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New Estimates on Hermite–Hadamard Type Inequalities via Generalized Tempered Fractional Integrals for Convex Functions with Applications

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Abstract: This paper presents a novel approach by introducing a set of operators known as the left and right generalized tempered fractional integral operators. These operators are utilized to establish new Hermite–Hadamard inequalities for convex functions as well as the multiplication of two convex functions. Additionally, this paper gives two useful identities involving the generalized tempered fractional integral operator for differentiable functions. By leveraging these identities, our results consist of integral inequalities of the Hermite–Hadamard type, which are specifically designed to accommodate convex functions. Furthermore, this study encompasses the identification of several special cases and the recovery of specific known results through comprehensive research. Lastly, this paper offers a range of applications in areas such as matrices, modified Bessel functions and q-digamma functions.

Keywords: Hermite–Hadamard inequality; convex functions; ζ-incomplete gamma functions; Hölder's inequality; power mean inequality; fractional integrals; matrices; modified Bessel functions; q-digamma function

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1. Introduction

Convexity theory is a fundamental concept in mathematical analysis that plays a crucial role in various fields, including optimization, economics, and geometry. It provides a powerful framework for studying and analyzing functions and their properties. In particular, convex functions possess significant characteristics that make them exceptionally useful in many applications.

The Hermite–Hadamard inequality [1] (H–H for short) stands as a fundamental outcome within the realm of convexity theory that establishes a relationship between the average value of a convex function and its endpoint values. Named after Charles Hermite and Jacques Hadamard, this inequality provides a powerful tool for studying the behaviour of convex functions and their integrals. It has been widely investigated and extended to various contexts, making it an essential tool in mathematical analysis. Over the years, researchers have made significant contributions to the development and generalization of the H–H inequality. These efforts have led to the exploration of various generalizations, extensions, and refinements of the original inequality, involving different types of functions, operators, and integral formulations; for example, Bayraktar et al. [2] proved Mercer versions, Sahoo et al. [3] established H–H inequalities via Atangana–Baleanu fractional operators, Tariq et al. [4] presented Simpson–Merer-type inequalities with the help of Atangana–Baleanu fractional operators, and for new versions of H–H results involving exponential kernels, one can refer to [5] and Bayraktar et al. [6], who employed a modified (h,m,s) convex function to establish weighted H–H inequalities. These advancements have broadened the scope of the H–H inequality and deepened our understanding of convex functions.

In this research paper, we aim to contribute to the ongoing development of convexity theory and the H–H inequality. Our focus will be on introducing new operators, namely the left and right generalized tempered fractional integral operators, and their applications in establishing novel H–H inequalities. These operators offer a fresh perspective on the interplay between convex functions and integral inequalities.

Additionally, we will derive two useful identities for differentiable functions that involve the generalized tempered fractional integral operator. We will explore various special cases and demonstrate how our general results recover known results. Moreover, we will showcase the practical relevance and applications of our findings. In particular, we will delve into applications encompassing matrices, modified Bessel functions, and q-digamma functions. By examining these applications, we aim to highlight the significance and versatility of convexity theory and the H–H inequality in solving real-world problems.

Overall, this research paper aims to contribute to the advancement of convexity theory and the H–H inequality by introducing new operators, establishing novel inequalities, deriving interesting identities, exploring special cases, and demonstrating practical applications. Through our work, we hope to deepen our understanding of convex functions and their properties, paving the way for further advancements in this research area of mathematical analysis. Let us denote with I an interval that is a subset of the set of real numbers R.

Definition 1. *A function $\mathcal{S} : \mathrm{I} \to \mathrm{R}$ is called convex, if*

$$\mathcal{S}(\eth\mu_1 + (1-\eth)\mu_2) \leq \eth\mathcal{S}(\mu_1) + (1-\eth)\mathcal{S}(\mu_2), \tag{1}$$

holds for all $\mu_1, \mu_2 \in \mathrm{I}$ and $\eth \in [0,1]$.

Theorem 1 (H–H inequality)**.** *For a convex function $\mathcal{S} : \mathrm{I} \to \mathrm{R}$ and two points $\mu_1, \mu_2 \in \mathrm{I}$ with $\mu_1 < \mu_2$, the following double inequality is valid:*

$$\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\mathcal{S}(\mathrm{x})\mathrm{dx} \leq \frac{\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)}{2}. \tag{2}$$

In recent decades, researchers have investigated inequality (2) using newly formulated definitions motivated by convex functions. For further exploration, interested readers can refer to the following sources: [7–22].

Definition 2. *For any real number $\alpha > 0$ and $\mathrm{x}, \zeta \geq 0$, the ζ-incomplete gamma function is defined as*

$$\gamma_\zeta(\alpha, \mathrm{x}) := \int_0^{\mathrm{x}} \mathrm{u}^{\alpha-1}\,\mathrm{e}^{-\zeta\mathrm{u}}\mathrm{du}.$$

If $\zeta = 1$, then the above function reduces to the incomplete gamma function

$$\gamma(\alpha, \mathrm{x}) := \int_0^{\mathrm{x}} \mathrm{u}^{\alpha-1}\,\mathrm{e}^{-\mathrm{u}}\mathrm{du}.$$

In the following sections, we will revisit the definition of tempered fractional integral operators.

Definition 3 ([22])**.** *Let $\mathcal{S} \in \mathrm{L}[\mu_1, \mu_2]$ (the set of all Lebesgue integrable functions on $[\mu_1, \mu_2]$), where $0 \leq \mu_1 < \mu_2$. Then for $\zeta \geq 0$, the tempered fractional integral operators ${}_{\mu_1^+}\mathrm{I}^{\alpha,\zeta}\mathcal{S}$ and ${}_{\mu_2^-}\mathrm{I}^{\alpha,\zeta}\mathcal{S}$ of order $\alpha > 0$ are defined as*

$${}_{\mu_1^+}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mathrm{x}) := \frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mathrm{x}}(\mathrm{x}-\sigma)^{\alpha-1}\,\mathrm{e}^{-\zeta(\mathrm{x}-\sigma)}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} > \mu_1 \tag{3}$$

and

$$ {}_{\mu_2^-}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mathrm{x}) := \frac{1}{\Gamma(\alpha)}\int_{\mathrm{x}}^{\mu_2}(\sigma-\mathrm{x})^{\alpha-1}\,\mathrm{e}^{-\zeta(\sigma-\mathrm{x})}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x}<\mu_2, \tag{4}$$

respectively.

Taking $\zeta = 0$ in Definition 3, we obtain Riemann–Liouville fractional integral operators defined by

$$ {}_{\mu_1^+}\mathrm{J}^{\alpha}\mathcal{S}(\mathrm{x}) := \frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mathrm{x}}(\mathrm{x}-\sigma)^{\alpha-1}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x}>\mu_1 \tag{5}$$

and

$$ {}_{\mu_2^-}\mathrm{J}^{\alpha}\mathcal{S}(\mathrm{x}) := \frac{1}{\Gamma(\alpha)}\int_{\mathrm{x}}^{\mu_2}(\sigma-\mathrm{x})^{\alpha-1}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x}<\mu_2. \tag{6}$$

The function $\Upsilon : [0,+\infty) \to [0,+\infty)$ was introduced by Sarikaya et al. in [18], characterized by the following conditions:

$$\int_0^1 \frac{\Upsilon(\eth)}{\eth}\mathrm{d}\eth < +\infty, \tag{7}$$

$$\frac{1}{\mathrm{A}_1} \le \frac{\Upsilon(\eth_1)}{\Upsilon(\eth_2)} \le \mathrm{A}_1 \text{ for } \frac{1}{2} \le \frac{\eth_1}{\eth_2} \le 2, \tag{8}$$

$$\frac{\Upsilon(\eth_2)}{\eth_2^2} \le \mathrm{A}_2\frac{\Upsilon(\eth_1)}{\eth_1^2} \text{ for } \eth_1 \le \eth_2, \tag{9}$$

$$\left|\frac{\Upsilon(\eth_2)}{\eth_2^2} - \frac{\Upsilon(\eth_1)}{\eth_1^2}\right| \le \mathrm{A}_3|\eth_2-\eth_1|\frac{\Upsilon(\eth_2)}{\eth_2^2} \text{ for } \frac{1}{2} \le \frac{\eth_1}{\eth_2} \le 2, \tag{10}$$

where $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 > 0$ are independent of $\eth_1, \eth_2 > 0$. If there exists a non-negative value of ξ such that $\Upsilon(\eth_2)\eth_2^{\xi}$ is an increasing function, and there exists a non-negative value of ζ such that $\frac{\Upsilon(\eth_2)}{\eth_2^{\zeta}}$ is a decreasing function, then it can be deduced from [23] that Υ satisfies (7)–(10). Based on this, the definitions of the left-sided and right-sided generalized integral operators are given as follows:

$$ {}_{\mu_1^+}\mathrm{I}_{\Upsilon}\mathcal{S}(\mathrm{x}) := \int_{\mu_1}^{\mathrm{x}}\frac{\Upsilon(\mathrm{x}-\sigma)}{\mathrm{x}-\sigma}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x}>\mu_1, \tag{11}$$

$$ {}_{\mu_2^-}\mathrm{I}_{\Upsilon}\mathcal{S}(\mathrm{x}) := \int_{\mathrm{x}}^{\mu_2}\frac{\Upsilon(\sigma-\mathrm{x})}{\sigma-\mathrm{x}}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x}<\mu_2. \tag{12}$$

A notable feature of generalized integrals is their ability to yield Riemann–Liouville fractional integrals, Katugampola fractional integrals, and other types as well. These results can be found in references such as [18,20,23,24].

In recent years, there has been a growing interest in extending the H–H inequality to the context of fractional calculus. Fractional calculus deals with the generalization of derivatives and integrals to non-integer orders, and it has emerged as a powerful mathematical tool for modeling and analyzing complex phenomena in physics, engineering, and other fields. The fractional H–H inequality explores the behavior of fractional integrals of convex functions. By considering fractional operators of different orders, such as the Riemann–Liouville or Caputo fractional operators, the inequality provides a bridge between the concepts of convexity and fractional calculus. It establishes relationships between the fractional integral of a convex function and its endpoint values, revealing intriguing properties and insights into the behavior of these functions. The study of fractional H–H inequalities has attracted significant attention from researchers in recent years. Many mathematicians have worked on deriving new inequalities, investigating their properties, and exploring their applications in various areas [25,26]. These developments have led to a deeper understanding of the interplay between convex functions and fractional calculus.

In Section 2, we begin by introducing the left and right generalized tempered fractional integral operators as novel tools in the study of H–H inequalities. By utilizing these operators, we establish novel inequalities for convex functions as well as for products of two convex functions. These results extend the existing literature and provide valuable insights into the behavior of convex functions.

Moving on to Section 3, we delve into the exploration of two interesting identities involving the generalized tempered fractional integral operators for differentiable functions. Through the utilization of these identities, we obtain supplementary integral inequalities of H–H and midpoint types for convex functions. This further expands the scope and applicability of our findings.

Section 4 takes a practical turn as we delve into the application of our results. We discuss their relevance and implications in various domains, such as matrices, modified Bessel functions and q-digamma functions.

Finally, in Section 5, we wrap up the paper with a concise conclusion summarizing the key findings and their implications. We also provide future recommendations, suggesting potential directions for further research.

Through this structured organization, our paper presents a comprehensive analysis of the introduced operators, establishes new inequalities, explores intriguing identities, demonstrates practical applications, and offers valuable insights for future research.

2. Generalized Tempered Fractional Hermite–Hadamard-Type Inequalities

At this point, we can present the left and right generalized tempered fractional integral operators with $\zeta \geq 0$, where $\Upsilon : [0, +\infty) \to [0, +\infty)$ satisfies the given conditions (7)–(10).

Throughout this study, let $\zeta \geq 0$ and $0 \leq \mu_1 < \mu_2$. Then, for all $\mathfrak{d} \in [0, 1]$, we define the following definitions:

Definition 4. *The operator that defines the left generalized tempered fractional integral is given by*

$$ {}_{\mu_1^+}\mathrm{T}^{\zeta}_{\Upsilon}\,\mathcal{S}(\mathrm{x}) := \int_{\mu_1}^{\mathrm{x}} \frac{\Upsilon(\mathrm{x}-\sigma)}{\mathrm{x}-\sigma}\, \mathrm{e}^{-\zeta(\mathrm{x}-\sigma)}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} > \mu_1. \tag{13} $$

Analogously, the right generalized tempered fractional integral operator is defined by

$$ {}_{\mu_2^-}\mathrm{T}^{\zeta}_{\Upsilon}\,\mathcal{S}(\mathrm{x}) := \int_{\mathrm{x}}^{\mu_2} \frac{\Upsilon(\sigma-\mathrm{x})}{\sigma-\mathrm{x}}\, \mathrm{e}^{-\zeta(\sigma-\mathrm{x})}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} < \mu_2. \tag{14} $$

From Definition 4, we can recapture some known integral operators such as Riemann integrals, Riemann–Liouville fractional integrals, generalized integral operators, and tempered fractional integral operators, and we can present some new integral operators as well. Some of them are given explicitly in the following Remark 1. Moreover, the importance of these new integral operators depends on the choices of the parameter ζ, since it takes values in the domain $[0, \infty)$. That means that interested readers that will do numerical computations will see clearly the advantage of this new operator compared with other known operators for suitably choosing this parameter ζ. For $\zeta = 0$, these operators attains maximum values. For other values of ζ, their values decrease. We believe that these new operators will open a new door for investigating various variational problems for different kinds of convexity functions. In our study, we are not focused on numerical analysis.

Remark 1.

(a) *Taking $\zeta = 0$ in Definition 4, we have generalized fractional integral operators given from (11) and (12). Choosing $\Upsilon(\mathfrak{d}) = \frac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha)}$ in Definition 4, we get Definition 3.*

(b) *Choosing* $\Upsilon(\mathfrak{d}) = \mathfrak{d}(\mu_2 - \mathfrak{d})^{\alpha-1}$ *for* $\alpha \in (0,1)$ *in Definition 4, we obtain the following conformable left and right tempered fractional integral operators:*

$$
\begin{aligned}
{}_{\mu_1^+}\mathrm{C}^{\alpha,\zeta}\,\mathcal{S}(\mathrm{x}) &:= \int_{\mu_1}^{\mathrm{x}} (\mu_2 - \mathrm{x} + \sigma)^{\alpha-1}\, \mathrm{e}^{-\zeta(\mathrm{x}-\sigma)}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} > \mu_1,\\
{}_{\mu_2^-}\mathrm{C}^{\alpha,\zeta}\,\mathcal{S}(\mathrm{x}) &:= \int_{\mathrm{x}}^{\mu_2} (\mu_2 + \mathrm{x} - \sigma)^{\alpha-1}\, \mathrm{e}^{-\zeta(\sigma-\mathrm{x})}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} < \mu_2.
\end{aligned}
$$

Taking $\Upsilon(\mathfrak{d}) = \frac{\mathfrak{d}}{\alpha}\exp(-\mathrm{A}\mathfrak{d})$, *where* $\mathrm{A} = \frac{1-\alpha}{\alpha}$ *for* $\alpha \in (0,1)$ *in Definition 4, we have the following exponential left and right tempered fractional integral operators:*

$$
\begin{aligned}
{}_{\mu_1^+}\mathrm{E}^{\alpha,\zeta}\,\mathcal{S}(\mathrm{x}) &:= \frac{1}{\alpha}\int_{\mu_1}^{\mathrm{x}} \mathrm{e}^{-(\mathrm{A}+\zeta)(\mathrm{x}-\sigma)}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} > \mu_1,\\
{}_{\mu_2^-}\mathrm{E}^{\alpha,\zeta}\,\mathcal{S}(\mathrm{x}) &:= \frac{1}{\alpha}\int_{\mathrm{x}}^{\mu_2} \mathrm{e}^{-(\mathrm{A}+\zeta)(\sigma-\mathrm{x})}\mathcal{S}(\sigma)\mathrm{d}\sigma, \quad \mathrm{x} < \mu_2.
\end{aligned}
$$

Furthermore, we define the following expressions that will be used in the sequel.

$$\Omega_{\Upsilon}^{\zeta}(\mathfrak{d}) := \int_0^{\mathfrak{d}} \frac{\Upsilon(\mathrm{u}(\mu_2-\mu_1))}{\mathrm{u}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathrm{u}}\,\mathrm{du} < +\infty, \quad \mu_1 < \mu_2, \tag{15}$$

$$\Sigma_{\Upsilon,1}^{\zeta}(\mathrm{x},\mathfrak{d}) := \int_0^{\mathfrak{d}} \frac{\Upsilon(\mathrm{u}(\mathrm{x}-\mu_1))}{\mathrm{u}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathrm{u}}\,\mathrm{du} < +\infty, \quad \mathrm{x} > \mu_1, \tag{16}$$

$$\Sigma_{\Upsilon,2}^{\zeta}(\mathrm{x},\mathfrak{d}) := \int_0^{\mathfrak{d}} \frac{\Upsilon(\mathrm{u}(\mu_2-\mathrm{x}))}{\mathrm{u}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathrm{u}}\,\mathrm{du} < +\infty, \quad \mathrm{x} < \mu_2 \tag{17}$$

and

$$\Psi_{\Upsilon}^{\zeta} := \int_0^1 \frac{\Upsilon(\mathrm{u}(\mu_2-\mu_1))}{\mathrm{u}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathrm{u}}\mathrm{du} = \int_0^{\mu_2-\mu_1} \frac{\Upsilon(\mathrm{u})}{\mathrm{u}}\, \mathrm{e}^{-\zeta\mathrm{u}}\mathrm{du}. \tag{18}$$

Let us now represent the generalized tempered fractional H–H-type inequality for convex functions as follows.

Theorem 2. *Consider a function* $\mathcal{S}$ *defined on the interval* $[\mu_1, \mu_2]$ *and* $\mathcal{S} \in \mathrm{L}[\mu_1,\mu_2]$. *If* $\mathcal{S}$ *is convex on* $[\mu_1,\mu_2]$, *then the following double inequality holds true for generalized tempered fractional integral operators:*

$$\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \le \frac{1}{2\,\Psi_{\Upsilon}^{\zeta}}\left[{}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2) + {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\right] \le \frac{\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)}{2}, \tag{19}$$

where Ψ_{Υ}^{ζ} *is defined from (18).*

Proof. Let $\mathrm{u}, \mathrm{v} \in [\mu_1,\mu_2]$. Since $\mathcal{S}$ is convex on $[\mu_1,\mu_2]$, we have

$$\mathcal{S}\left(\frac{\mathrm{u}+\mathrm{v}}{2}\right) \le \frac{\mathcal{S}(\mathrm{u})+\mathcal{S}(\mathrm{v})}{2}.$$

Taking $\mathrm{u} = \mathfrak{d}\mu_1 + (1-\mathfrak{d})\mu_2$ and $\mathrm{v} = (1-\mathfrak{d})\mu_1 + \mathfrak{d}\mu_2$, we get

$$2\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \le \mathcal{S}(\mathfrak{d}\mu_1 + (1-\mathfrak{d})\mu_2) + \mathcal{S}((1-\mathfrak{d})\mu_1 + \mathfrak{d}\mu_2). \tag{20}$$

By multiplying both sides of inequality (20) with the expression $\dfrac{\Upsilon(\mathfrak{d}(\mu_2-\mu_1))}{\mathfrak{d}}\mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathfrak{d}}$ and integrating the resulting inequality over the interval $[0,1]$ with respect to $\mathfrak{d}$, we derive the following result

$$
\begin{aligned}
&2\mathcal{S}\left(\tfrac{\mu_1+\mu_2}{2}\right)\int_0^1 \tfrac{\Upsilon(\mathfrak{d}(\mu_2-\mu_1))}{\mathfrak{d}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathfrak{d}}\mathrm{d}\mathfrak{d}\\
\le &\int_0^1 \tfrac{\Upsilon(\mathfrak{d}(\mu_2-\mu_1))}{\mathfrak{d}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathfrak{d}}\mathcal{S}(\mathfrak{d}\mu_1 + (1-\mathfrak{d})\mu_2)\mathrm{d}\mathfrak{d}\\
&+ \int_0^1 \tfrac{\Upsilon(\mathfrak{d}(\mu_2-\mu_1))}{\mathfrak{d}}\, \mathrm{e}^{-\zeta(\mu_2-\mu_1)\mathfrak{d}}\mathcal{S}((1-\mathfrak{d})\mu_1 + \mathfrak{d}\mu_2)\mathrm{d}\mathfrak{d}.
\end{aligned}
$$

So, we have

$$2\,\Psi_{\Upsilon}^{\zeta}\,\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \le \left[{}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2) + {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\right]. \tag{21}$$

This implies that the left-hand side of inequality (19) has been established. In order to prove the right-hand side of inequality (19), we utilize the fact that $\mathcal{S}$ is a convex function on $[\mu_1, \mu_2]$. This allows us to derive the following inequalities:

$$\mathcal{S}(\eth\mu_1 + (1-\eth)\mu_2) \le \eth\mathcal{S}(\mu_1) + (1-\eth)\mathcal{S}(\mu_2) \tag{22}$$

and

$$\mathcal{S}((1-\eth)\mu_1 + \eth\mu_2) \le (1-\eth)\mathcal{S}(\mu_1) + \eth\mathcal{S}(\mu_2). \tag{23}$$

Adding (22) and (23)), we get

$$\mathcal{S}(\eth\mu_1 + (1-\eth)\mu_2) + \mathcal{S}((1-\eth)\mu_1 + \eth\mu_2) \le \mathcal{S}(\mu_1) + \mathcal{S}(\mu_2). \tag{24}$$

Multiplying both sides of inequality (24) with $\dfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathrm{d}\eth \\
&+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathrm{d}\eth \\
\le\ &\left[\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)\right]\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathrm{d}\eth.
\end{aligned}$$

Therefore, we have the following inequality:

$$\left[{}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2) + {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\right] \le \left[\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)\right]\Psi_{\Upsilon}^{\zeta}. \tag{25}$$

This confirms that the right-hand side of inequality (19) has been established. Therefore, the proof of Theorem 2 is concluded. □

Corollary 1. *By taking* $\Upsilon(\eth) = \dfrac{\eth^{\alpha}}{\Gamma(\alpha)}$ *in Theorem 2, we obtain the following double inequality for convex functions using tempered fractional integral operators:*

$$\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \le \frac{\Gamma(\alpha)}{2\gamma_{\zeta}(\alpha,\mu_2-\mu_1)}\left[{}_{\mu_1^+}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mu_2) + {}_{\mu_2^-}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mu_1)\right] \le \frac{\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)}{2}. \tag{26}$$

Corollary 2. *Choosing* $\alpha = 1$ *and* $\zeta = 0$ *in Corollary 1, we get Theorem 1.*

Theorem 3. *Let* $\mathcal{S}, \mathcal{R} : [\mu_1, \mu_2] \to \mathrm{R}$ *be two functions and* $\mathcal{S}, \mathcal{R} \in \mathrm{L}[\mu_1, \mu_2]$. *If* $\mathcal{S}$ *and* $\mathcal{R}$ *are convex on* $[\mu_1, \mu_2]$, *then the following double inequality for generalized tempered fractional integral operators holds true:*

$$\begin{aligned}
&\mathcal{S}\left(\tfrac{\mu_1+\mu_2}{2}\right)\mathcal{R}\left(\tfrac{\mu_1+\mu_2}{2}\right) - \tfrac{1}{4\Psi_{\Upsilon}^{\zeta}}\left[{}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2)\mathcal{R}(\mu_2) + {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\mathcal{R}(\mu_1)\right] \\
\le\ &\tfrac{1}{4\Psi_{\Upsilon}^{\zeta}}\Theta_{\Upsilon}^{\zeta}(\mathcal{S},\mathcal{R}) \\
\le\ &\tfrac{1}{2}\mathrm{M}(\mu_1,\mu_2)\tfrac{\Delta_{\Upsilon}^{\zeta}}{\Psi_{\Upsilon}^{\zeta}} + \tfrac{1}{4}\mathrm{N}(\mu_1,\mu_2)\left(1-2\tfrac{\Delta_{\Upsilon}^{\zeta}}{\Psi_{\Upsilon}^{\zeta}}\right),
\end{aligned} \tag{27}$$

where

$$\mathrm{M}(\mu_1,\mu_2) := \mathcal{S}(\mu_1)\mathcal{R}(\mu_1) + \mathcal{S}(\mu_2)\mathcal{R}(\mu_2),$$
$$\mathrm{N}(\mu_1,\mu_2) := \mathcal{S}(\mu_1)\mathcal{R}(\mu_2) + \mathcal{S}(\mu_2)\mathcal{R}(\mu_1),$$

$$\Theta_{\Upsilon}^{\zeta}(\mathcal{S},\mathcal{R}) := \int_0^{\mu_2-\mu_1} \frac{\Upsilon(\eth)}{\eth}\, e^{-\zeta\eth}\big[\mathcal{R}(\mu_2-\eth)\mathcal{S}(\eth-\mu_1)+\mathcal{R}(\eth-\mu_1)\mathcal{S}(\mu_2-\eth)\big]d\eth,$$

$$\Delta_{\Upsilon}^{\zeta} := \frac{1}{(\mu_2-\mu_1)^2}\int_0^{\mu_2-\mu_1}\Upsilon(\eth)\, e^{-\zeta\eth}(\mu_2-\mu_1-\eth)d\eth$$

and Ψ_{Υ}^{ζ} *is defined from* (18).

Proof. Let $u, v \in [\mu_1,\mu_2]$. Since $\mathcal{S}$ and $\mathcal{R}$ are convex on $[\mu_1,\mu_2]$, we have

$$\mathcal{S}\left(\frac{u+v}{2}\right) \le \frac{\mathcal{S}(u)+\mathcal{S}(v)}{2}, \quad \mathcal{R}\left(\frac{u+v}{2}\right) \le \frac{\mathcal{R}(u)+\mathcal{R}(v)}{2}.$$

Taking $u = \eth\mu_1+(1-\eth)\mu_2$ and $v = (1-\eth)\mu_1+\eth\mu_2$, we get

$$2\mathcal{S}\left(\frac{\mu_1+\mu_2}{2}\right) \le \mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2) \tag{28}$$

and

$$2\mathcal{R}\left(\frac{\mu_1+\mu_2}{2}\right) \le \mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1). \tag{29}$$

By multiplying both sides of inequalities (28) and (29), we obtain

$$\begin{aligned}
&4\mathcal{S}\left(\tfrac{\mu_1+\mu_2}{2}\right)\mathcal{R}\left(\tfrac{\mu_1+\mu_2}{2}\right)\\
&\le \mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\\
&+\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2).
\end{aligned} \tag{30}$$

Multiplying both sides of inequality (30) with $\dfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}$ and integrating the resulting inequality with respect to $\eth$ over $[0,1]$, we have

$$\begin{aligned}
&4\mathcal{S}\left(\tfrac{\mu_1+\mu_2}{2}\right)\mathcal{R}\left(\tfrac{\mu_1+\mu_2}{2}\right)\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}d\eth\\
&\le \int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_2+\eth\mu_1)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)d\eth\\
&\quad+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)d\eth\\
&\quad+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_2+\eth\mu_1)\mathcal{R}((1-\eth)\mu_1+\eth\mu_2)d\eth\\
&\quad+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\, e^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)d\eth.
\end{aligned}$$

So, we get

$$\begin{aligned}
&4\,\Psi_{\Upsilon}^{\zeta}\,\mathcal{S}\left(\tfrac{\mu_1+\mu_2}{2}\right)\mathcal{R}\left(\tfrac{\mu_1+\mu_2}{2}\right) \le {}_{\mu_1^+}T_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2)\mathcal{R}(\mu_2) + {}_{\mu_2^-}T_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\mathcal{R}(\mu_1)\\
&+\int_0^{\mu_2-\mu_1}\tfrac{\Upsilon(\eth)}{\eth}\, e^{-\zeta\eth}\big[\mathcal{S}(\eth-\mu_1)\mathcal{R}(\mu_2-\eth)+\mathcal{S}(\mu_2-\eth)\mathcal{R}(\eth-\mu_1)\big]d\eth.
\end{aligned} \tag{31}$$

This implies that the left-hand side of inequality (27) has been established. In order to prove the right-hand side of inequality (27), we utilize the fact that $\mathcal{S}$ and $\mathcal{R}$ are both convex functions on $[\mu_1,\mu_2]$. This allows us to derive the following inequalities:

$$\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2) \le \eth\mathcal{S}(\mu_1)+(1-\eth)\mathcal{S}(\mu_2), \tag{32}$$

$$\mathcal{S}((1-\eth)\mu_1+\eth\mu_2) \le (1-\eth)\mathcal{S}(\mu_1)+\eth\mathcal{S}(\mu_2) \tag{33}$$

and

$$\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2) \le \eth\mathcal{R}(\mu_1)+(1-\eth)\mathcal{R}(\mu_2), \tag{34}$$

$$\mathcal{R}((1-\eth)\mu_1+\eth\mu_2) \le (1-\eth)\mathcal{R}(\mu_1)+\eth\mathcal{R}(\mu_2). \tag{35}$$

Utilizing inequalities (32)–(35), we have

$$
\begin{aligned}
&\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\\
&+\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}((1-\eth)\mu_1+\eth\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)\\
&\le \mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}((1-\eth)\mu_1+\eth\mu_2)\\
&+\big[\eth\mathcal{S}(\mu_1)+(1-\eth)\mathcal{S}(\mu_2)\big]\cdot\big[(1-\eth)\mathcal{R}(\mu_1)+\eth\mathcal{R}(\mu_2)\big]\\
&+\big[(1-\eth)\mathcal{S}(\mu_1)+\eth\mathcal{S}(\mu_2)\big]\cdot\big[\eth\mathcal{R}(\mu_1)+(1-\eth)\mathcal{R}(\mu_2)\big]\\
&=\mathcal{S}(\eth\mu_1+(1-\eth)\mu_2)\mathcal{R}(\eth\mu_1+(1-\eth)\mu_2)+\mathcal{S}((1-\eth)\mu_1+\eth\mu_2)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\\
&+2\eth(1-\eth)\mathrm{M}(\mu_1,\mu_2)+\big[\eth^2+(1-\eth)^2\big]\mathrm{N}(\mu_1,\mu_2).
\end{aligned}
\tag{36}
$$

Multiplying both sides of inequality (36) with $\dfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}$ and integrating the resulting inequality with respect to $\eth$ over $[0,1]$, we obtain

$$
\begin{aligned}
&\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_2+\eth\mu_1)\mathcal{R}((1-\eth)\mu_2+\eth\mu_1)\mathrm{d}\eth\\
&+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}(\eth\mu_2+(1-\eth)\mu_1)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\mathrm{d}\eth\\
&+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_2+\eth\mu_1)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\mathrm{d}\eth\\
&+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}(\eth\mu_2+(1-\eth)\mu_1)\mathcal{R}((1-\eth)\mu_2+\eth\mu_1)\mathrm{d}\eth\\
\le&\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}((1-\eth)\mu_2+\eth\mu_1)\mathcal{R}((1-\eth)\mu_2+\eth\mu_1)\mathrm{d}\eth\\
&+\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\mathcal{S}(\eth\mu_2(1-\eth)\mu_1)\mathcal{R}(\eth\mu_2+(1-\eth)\mu_1)\mathrm{d}\eth\\
&+2\mathrm{M}(\mu_1,\mu_2)\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\eth(1-\eth)\mathrm{d}\eth\\
&+\mathrm{N}(\mu_1,\mu_2)\int_0^1 \tfrac{\Upsilon(\eth(\mu_2-\mu_1))}{\eth}\,\mathrm{e}^{-\zeta(\mu_2-\mu_1)\eth}\big[\eth^2+(1-\eth)^2\big]\mathrm{d}\eth.
\end{aligned}
$$

So, we get

$$
\begin{aligned}
&\Big[{}_{\mu_1^+}\mathrm{T}_\Upsilon^\zeta\mathcal{S}(\mu_2)\mathcal{R}(\mu_2)\;+\;{}_{\mu_2^-}\mathrm{T}_\Upsilon^\zeta\mathcal{S}(\mu_1)\mathcal{R}(\mu_1)\Big]+\Theta_\Upsilon^\zeta(\mathcal{S},\mathcal{R})\\
\le&\Big[{}_{\mu_1^+}\mathrm{T}_\Upsilon^\zeta\mathcal{S}(\mu_2)\mathcal{R}(\mu_2)\;+\;{}_{\mu_2^-}\mathrm{T}_\Upsilon^\zeta\mathcal{S}(\mu_1)\mathcal{R}(\mu_1)\Big]+2\mathrm{M}(\mu_1,\mu_2)\Delta_\Upsilon^\zeta\\
&+\mathrm{N}(\mu_1,\mu_2)\Big(\Psi_\Upsilon^\zeta-2\Delta_\Upsilon^\zeta\Big).
\end{aligned}
\tag{37}
$$

This implies that the right-hand side of inequality (27) has been established. Therefore, the proof of Theorem 3 is concluded. □

Corollary 3. *By substituting $\mathcal{S}=\mathcal{R}$ into Theorem 3, we obtain the following result:*

$$
\begin{aligned}
&\mathcal{S}^2\Big(\tfrac{\mu_1+\mu_2}{2}\Big)\;-\;\tfrac{1}{4\Psi_\Upsilon^\zeta}\Big[{}_{\mu_1^+}\mathrm{T}_\Upsilon^\zeta\mathcal{S}^2(\mu_2)\;+\;{}_{\mu_2^-}\mathrm{T}_\Upsilon^\zeta\mathcal{S}^2(\mu_1)\Big]\\
\le&\tfrac{1}{2\Psi_\Upsilon^\zeta}\Theta_\Upsilon^\zeta(\mathcal{S})\\
\le&\tfrac{1}{2}\mathrm{P}(\mu_1,\mu_2)\tfrac{\Delta_\Upsilon^\zeta}{\Psi_\Upsilon^\zeta}+\tfrac{1}{4}\mathrm{Q}(\mu_1,\mu_2)\Big(1-2\tfrac{\Delta_\Upsilon^\zeta}{\Psi_\Upsilon^\zeta}\Big),
\end{aligned}
\tag{38}
$$

where

$$\Theta_\Upsilon^\zeta(\mathcal{S}):=\int_0^{\mu_2-\mu_1}\frac{\Upsilon(\eth)}{\eth}\,\mathrm{e}^{-\zeta\eth}\mathcal{S}(\eth-\mu_1)\mathcal{S}(\mu_2-\eth)\mathrm{d}\eth$$

and

$$\mathrm{P}(\mu_1,\mu_2):=\mathcal{S}^2(\mu_1)+\mathcal{S}^2(\mu_2),\quad \mathrm{Q}(\mu_1,\mu_2):=2\mathcal{S}(\mu_1)\mathcal{S}(\mu_2).$$

Corollary 4. *By selecting $\Upsilon(\eth)=\dfrac{\eth^\alpha}{\Gamma(\alpha)}$ in Theorem 3, we derive the following double inequality for the product of two convex functions employing tempered fractional integral operators:*

$$
\begin{aligned}
&\mathcal{S}\Big(\tfrac{\mu_1+\mu_2}{2}\Big)\mathcal{R}\Big(\tfrac{\mu_1+\mu_2}{2}\Big)\;-\;\tfrac{\Gamma(\alpha)}{4\gamma_\zeta(\alpha,\mu_2-\mu_1)}\Big[{}_{\mu_1^+}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mu_2)\mathcal{R}(\mu_2)\;+\;{}_{\mu_2^-}\mathrm{I}^{\alpha,\zeta}\mathcal{S}(\mu_1)\mathcal{R}(\mu_1)\Big]\\
\le&\tfrac{1}{4\gamma_\zeta(\alpha,\mu_2-\mu_1)}\Theta^{\alpha,\zeta}(\mathcal{S},\mathcal{R})\\
\le&\tfrac{1}{2}\mathrm{M}(\mu_1,\mu_2)\tfrac{\Delta^{\alpha,\zeta}}{\gamma_\zeta(\alpha,\mu_2-\mu_1)}+\tfrac{1}{4}\mathrm{N}(\mu_1,\mu_2)\Big(1-2\tfrac{\Delta^{\alpha,\zeta}}{\gamma_\zeta(\alpha,\mu_2-\mu_1)}\Big),
\end{aligned}
\tag{39}
$$

where

$$\Theta^{\alpha,\zeta}(\mathcal{S},\mathcal{R}) := \int_0^{\mu_2-\mu_1} \mathfrak{d}^{\alpha-1}\, e^{-\zeta\mathfrak{d}}\left[\mathcal{S}(\mathfrak{d}-\mu_1)\mathcal{R}(\mu_2-\mathfrak{d})+\mathcal{S}(\mu_2-\mathfrak{d})\mathcal{R}(\mathfrak{d}-\mu_1)\right]d\mathfrak{d}$$

and

$$\Delta^{\alpha,\zeta} := \frac{1}{(\mu_2-\mu_1)^2}\int_0^{\mu_2-\mu_1} \mathfrak{d}^{\alpha}\, e^{-\zeta\mathfrak{d}}(\mu_2-\mu_1-\mathfrak{d})d\mathfrak{d}.$$

3. Further Results Related to Generalized Tempered Fractional Integral Operators

To establish the results of this section regarding generalized tempered fractional integral operators, we commence by proving the following two lemmas.

Lemma 1. *Consider a differentiable function $\mathcal{S}$ defined on the interval $[\mu_1,\mu_2]$. If $\mathcal{S}' \in L[\mu_1,\mu_2]$, then the following identity holds true for generalized tempered fractional integral operators:*

$$\begin{aligned}&\tfrac{\mathcal{S}(\mu_1)+\mathcal{S}(\mu_2)}{2} - \tfrac{1}{2\Psi_{\Upsilon}^{\zeta}}\left[{}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2) + {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\right]\\ &= \tfrac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\left[\Omega_{\Upsilon}^{\zeta}(1-\mathfrak{d})-\Omega_{\Upsilon}^{\zeta}(\mathfrak{d})\right]\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)d\mathfrak{d},\end{aligned} \tag{40}$$

where $\Omega_{\Upsilon}^{\zeta}(\mathfrak{d})$ and Ψ_{Υ}^{ζ} are defined respectively, from (15) and (18). We denote

$$\mathrm{I}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2) := \frac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\left[\Omega_{\Upsilon}^{\zeta}(1-\mathfrak{d})-\Omega_{\Upsilon}^{\zeta}(\mathfrak{d})\right]\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)d\mathfrak{d}. \tag{41}$$

Proof. We write (41) in the following form

$$\mathrm{I}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2) = \frac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left[\mathrm{I}^{(1)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)-\mathrm{I}^{(2)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)\right], \tag{42}$$

where

$$\mathrm{I}^{(1)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2) := \int_0^1 \Omega_{\Upsilon}^{\zeta}(1-\mathfrak{d})\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)d\mathfrak{d} \tag{43}$$

and

$$\mathrm{I}^{(2)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2) := \int_0^1 \Omega_{\Upsilon}^{\zeta}(\mathfrak{d})\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)d\mathfrak{d}. \tag{44}$$

By performing integration by parts on Equation (43) and making a change of variables in the integration, we obtain

$$\begin{aligned}&\mathrm{I}^{(1)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)\\ &= \Omega_{\Upsilon}^{\zeta}(1-\mathfrak{d})\tfrac{\mathcal{S}(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)}{\mu_1-\mu_2}\Big|_0^1\\ &+\tfrac{1}{\mu_1-\mu_2}\int_0^1\tfrac{\Upsilon((1-\mathfrak{d})(\mu_2-\mu_1))}{1-\mathfrak{d}}\, e^{-\zeta(\mu_2-\mu_1)(1-\mathfrak{d})}\mathcal{S}(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)d\mathfrak{d}\\ &= \Psi_{\Upsilon}^{\zeta}\tfrac{\mathcal{S}(\mu_2)}{\mu_2-\mu_1} - \tfrac{1}{\mu_2-\mu_1}\times {}_{\mu_2^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1).\end{aligned} \tag{45}$$

Similarly, using (44), we get

$$\mathrm{I}^{(2)}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2) = -\Psi_{\Upsilon}^{\zeta}\frac{\mathcal{S}(\mu_1)}{\mu_2-\mu_1} + \frac{1}{\mu_2-\mu_1}\times {}_{\mu_1^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2). \tag{46}$$

Substituting (45) and (46) in (42), we obtain the desired equality (40). □

Remark 2. *Taking $\Upsilon(\mathfrak{d}) = \dfrac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha)}$ and $\zeta = 0$ in Lemma 1, we get ([19], Lemma 3.1).*

Lemma 2. *Suppose $\mathcal{S}:[\mu_1,\mu_2]\to\mathrm{R}$ is a differentiable function on the open interval (μ_1,μ_2). If $\mathcal{S}'\in\mathrm{L}[\mu_1,\mu_2]$, then the following identity holds true for generalized tempered fractional integral operators:*

$$\begin{aligned}&\mathcal{S}(\mathrm{x})-\tfrac{1}{2\Psi_{\Upsilon}^{\zeta}}\left[{}_{\mathrm{x}^+}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_2)+{}_{\mathrm{x}^-}\mathrm{T}_{\Upsilon}^{\zeta}\mathcal{S}(\mu_1)\right]\\&=\tfrac{(\mathrm{x}-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\Sigma_{\Upsilon,1}^{\zeta}(\mathrm{x},\eth)\mathcal{S}'(\eth\mathrm{x}+(1-\eth)\mu_1)d\eth\\&-\tfrac{(\mu_2-\mathrm{x})}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\Sigma_{\Upsilon,2}^{\zeta}(\mathrm{x},\eth)\mathcal{S}'(\eth\mathrm{x}+(1-\eth)\mu_2)d\eth,\end{aligned}\tag{47}$$

where $\Sigma_{\Upsilon,1}^{\zeta}(\mathrm{x},\eth)$ and $\Sigma_{\Upsilon,2}^{\zeta}(\mathrm{x},\eth)$ are defined respectively, from (16) and (17). We denote

$$\begin{aligned}&\mathrm{I}_{\mathcal{S},\Sigma_{\Upsilon,1}^{\zeta},\Sigma_{\Upsilon,2}^{\zeta}}(\mathrm{x};\mu_1,\mu_2):=\tfrac{(\mathrm{x}-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\Sigma_{\Upsilon,1}^{\zeta}(\mathrm{x},\eth)\mathcal{S}'(\eth\mathrm{x}+(1-\eth)\mu_1)d\eth\\&-\tfrac{(\mu_2-\mathrm{x})}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\Sigma_{\Upsilon,2}^{\zeta}(\mathrm{x},\eth)\mathcal{S}'(\eth\mathrm{x}+(1-\eth)\mu_2)d\eth.\end{aligned}\tag{48}$$

Proof. The proof is similar to the proof of Lemma 1. □

Theorem 4. *Assume that $\mathcal{S}$ is a differentiable function on the interval $[\mu_1,\mu_2]$. If $\mathcal{S}'\in\mathrm{L}[\mu_1,\mu_2]$ and $|\mathcal{S}'|^{\mathrm{q}}$ is a convex function, then, under the conditions $\mathrm{q}>1$ and $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1$, the following inequality holds true for generalized tempered fractional integral operators:*

$$\left|\mathrm{I}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)\right|\leq\frac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left[\Pi_{\Upsilon}^{\zeta}(\mathrm{p})\right]^{\frac{1}{\mathrm{p}}}\left[\frac{|\mathcal{S}'(\mu_1)|^{\mathrm{q}}+|\mathcal{S}'(\mu_2)|^{\mathrm{q}}}{2}\right]^{\frac{1}{\mathrm{q}}},\tag{49}$$

where

$$\begin{aligned}&\Pi_{\Upsilon}^{\zeta}(\mathrm{p}):=\int_0^1\left|\Omega_{\Upsilon}^{\zeta}(1-\eth)-\Omega_{\Upsilon}^{\zeta}(\eth)\right|^{\mathrm{p}}d\eth\\&=\int_0^{\frac{1}{2}}\left[\Omega_{\Upsilon}^{\zeta}(1-\eth)-\Omega_{\Upsilon}^{\zeta}(\eth)\right]^{\mathrm{p}}d\eth+\int_{\frac{1}{2}}^1\left[\Omega_{\Upsilon}^{\zeta}(\eth)-\Omega_{\Upsilon}^{\zeta}(1-\eth)\right]^{\mathrm{p}}d\eth.\end{aligned}\tag{50}$$

Proof. Based on the properties of modulus, the convexity of $|\mathcal{S}'|^{\mathrm{q}}$ and applying Hölder's inequality, we can deduce from Lemma 1 the following results:

$$\begin{aligned}\left|\mathrm{I}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)\right|&\leq\tfrac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\int_0^1\left|\Omega_{\Upsilon}^{\zeta}(1-\eth)-\Omega_{\Upsilon}^{\zeta}(\eth)\right|\left|\mathcal{S}'(\eth\mu_1+(1-\eth)\mu_2)\right|d\eth\\&\leq\tfrac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left(\int_0^1\left|\Omega_{\Upsilon}^{\zeta}(1-\eth)-\Omega_{\Upsilon}^{\zeta}(\eth)\right|^{\mathrm{p}}d\eth\right)^{\frac{1}{\mathrm{p}}}\left(\int_0^1|\mathcal{S}'(\eth\mu_1+(1-\eth)\mu_2)|^{\mathrm{q}}d\eth\right)^{\frac{1}{\mathrm{q}}}\\&\leq\tfrac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left[\Pi_{\Upsilon}^{\zeta}(\mathrm{p})\right]^{\frac{1}{\mathrm{p}}}\left(\int_0^1\left[\eth|\mathcal{S}'(\mu_1)|^{\mathrm{q}}+(1-\eth)|\mathcal{S}'(\mu_2)|^{\mathrm{q}}\right]d\eth\right)^{\frac{1}{\mathrm{q}}}\\&=\tfrac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left[\Pi_{\Upsilon}^{\zeta}(\mathrm{p})\right]^{\frac{1}{\mathrm{p}}}\left[\tfrac{|\mathcal{S}'(\mu_2)|^{\mathrm{q}}+|\mathcal{S}'(\mu_1)|^{\mathrm{q}}}{2}\right]^{\frac{1}{\mathrm{q}}}.\end{aligned}$$

We have successfully completed the proof of Theorem 4. □

Let us highlight a few specific scenarios that arise as special cases of Theorem 4.

Corollary 5. *Taking $\Upsilon(\eth)=\dfrac{\eth^{\alpha}}{\Gamma(\alpha)}$ and $\zeta=0$ in Theorem 4, we get ([19], Corollary 3.3).*

Corollary 6. *Choosing $|\mathcal{S}'|\leq\mathrm{K}$ in Theorem 4, we obtain*

$$\left|\mathrm{I}_{\mathcal{S},\Omega_{\Upsilon}^{\zeta}}(\mu_1,\mu_2)\right|\leq\mathrm{K}\frac{(\mu_2-\mu_1)}{2\Psi_{\Upsilon}^{\zeta}}\left[\Pi_{\Upsilon}^{\zeta}(\mathrm{p})\right]^{\frac{1}{\mathrm{p}}}.\tag{51}$$

Theorem 5. *Consider a differentiable function $\mathcal{S}$ defined on the interval $[\mu_1, \mu_2]$. Suppose $\mathcal{S}' \in L[\mu_1, \mu_2]$ and $|\mathcal{S}'|^{q}$ is a convex function. Then, for $q \geq 1$, the following inequality holds true for generalized tempered fractional integral operators:*

$$\begin{aligned} &\left|I_{\mathcal{S},\Omega_{Y}^{\zeta}}(\mu_1,\mu_2)\right| \leq \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}} \\ &\times\left\{\left[A_{Y,1}^{\zeta}\right]^{1-\frac{1}{q}}\left[A_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|^{q}+\left(A_{Y,1}^{\zeta}-A_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}\right. \\ &\left.+\left[B_{Y,1}^{\zeta}\right]^{1-\frac{1}{q}}\left[B_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|^{q}+\left(B_{Y,1}^{\zeta}-B_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}\right\}, \end{aligned} \tag{52}$$

where

$$\begin{aligned} A_{Y,1}^{\zeta} &:= \int_0^{\frac{1}{2}}\left[\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right]d\mathfrak{d}, \\ A_{Y,2}^{\zeta} &:= \int_0^{\frac{1}{2}}\mathfrak{d}\left[\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right]d\mathfrak{d}, \\ B_{Y,1}^{\zeta} &:= \int_{\frac{1}{2}}^{1}\left[\Omega_{Y}^{\zeta}(\mathfrak{d})-\Omega_{Y}^{\zeta}(1-\mathfrak{d})\right]d\mathfrak{d}, \\ B_{Y,2}^{\zeta} &:= \int_{\frac{1}{2}}^{1}\mathfrak{d}\left[\Omega_{Y}^{\zeta}(\mathfrak{d})-\Omega_{Y}^{\zeta}(1-\mathfrak{d})\right]d\mathfrak{d} \end{aligned}$$

and Ψ_{Y}^{ζ} is defined from (18).

Proof. From Lemma 1, convexity of $|\mathcal{S}'|^{q}$, power mean inequality and properties of the modulus, we have

$$\begin{aligned} &\left|I_{\mathcal{S},\Omega_{Y}^{\zeta}}(\mu_1,\mu_2)\right| \\ &\leq \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}}\int_0^1\left|\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right||\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)|d\mathfrak{d} \\ &\leq \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}}\left(\int_0^1\left|\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right|d\mathfrak{d}\right)^{1-\frac{1}{q}} \\ &\quad\times\left(\int_0^1\left|\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right||\mathcal{S}'(\mathfrak{d}\mu_1+(1-\mathfrak{d})\mu_2)|^{q}d\mathfrak{d}\right)^{\frac{1}{q}} \\ &\leq \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}}\left\{\left(\int_0^{\frac{1}{2}}\left[\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right]d\mathfrak{d}\right)^{1-\frac{1}{q}}\right. \\ &\quad\times\left(\int_0^{\frac{1}{2}}\left[\Omega_{Y}^{\zeta}(1-\mathfrak{d})-\Omega_{Y}^{\zeta}(\mathfrak{d})\right]\left[\mathfrak{d}|\mathcal{S}'(\mu_1)|^{q}+(1-\mathfrak{d})|\mathcal{S}'(\mu_2)|^{q}\right]d\mathfrak{d}\right)^{\frac{1}{q}} \\ &\quad+\left(\int_{\frac{1}{2}}^{1}\left[\Omega_{Y}^{\zeta}(\mathfrak{d})-\Omega_{Y}^{\zeta}(1-\mathfrak{d})\right]d\mathfrak{d}\right)^{1-\frac{1}{q}} \\ &\quad\left.\times\left(\int_{\frac{1}{2}}^{1}\left[\Omega_{Y}^{\zeta}(\mathfrak{d})-\Omega_{Y}^{\zeta}(1-\mathfrak{d})\right]\left[\mathfrak{d}|\mathcal{S}'(\mu_1)|^{q}+(1-\mathfrak{d})|\mathcal{S}'(\mu_2)|^{q}\right]d\mathfrak{d}\right)^{\frac{1}{q}}\right\} \\ &= \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}} \\ &\quad\times\left\{\left[\phi_{Y,1}^{\zeta}\right]^{1-\frac{1}{q}}\left[\phi_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|^{q}+\left(\phi_{Y,1}^{\zeta}-\phi_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}\right. \\ &\quad\left.+\left[B_{Y,1}^{\zeta}\right]^{1-\frac{1}{q}}\left[B_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|^{q}+\left(B_{Y,1}^{\zeta}-B_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}\right\}. \end{aligned}$$

We have successfully completed the proof of Theorem 5. □

Now, let us consider some specific instances that arise as special cases of Theorem 5.

Corollary 7. *Taking $q = 1$ in Theorem 5, we have*

$$\begin{aligned}&\left|I_{\mathcal{S},\Omega_{Y}^{\zeta}}(\mu_1,\mu_2)\right| \leq \frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}}\\&\times\left\{\left[A_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|+\left(A_{Y,1}^{\zeta}-A_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|\right]+\left[B_{Y,2}^{\zeta}|\mathcal{S}'(\mu_1)|+\left(B_{Y,1}^{\zeta}-B_{Y,2}^{\zeta}\right)|\mathcal{S}'(\mu_2)|\right]\right\}.\end{aligned} \tag{53}$$

Corollary 8. *Choosing* $Y(\mathfrak{d})=\dfrac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha)}$ *and* $\zeta=0$ *in Corollary 7, we get ([19], Theorem 1.4).*

Corollary 9. *By employing the inequality* $|\mathcal{S}'|\leq K$ *in Theorem 5, we can derive the following result,*

$$\left|I_{\mathcal{S},\Omega_{Y}^{\zeta}}(\mu_1,\mu_2)\right| \leq K\frac{(\mu_2-\mu_1)}{2\Psi_{Y}^{\zeta}}\left[A_{Y,1}^{\zeta}+B_{Y,1}^{\zeta}\right]. \tag{54}$$

Theorem 6. *Consider a differentiable function* $\mathcal{S}$ *defined on the interval* $[\mu_1,\mu_2]$. *Suppose* $\mathcal{S}'\in L[\mu_1,\mu_2]$ *and* $|\mathcal{S}'|^{q}$ *is a convex function. Then, under the conditions* $q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *the following inequality holds true for the generalized tempered fractional integral operators:*

$$\begin{aligned}&\left|I_{\mathcal{S},\Sigma_{Y,1}^{\zeta},\Sigma_{Y,2}^{\zeta}}(x;\mu_1,\mu_2)\right| \leq \frac{(x-\mu_1)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,1}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left[\frac{|\mathcal{S}'(x)|^{q}+|\mathcal{S}'(\mu_1)|^{q}}{2}\right]^{\frac{1}{q}}\\&+\frac{(\mu_2-x)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,2}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left[\frac{|\mathcal{S}'(x)|^{q}+|\mathcal{S}'(\mu_2)|^{q}}{2}\right]^{\frac{1}{q}},\end{aligned} \tag{55}$$

where

$$\Xi_{Y,1}^{\zeta}(x,p):=\int_0^1\left[\Sigma_{Y,1}^{\zeta}(x,\mathfrak{d})\right]^{p}d\mathfrak{d},\quad \Xi_{Y,2}^{\zeta}(x,p):=\int_0^1\left[\Sigma_{Y,2}^{\zeta}(x,\mathfrak{d})\right]^{p}d\mathfrak{d}.$$

Proof. By utilizing Lemma 2, convexity of $|\mathcal{S}'|^{q}$, Hölder's inequality and considering the properties of the modulus, we can derive the following results:

$$\begin{aligned}&\left|I_{\mathcal{S},\Sigma_{Y,1}^{\zeta},\Sigma_{Y,2}^{\zeta}}(x;\mu_1,\mu_2)\right| \leq \frac{(x-\mu_1)}{2\Psi_{Y}^{\zeta}}\int_0^1\Sigma_{Y,1}^{\zeta}(x,\mathfrak{d})|\mathcal{S}'(\mathfrak{d}x+(1-\mathfrak{d})\mu_1)|d\mathfrak{d}\\&+\frac{(\mu_2-x)}{2\Psi_{Y}^{\zeta}}\int_0^1\Sigma_{Y,2}^{\zeta}(x,\mathfrak{d})|\mathcal{S}'(\mathfrak{d}x+(1-\mathfrak{d})\mu_2)|d\mathfrak{d}\\&\leq\frac{(x-\mu_1)}{2\Psi_{Y}^{\zeta}}\left(\int_0^1\left[\Sigma_{Y,1}^{\zeta}(x,\mathfrak{d})\right]^{p}d\mathfrak{d}\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{S}'(\mathfrak{d}x+(1-\mathfrak{d})\mu_1)|^{q}d\mathfrak{d}\right)^{\frac{1}{q}}\\&+\frac{(\mu_2-x)}{2\Psi_{Y}^{\zeta}}\left(\int_0^1\left[\Sigma_{Y,2}^{\zeta}(x,\mathfrak{d})\right]^{p}d\mathfrak{d}\right)^{\frac{1}{p}}\left(\int_0^1|\mathcal{S}'(\mathfrak{d}x+(1-\mathfrak{d})\mu_2)|^{q}d\mathfrak{d}\right)^{\frac{1}{q}}\\&\leq\frac{(x-\mu_1)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,1}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left(\int_0^1\left[\mathfrak{d}|\mathcal{S}'(x)|^{q}+(1-\mathfrak{d})|\mathcal{S}'(\mu_1)|^{q}\right]d\mathfrak{d}\right)^{\frac{1}{q}}\\&+\frac{(\mu_2-x)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,2}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left(\int_0^1\left[\mathfrak{d}|\mathcal{S}'(x)|^{q}+(1-\mathfrak{d})|\mathcal{S}'(\mu_2)|^{q}\right]d\mathfrak{d}\right)^{\frac{1}{q}}\\&=\frac{(x-\mu_1)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,1}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left[\frac{|\mathcal{S}'(x)|^{q}+|\mathcal{S}'(\mu_1)|^{q}}{2}\right]^{\frac{1}{q}}\\&+\frac{(\mu_2-x)}{2\Psi_{Y}^{\zeta}}\left[\Xi_{Y,2}^{\zeta}(x,p)\right]^{\frac{1}{p}}\left[\frac{|\mathcal{S}'(x)|^{q}+|\mathcal{S}'(\mu_2)|^{q}}{2}\right]^{\frac{1}{q}}.\end{aligned}$$

The proof of Theorem 6 has been successfully concluded. □

Let us highlight certain specific instances of Theorem 6.

Corollary 10. *By substituting* $x = \dfrac{\mu_1+\mu_2}{2}$ *into Theorem 6, we obtain the following midpoint inequality using generalized tempered fractional integral operators:*

$$\begin{aligned}&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}\left(\tfrac{\mu_1+\mu_2}{2};\mu_1,\mu_2\right)\right| \le \tfrac{(\mu_2-\mu_1)}{2^{2+\frac{1}{q}}\Psi^{\zeta}_{Y}}\left[\Xi^{\zeta}_{Y}(p)\right]^{\frac{1}{p}}\\&\times\left\{\left[|\mathcal{S}'(\mu_1)|^q+\left|\mathcal{S}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q\right]^{\frac{1}{q}}+\left[\left|\mathcal{S}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^q+|\mathcal{S}'(\mu_2)|^q\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{56}$$

where

$$\Xi^{\zeta}_{Y}(p) := \int_0^1\left(\int_0^{\eth}\frac{Y\left(u\left(\frac{\mu_2-\mu_1}{2}\right)\right)}{u}e^{-\zeta(\mu_2-\mu_1)u}du\right)^p d\eth.$$

Corollary 11. *Choosing* $Y(\eth) = \dfrac{\eth^{\alpha}}{\Gamma(\alpha)}$ *in Theorem 6, we get*

$$\begin{aligned}&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \le \tfrac{1}{2\gamma_{\zeta}(\alpha,\mu_2-\mu_1)}\left[\eth_{\gamma}(\alpha,p)\right]^{\frac{1}{p}}\\&\times\left\{(x-\mu_1)^{\alpha+1}\left[\tfrac{|\mathcal{S}'(x)|^q+|\mathcal{S}'(\mu_1)|^q}{2}\right]^{\frac{1}{q}}+(\mu_2-x)^{\alpha+1}\left[\tfrac{|\mathcal{S}'(x)|^q+|\mathcal{S}'(\mu_2)|^q}{2}\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{57}$$

where

$$\eth_{\gamma}(\alpha,p) := \int_0^1\left[\gamma_{\zeta(\mu_2-\mu_1)}(\alpha,\eth)\right]^p d\eth.$$

Corollary 12. *By substituting* $|\mathcal{S}'| \le K$ *into Theorem 6, we derive the following result:*

$$\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \le \frac{K}{2\Psi^{\zeta}_{Y}}\left\{(x-\mu_1)\left[\Xi^{\zeta}_{Y,1}(x,p)\right]^{\frac{1}{p}}+(\mu_2-x)\left[\Xi^{\zeta}_{Y,2}(x,p)\right]^{\frac{1}{p}}\right\}. \tag{58}$$

Theorem 7. *For a differentiable function* $\mathcal{S} : [\mu_1,\mu_2] \to R$ *defined on the interval* $[\mu_1,\mu_2]$*, if* $\mathcal{S}' \in L[\mu_1,\mu_2]$ *and the function* $|\mathcal{S}'|^q$ *is convex, where* $q \ge 1$*, then the following inequality holds true for the generalized tempered fractional integral operators:*

$$\begin{aligned}&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \le \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}}\\&\times\left[M^{\zeta}_{Y,1}(x)\right]^{1-\frac{1}{q}}\left[M^{\zeta}_{Y,3}(x)\,|\mathcal{S}'(x)|^q+\left(M^{\zeta}_{Y,1}(x)-M^{\zeta}_{Y,3}(x)\right)|\mathcal{S}'(\mu_1)|^q\right]^{\frac{1}{q}}\\&+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,2}(x)\right]^{1-\frac{1}{q}}\left[M^{\zeta}_{Y,4}(x)\,|\mathcal{S}'(x)|^q+\left(M^{\zeta}_{Y,2}(x)-M^{\zeta}_{Y,4}(x)\right)|\mathcal{S}'(\mu_2)|^q\right]^{\frac{1}{q}},\end{aligned} \tag{59}$$

where

$$\begin{aligned}M^{\zeta}_{Y,1}(x) &:= \textstyle\int_0^1\Sigma^{\zeta}_{Y,1}(x,\eth)d\eth,\\M^{\zeta}_{Y,2}(x) &:= \textstyle\int_0^1\Sigma^{\zeta}_{Y,2}(x,\eth)d\eth,\\M^{\zeta}_{Y,3}(x) &:= \textstyle\int_0^1\eth\Sigma^{\zeta}_{Y,1}(x,\eth)d\eth,\\M^{\zeta}_{Y,4}(x) &:= \textstyle\int_0^1\eth\Sigma^{\zeta}_{Y,2}(x,\eth)d\eth\end{aligned}$$

and Ψ^{ζ}_{Y} *is defined from (18).*

Proof. From Lemma 2, convexity of $|\mathcal{S}'|^{q}$, power mean inequality and properties of the modulus, we have

$$
\begin{aligned}
&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \leq \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}}\int_0^1 \Sigma^{\zeta}_{Y,1}(x,\eth)|\mathcal{S}'(\eth x+(1-\eth)\mu_1)|d\eth \\
&\quad+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\int_0^1 \Sigma^{\zeta}_{Y,2}(x,\eth)|\mathcal{S}'(\eth x+(1-\eth)\mu_2)|d\eth \\
&\leq \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}}\left(\int_0^1 \Sigma^{\zeta}_{Y,1}(x,\eth)d\eth\right)^{1-\frac{1}{q}}\left(\int_0^1 \Sigma^{\zeta}_{Y,1}(x,\eth)|\mathcal{S}'(\eth x+(1-\eth)\mu_1)|^{q}d\eth\right)^{\frac{1}{q}} \\
&\quad+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\left(\int_0^1 \Sigma^{\zeta}_{Y,2}(x,\eth)d\eth\right)^{1-\frac{1}{q}}\left(\int_0^1 \Sigma^{\zeta}_{Y,2}(x,\eth)|\mathcal{S}'(\eth x+(1-\eth)\mu_2)|^{q}d\eth\right)^{\frac{1}{q}} \\
&\leq \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,1}(x)\right]^{1-\frac{1}{q}}\left(\int_0^1 \Sigma^{\zeta}_{Y,1}(x,\eth)\left[\eth|\mathcal{S}'(x)|^{q}+(1-\eth)|\mathcal{S}'(\mu_1)|^{q}\right]d\eth\right)^{\frac{1}{q}} \\
&\quad+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,2}(x)\right]^{1-\frac{1}{q}}\left(\int_0^1 \Sigma^{\zeta}_{Y,2}(x,\eth)\left[\eth|\mathcal{S}'(x)|^{q}+(1-\eth)|\mathcal{S}'(\mu_2)|^{q}\right]d\eth\right)^{\frac{1}{q}} \\
&= \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,1}(x)\right]^{1-\frac{1}{q}}\left[M^{\zeta}_{Y,3}(x)\,|\mathcal{S}'(x)|^{q}+\left(M^{\zeta}_{Y,1}(x)-M^{\zeta}_{Y,3}(x)\right)|\mathcal{S}'(\mu_1)|^{q}\right]^{\frac{1}{q}} \\
&\quad+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,2}(x)\right]^{1-\frac{1}{q}}\left[M^{\zeta}_{Y,4}(x)\,|\mathcal{S}'(x)|^{q}+\left(M^{\zeta}_{Y,2}(x)-M^{\zeta}_{Y,4}(x)\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}.
\end{aligned}
$$

The proof of Theorem 7 is completed. □

We point out some special cases of Theorem 7.

Corollary 13. *Taking* $q=1$ *in Theorem 7, we have*

$$
\begin{aligned}
&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \leq \tfrac{(x-\mu_1)}{2\Psi^{\zeta}_{Y}} \\
&\times\left[M^{\zeta}_{Y,3}(x)\,|\mathcal{S}'(x)|+\left(M^{\zeta}_{Y,1}(x)-M^{\zeta}_{Y,3}(x)\right)|\mathcal{S}'(\mu_1)|\right] \\
&+\tfrac{(\mu_2-x)}{2\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y,4}(x)\,|\mathcal{S}'(x)|+\left(M^{\zeta}_{Y,2}(x)-M^{\zeta}_{Y,4}(x)\right)|\mathcal{S}'(\mu_2)|\right].
\end{aligned}
\tag{60}
$$

Corollary 14. *Choosing* $x=\dfrac{\mu_1+\mu_2}{2}$ *in Theorem 7 and utilizing the generalized tempered fractional integral operators, we obtain the following midpoint inequality:*

$$
\begin{aligned}
&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}\left(\tfrac{\mu_1+\mu_2}{2};\mu_1,\mu_2\right)\right| \leq \tfrac{(\mu_2-\mu_1)}{4\Psi^{\zeta}_{Y}}\left[M^{\zeta}_{Y}\right]^{1-\frac{1}{q}} \\
&\times\left\{\left[N^{\zeta}_{Y}\left|\mathcal{S}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^{q}+\left(M^{\zeta}_{Y}-N^{\zeta}_{Y}\right)|\mathcal{S}'(\mu_1)|^{q}\right]^{\frac{1}{q}}\right. \\
&\left.+\left[N^{\zeta}_{Y}\left|\mathcal{S}'\left(\tfrac{\mu_1+\mu_2}{2}\right)\right|^{q}+\left(M^{\zeta}_{Y}-N^{\zeta}_{Y}\right)|\mathcal{S}'(\mu_2)|^{q}\right]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{61}
$$

where

$$
\begin{aligned}
M^{\zeta}_{Y} &:= \int_0^1\left(\int_0^{\eth}\frac{Y\left(u\left(\frac{\mu_2-\mu_1}{2}\right)\right)}{u}\,e^{-\zeta(\mu_2-\mu_1)u}du\right)d\eth, \\
N^{\zeta}_{Y} &:= \int_0^1 \eth\left(\int_0^{\eth}\frac{Y\left(u\left(\frac{\mu_2-\mu_1}{2}\right)\right)}{u}\,e^{-\zeta(\mu_2-\mu_1)u}du\right)d\eth.
\end{aligned}
$$

Corollary 15. *Taking* $\Upsilon(\mathfrak{d}) = \dfrac{\mathfrak{d}^{\alpha}}{\Gamma(\alpha)}$ *in Theorem 7, we obtain*

$$\begin{aligned}
&\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \le \tfrac{1}{2\gamma_{\zeta}(\alpha,\mu_2-\mu_1)}\left[\mathfrak{d}_{\gamma}(\alpha)\right]^{1-\frac{1}{q}}\\
&\times\left\{(x-\mu_1)^{\alpha+1}\left[\mathfrak{d}_{\gamma,1}(\alpha)\,|\mathcal{S}'(x)|^q+\left(\mathfrak{d}_{\gamma}(\alpha)-\mathfrak{d}_{\gamma,1}(\alpha)\right)|\mathcal{S}'(\mu_1)|^q\right]^{\frac{1}{q}}\right.\\
&\left.+(\mu_2-x)^{\alpha+1}\left[\mathfrak{d}_{\gamma,1}(\alpha)\,|\mathcal{S}'(x)|^q+\left(\mathfrak{d}_{\gamma}(\alpha)-\mathfrak{d}_{\gamma,1}(\alpha)\right)|\mathcal{S}'(\mu_2)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{62}$$

where

$$\begin{aligned}
\mathfrak{d}_{\gamma}(\alpha) &:= \textstyle\int_0^1 \gamma_{\zeta(\mu_2-\mu_1)}(\alpha,\mathfrak{d})d\mathfrak{d},\\
\mathfrak{d}_{\gamma,1}(\alpha) &:= \textstyle\int_0^1 \mathfrak{d}\,\gamma_{\zeta(\mu_2-\mu_1)}(\alpha,\mathfrak{d})d\mathfrak{d}.
\end{aligned}$$

Corollary 16. *Choosing* $|\mathcal{S}'| \le K$ *in Theorem 7, we have*

$$\left|I_{\mathcal{S},\Sigma^{\zeta}_{Y,1},\Sigma^{\zeta}_{Y,2}}(x;\mu_1,\mu_2)\right| \le \frac{K}{2\Psi^{\zeta}_{Y}}\left\{(x-\mu_1)M^{\zeta}_{Y,1}(x)+(\mu_2-x)M^{\zeta}_{Y,2}(x)\right\}. \tag{63}$$

Remark 3. *Applying our results for suitable choices of function* $\Upsilon(\mathfrak{d}) = \mathfrak{d}(\mu_2-\mathfrak{d})^{\alpha-1}$ *and* $\Upsilon(\mathfrak{d}) = \frac{\mathfrak{d}}{\alpha}\exp(-A\mathfrak{d})$*, where* $A := \frac{1-\alpha}{\alpha}$ *for* $\alpha \in (0,1)$ *such that* $|\mathcal{S}'|^q$ *to be convex function, we can construct some new tempered fractional integral type inequalities. We omit their proofs and the details are left to the interested readers.*

4. Applications

In the final section, we provide several examples that illustrate the results we have established in relation to matrices, modified Bessel functions and q-digamma function.

4.1. Matrices

Example 1. *In this context, the set of* $n\times n$ *complex matrices is denoted by* C^n*. Similarly,* M_n *represents the algebra of* $n\times n$ *complex matrices and* M_n^+ *refers to the subset of strictly positive matrices within* M_n*. Consequently, for a matrix* $A \in M_n^+$*, it holds that* $\langle Au,u\rangle > 0$ *for all nonzero* $u \in C^n$*.*

Incorporating the concepts of matrices and convexity, Sababheh [27] introduced the function $\mathcal{S}(\mathfrak{d}) := \|A^{\mathfrak{d}}XB^{1-\mathfrak{d}} + A^{1-\mathfrak{d}}XB^{\mathfrak{d}}\|$*, where* $A, B \in M_n^+$*, and* $X \in M_n$*. It was shown that this function is convex for all* $\mathfrak{d} \in [0,1]$*. Consequently, by utilizing Theorem 2 for any* $0 \le \mu_1 < \mu_2 \le 1$*, we obtain*

$$\begin{aligned}
&\|A^{\frac{\mu_1+\mu_2}{2}}XB^{1-\frac{\mu_1+\mu_2}{2}} + A^{1-\frac{\mu_1+\mu_2}{2}}XB^{\frac{\mu_1+\mu_2}{2}}\|\\
&\le \tfrac{1}{2\Psi^{\zeta}_{Y}}\left[{}_{\mu_1^+}T^{\zeta}_{Y}\|A^{\mu_2}XB^{1-\mu_2}+A^{1-\mu_2}XB^{\mu_2}\| + {}_{\mu_2^-}T^{\zeta}_{Y}\|A^{\mu_1}XB^{1-\mu_1}+A^{1-\mu_1}XB^{\mu_1}\|\right]\\
&\le \tfrac{\|A^{\mu_1}XB^{1-\mu_1}+A^{1-\mu_1}XB^{\mu_1}\|+\|A^{\mu_2}XB^{1-\mu_2}+A^{1-\mu_2}XB^{\mu_2}\|}{2}.
\end{aligned} \tag{64}$$

4.2. Modified Bessel Functions

The modified Bessel function of the first kind, denoted as $\mathfrak{d}_{\rho}(a)$, is a mathematical function defined as the following infinite sum:

$$\mathfrak{d}_{\rho}(a) = \sum_{n\ge 0}\frac{\left(\frac{a}{2}\right)^{\rho+2n}}{n!\Gamma(n+\rho+1)},$$

where $a > 0$ is a real number and $\rho > -1$.

The modified Bessel function of the second kind, denoted as $\mathfrak{B}_{\rho}(a)$, is defined in terms of $\mathfrak{d}_{\rho}(a)$ as follows:

$$\mathfrak{B}_{\rho}(a) := \frac{\pi}{2}\,\frac{\mathfrak{d}_{-\rho}(a)-\mathfrak{d}_{\rho}(a)}{\sin\rho\pi}.$$

Another function, $\mathrm{J}_\rho(\mathtt{a})$, can be defined as:

$$\mathrm{J}_\rho(\mathtt{a}) := 2^\rho \Gamma(\rho+1)\mathtt{a}^{-\rho}\mathfrak{B}_\rho(\mathtt{a}),$$

where $\Gamma(\cdot)$ represents the gamma function.

In the cited reference [28], two derivative formulas for $\mathrm{J}_\rho(\mathtt{a})$ are given. The first derivative is expressed as:

$$\mathrm{J}'_\rho(\mathtt{a}) = \frac{\mathtt{a}}{2(\rho+1)}\mathrm{J}_{\rho+1}(\mathtt{a}). \tag{65}$$

The second derivative is given by:

$$\mathrm{J}''_\rho(\mathtt{a}) = \frac{\mathtt{a}^2\mathrm{J}_{\rho+2}(\mathtt{a})}{4(\rho+1)(\rho+2)} + \frac{\mathrm{J}_{\rho+1}(\mathtt{a})}{2(\rho+1)}. \tag{66}$$

Example 2. *Applying Corollary 7 and using expressions (65) and (66), we have*

$$\begin{aligned}
&\left|\mu_1\mathrm{J}_{\rho+1}(\mu_1) + \mu_2\mathrm{J}_{\rho+1}(\mu_2) - \frac{1}{\Psi_{\mathrm{Y}}^{\zeta}}\left[\mu_2 \cdot {}_{\mu_1^+}\mathrm{T}_{\mathrm{Y}}^{\zeta}\mathrm{J}_{\rho+1}(\mu_2) + \mu_1 \cdot {}_{\mu_2^-}\mathrm{T}_{\mathrm{Y}}^{\zeta}\mathrm{J}_{\rho+1}(\mu_1)\right]\right| \le \frac{(\mu_2-\mu_1)}{2\Psi_{\mathrm{Y}}^{\zeta}} \\
&\times\left\{\left[\mathrm{A}_{\mathrm{Y},2}^{\zeta}\left|\frac{\mu_1^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_1) + 2\mathrm{J}_{\rho+1}(\mu_1)\right| + \left(\mathrm{A}_{\mathrm{Y},1}^{\zeta} - \mathrm{A}_{\mathrm{Y},2}^{\zeta}\right)\left|\frac{\mu_2^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_2) + 2\mathrm{J}_{\rho+1}(\mu_2)\right|\right]\right. \\
&\left.+\left[\mathrm{B}_{\mathrm{Y},2}^{\zeta}\left|\frac{\mu_1^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_1) + 2\mathrm{J}_{\rho+1}(\mu_1)\right| + \left(\mathrm{B}_{\mathrm{Y},1}^{\zeta} - \mathrm{B}_{\mathrm{Y},2}^{\zeta}\right)\left|\frac{\mu_2^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_2) + 2\mathrm{J}_{\rho+1}(\mu_2)\right|\right]\right\}.
\end{aligned}$$

Proof. By utilizing Corollary 7 and incorporating Equations (65) and (66) where the function $\mathcal{S}(\mathtt{a}) := \mathrm{J}'_\rho(\mathtt{a})$ with $\mathtt{a} > 0$, we can derive the final result, thus completing the proof. □

Example 3. *Using Corollary 13 and applying expressions (65) and (66), we get*

$$\begin{aligned}
&\left|2\mathrm{x}\mathrm{J}_{\rho+1}(\mathrm{x}) - \frac{1}{\Psi_{\mathrm{Y}}^{\zeta}}\left[\mu_2 \cdot {}_{\mathrm{x}^+}\mathrm{T}_{\mathrm{Y}}^{\zeta}\mathrm{J}_{\rho+1}(\mu_2) + \mu_1 \cdot {}_{\mathrm{x}^-}\mathrm{T}_{\mathrm{Y}}^{\zeta}\mathrm{J}_{\rho+1}(\mu_1)\right]\right| \\
&\le \frac{(\mathrm{x}-\mu_1)}{2\Psi_{\mathrm{Y}}^{\zeta}}\left[\mathrm{M}_{\mathrm{Y},3}^{\zeta}(\mathrm{x})\left|\frac{\mathrm{x}^2}{\rho+2}\mathrm{J}_{\rho+2}(\mathrm{x}) + 2\mathrm{J}_{\rho+1}(\mathrm{x})\right|\right. \\
&\left.+\left(\mathrm{M}_{\mathrm{Y},1}^{\zeta}(\mathrm{x}) - \mathrm{M}_{\mathrm{Y},3}^{\zeta}(\mathrm{x})\right)\left|\frac{\mu_1^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_1) + 2\mathrm{J}_{\rho+1}(\mu_1)\right|\right] \\
&+\frac{(\mu_2-\mathrm{x})}{2\Psi_{\mathrm{Y}}^{\zeta}}\left[\mathrm{M}_{\mathrm{Y},4}^{\zeta}(\mathrm{x})\left|\frac{\mathrm{x}^2}{\rho+2}\mathrm{J}_{\rho+2}(\mathrm{x}) + 2\mathrm{J}_{\rho+1}(\mathrm{x})\right|\right. \\
&\left.+\left(\mathrm{M}_{\mathrm{Y},2}^{\zeta}(\mathrm{x}) - \mathrm{M}_{\mathrm{Y},4}^{\zeta}(\mathrm{x})\right)\left|\frac{\mu_2^2}{\rho+2}\mathrm{J}_{\rho+2}(\mu_2) + 2\mathrm{J}_{\rho+1}(\mu_2)\right|\right].
\end{aligned}$$

Proof. By employing Corollary 13 and incorporating Equations (65) and (66), where the function $\mathcal{S}(\mathtt{a}) := \mathrm{J}'_\rho(\mathtt{a})$ with $\mathtt{a} > 0$, we can deduce the outcome, thereby concluding the proof. □

4.3. q*-Digamma Function*

The q-digamma function, denoted as ϱ_{q}, is a function that serves as the q-analogue of the digamma function ϱ. The formula for ϱ_{q} is provided in the references [28,29] and is given as follows:

$$\varrho_{\mathrm{q}}(\gamma) = -\ln(1-\mathrm{q}) + \ln \mathrm{q} \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{q}^{\mathrm{k}+\gamma}}{1-\mathrm{q}^{\mathrm{k}+\gamma}}$$

or alternatively,

$$\varrho_{\mathrm{q}}(\gamma) = -\ln(1-\mathrm{q}) + \ln \mathrm{q} \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{q}^{\mathrm{k}\gamma}}{1-\mathrm{q}^{\mathrm{k}\gamma}}.$$

This expression holds true when $\mathrm{q} > 1$ and $\gamma > 0$. The q-digamma function ϱ_{q} can also be expressed in the following form:

$$\varrho_{\mathrm{q}}(\gamma) = -\ln(\mathrm{q}-1) + \ln \mathrm{q}\left[\gamma - \frac{1}{2} - \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{q}^{-(\mathrm{k}+\gamma)}}{1-\mathrm{q}^{-(\mathrm{k}+\gamma)}}\right]$$

or alternatively,

$$\varrho_{\mathrm{q}}(\gamma) = -\ln(\mathrm{q}-1) + \ln \mathrm{q}\left[\gamma - \frac{1}{2} - \sum_{\mathrm{k}=0}^{\infty} \frac{\mathrm{q}^{-\mathrm{k}\gamma}}{1-\mathrm{q}^{-\mathrm{k}\gamma}}\right].$$

Example 4. *Applying Corollary 7, we have*

$$\begin{aligned}
&\left|\frac{\varrho'_{\mathrm{q}}(\mu_1)+\varrho'_{\mathrm{q}}(\mu_2)}{2} - \frac{1}{2\Psi^{\zeta}_{\mathrm{Y}}}\left[{}_{\mu_1^+}\mathrm{T}^{\zeta}_{\mathrm{Y}}\varrho'_{\mathrm{q}}(\mu_2) + {}_{\mu_2^-}\mathrm{T}^{\zeta}_{\mathrm{Y}}\varrho'_{\mathrm{q}}(\mu_1)\right]\right| \leq \frac{(\mu_2-\mu_1)}{2\Psi^{\zeta}_{\mathrm{Y}}}\\
&\times\left\{\left[\mathrm{A}^{\zeta}_{\mathrm{Y},2}|\varrho''_{\mathrm{q}}(\mu_1)| + \left(\mathrm{A}^{\zeta}_{\mathrm{Y},1} - \mathrm{A}^{\zeta}_{\mathrm{Y},2}\right)|\varrho''_{\mathrm{q}}(\mu_2)|\right] + \left[\mathrm{B}^{\zeta}_{\mathrm{Y},2}|\varrho''_{\mathrm{q}}(\mu_1)| + \left(\mathrm{B}^{\zeta}_{\mathrm{Y},1} - \mathrm{B}^{\zeta}_{\mathrm{Y},2}\right)|\varrho''_{\mathrm{q}}(\mu_2)|\right]\right\}.
\end{aligned}$$

Proof. The statement can be readily derived by utilizing Corollary 7, where $\mathcal{S}(\mathrm{a}) \to \varrho_{\mathrm{q}}(\mathrm{a})$ represents a completely monotone function on the interval $(0,\infty)$ for all $\mathrm{a} > 0$. As a result, $\mathcal{S}(\mathrm{a}) := \varrho'_{\mathrm{q}}(\mathrm{a})$ is a convex function. □

Example 5. *Using Corollary 13, we get*

$$\begin{aligned}
&\left|\varrho'_{\mathrm{q}}(\mathrm{x}) - \frac{1}{2\Psi^{\zeta}_{\mathrm{Y}}}\left[{}_{\mathrm{x}^+}\mathrm{T}^{\zeta}_{\mathrm{Y}}\varrho'_{\mathrm{q}}(\mu_2) + {}_{\mathrm{x}^-}\mathrm{T}^{\zeta}_{\mathrm{Y}}\varrho'_{\mathrm{q}}(\mu_1)\right]\right|\\
&\leq \frac{(\mathrm{x}-\mu_1)}{2\Psi^{\zeta}_{\mathrm{Y}}}\left[\mathrm{M}^{\zeta}_{\mathrm{Y},3}(\mathrm{x})\,|\varrho''_{\mathrm{q}}(\mathrm{x})| + \left(\mathrm{M}^{\zeta}_{\mathrm{Y},1}(\mathrm{x}) - \mathrm{M}^{\zeta}_{\mathrm{Y},3}(\mathrm{x})\right)|\varrho''_{\mathrm{q}}(\mu_1)|\right]\\
&+\frac{(\mu_2-\mathrm{x})}{2\Psi^{\zeta}_{\mathrm{Y}}}\left[\mathrm{M}^{\zeta}_{\mathrm{Y},4}(\mathrm{x})\,|\varrho''_{\mathrm{q}}(\mathrm{x})| + \left(\mathrm{M}^{\zeta}_{\mathrm{Y},2}(\mathrm{x}) - \mathrm{M}^{\zeta}_{\mathrm{Y},4}(\mathrm{x})\right)|\varrho''_{\mathrm{q}}(\mu_2)|\right].
\end{aligned}$$

Proof. The assertion can be immediately derived by utilizing Corollary 13, where $\mathcal{S}(\mathrm{a}) \to \varrho_{\mathrm{q}}(\mathrm{a})$ represents a completely monotone function on the interval $(0,\infty)$ for all $\mathrm{a} > 0$. Consequently, $\mathcal{S}(\mathrm{a}) := \varrho'_{\mathrm{q}}(\mathrm{a})$ is a convex function. □

5. Conclusions

Overall, this paper aimed to contribute to the expanding field of fractional H–H inequalities by presenting new results, exploring their properties, investigating their connections with fractional calculus, and demonstrating their applications. Through our work, we sought to advance the theoretical foundations of fractional calculus, enhance our understanding of convex functions in the fractional calculus framework, and inspire further research in this exciting and evolving area of mathematics.

This research paper introduces novel concepts of left and right generalized tempered fractional integral operators and establishes fresh H–H inequalities for convex functions and their products. Additionally, it derives two useful identities for differentiable functions that involve the generalized tempered fractional integral operator. These identities are then utilized to establish H–H and midpoint-type integral inequalities for convex functions. The paper also explores various special cases and demonstrates how the general findings recover known results. Furthermore, the paper presents compelling applications related to matrices, modified Bessel functions and q-digamma functions. By employing the newly introduced generalized tempered fractional integral operators, along with well-known inequalities such as Hölder–İşcan's inequality, Improved-Power-mean's inequality, Young's inequality, Minkowski's inequality, and Chebyshev's inequality, this paper establishes novel bounds for differentiable convex functions. Finally, the importance of these new integral operators depends on the choices of the parameter ζ, since it takes values in the domain $[0, \infty)$. In other words, interested readers that will do numerical computations will immediately see the advantage of this new operator compared with other known integral operators in terms of the suitable choices of parameter ζ. We believe that these new operators will be important tools for investigating various variational problems for different types of convexities.

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