|w_1|^4 + \frac{1}{8}|w_1|\left(1 - |w_1|^2 - \frac{|w_2|^2}{1+|w_1|}\right) + \frac{5}{16}|w_1|^2|w_2| + |w_2|^2, \\
&\le \frac{25}{128}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 - \frac{|w_1||w_2|^2}{8(1+|w_1|)} + \frac{5}{16}|w_1|^2|w_2| + |w_2|^2, \\
&\le \frac{25}{128}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 + |w_2|^2\left(1 - \frac{|w_1|}{8(1+|w_1|)}\right) + \frac{5}{16}|w_1|^2|w_2|.
\end{aligned} \tag{47}$$

Since $\left(1 - \frac{|w_1|}{8(1+|w_1|)}\right) > 0$, we can put $|w_2| \le 1 - |w_1|^2$ in (47), and we have

$$|M_2| \le \frac{25}{128}|w_1|^4 + \frac{1}{8}|w_1| - \frac{1}{8}|w_1|^3 + \left(1 - |w_1|^2\right)^2\left(1 - \frac{|w_1|}{8(1+|w_1|)}\right) + \frac{5}{16}|w_1|^2\left(1 - |w_1|^2\right).$$

After simple computation of maxima and minima, we obtain

$$|M_2| \leq 1.$$

Hence, we obtain that

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{288}M_1 + \frac{1}{288}M_2 \leq \frac{1}{144}.$$

The assertion of Theorem 18 is thus proved. □

6. Conclusions

Obtaining estimates of the coefficients appearing in analytic univalent functions is one of the major problems in the field of function theory. The basic idea behind finding the bounds of the coefficients in different families of univalent functions is to express their coefficients in the coefficients of Carathéodory functions. In order to study the coefficient functionals, one can use inequalities that are known for the class of Carathéodory functions. In our present investigation, we applied a novel approach to determine the bounds of various coefficient-related problems, including the Zalcman inequalities, the Feketo–Szegö inequalities, and the Hankel determinant of second order. The families for which we studied such coefficient-type problems were the families $\mathcal{SS}^*_{car}$ and $\mathcal{SK}_{car}$ of functions, which are starlike and convex with respect to symmetric points associated with a cardioid-shaped domain. All bounds were shown to be sharp. More research on the precise bounds of analytical functions defined by the convolution operator may be motivated by this work.

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