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# Discrete Octonion Linear Canonical Transform: Definition and Properties

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**Abstract:** In this paper, the discrete octonion linear canonical transform (DOCLCT) is defined. According to the definition of the DOCLCT, some properties associated with the DOCLCT are explored, such as linearity, scaling, boundedness, Plancherel theorem, inversion transform and shift transform. Then, the relationship between the DOCLCT and the three-dimensional (3-D) discrete linear canonical transform (DLCT) is obtained. Moreover, based on a new convolution operator, we derive the convolution theorem of the DOCLCT. Finally, the correlation theorem of the DOCLCT is established.

**Keywords:** linear canonical transform; discrete octonion Fourier transform; octonion linear canonical transform; convolution theorem; correlation theorem



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## 1. Introduction

The linear canonical transform (LCT) [1–3] is a generalized form of the fractional Fourier transform (FrFT). As a linear integral transform with three parameter class, the LCT is more flexible than the FrFT and is a widely used analytical and processing tool in applied mathematics and engineering [4–8]. For analyzing and processing the non-stationary spectrum of finite-duration signals, Pei and Ding [9] proposed the discrete linear canonical transform (DLCT). The DLCT is a very important tool for processing discrete data with a digital camera. Wei et al. [10] studied image encryption using the the random discrete linear canonical transform, which demonstrated that the proposed encryption method is a security-enhanced image encryption scheme. Sun and Li [11] proposed the sliding discrete linear canonical transform and obtained an adaptive method for the computation of the DLCT. Zhang and Li [12] proposed and designed the definition of the DLCT in graph settings. Based on different kinds of DLCTs, several scholars studied many properties and applications [13–17].

Recently, hypercomplex algebras [18,19] are increasingly receiving research interest from scholars. Quaternion algebras are hypercomplex algebras of order 4 and have been widely applied in optical and signal processing [20–22]. Urynbassarova et al. [23] extended the DLCT to the quaternion linear canonical transform domain, and proposed the discrete quaternion linear canonical transform (DQLCT). Some properties of the two-dimensional (2-D) DQLCT were derived, such as the shift, modulation, inversion formula and Plancherel theorem. Moreover, they studied the convolution theorem and fast algorithm for the 2-D DQLCT. Based on the 2-D DQLCT, some applications were illustrated by the simulations. Srivastava et al. [24] presented the discrete quadratic-phase Fourier transform. The convolution and correlation theorems for the discrete quadratic-phase Fourier transform were studied.

Octonion algebras [25] are another hypercomplex algebra with order 8 which is the generalized form of the quaternion algebra. Hahn and Snopek [26] proposed the octonion Fourier transform (OFT) and studied the properties. Several applications of the OFT in signal processing were studied in [27,28]. In order to analyze and process the octonion spectrum of finite-duration signals, Błaszczyk [29] exploited the discrete form for the OFT and obtained the discrete octonion Fourier transform (DOFT). According to the DOFT, the analysis of solutions for difference equations and 3-D discrete linear time invariant systems were discussed. Researchers have considered that the linear canonical transform is a general form of the Fourier transform and has good analytical and processing properties [30,31]. We [32] substituted the octonion Fourier kernel function with the octonion linear canonical kernel function and obtained the octonion linear canonical transform (OCLCT). Next, some papers [32,33] discussed many properties and uncertainty principles associated with the OCLCT. Moreover, many scholars [34,35] proposed different transform forms of the OCLCT.

So far, the OCLCT is mainly studied regarding the integral transform of non-stationary continuous signals. As far as we know, the discrete form of the OCLCT has never been published to date. In this paper, in order to study octonion finite-length signals, we propose the discrete octonion linear canonical transform (DOCLCT). The DOCLCT is obtained by replacing the Fourier transform kernel function with the linear canonical transform kernel function based on the octonion algebra setting. Then, several important properties of the DOCLCT are derived, such as linearity, scaling, boundedness, Plancherel theorem inversion transform and shift transform. Moreover, the relation between the DOCLCT and the 3-D DLCT is obtained. The convolution theorem associated with the DOCLCT is presented by a new convolution operator. Finally, the correlation theorem of the DOCLCT is exploited.

This paper is organized as follows: In Section 2, several basic properties of octonion algebra are presented. The definition and the properties of the DOCLCT are obtained in Section 3. In Section 4, the convolution theorem for the DOCLCT is derived. The correlation theorem of the DOCLCT is discussed in Section 5. In Section 6, the conclusions and potential applications are drawn.

## 2. Preliminaries

This section presents knowledge of octonion algebra [36]. This is the research foundation of this paper.

### *2.1. Octonion Algebra*

Octonion algebra is defined by $\mathbb{O}$ [36]. An arbitrary $o \in \mathbb{O}$ can be given by

$$o = o_0 + o_1e_1 + o_2e_2 + o_3e_3 + o_4e_4 + o_5e_5 + o_6e_6 + o_7e_7$$

where $o_0, o_1, \cdots, o_7 \in \mathbb{R}$. Octonion algebra is a non-associative and non-commutative algebra. Figure 1 presents the multiplication rules of octonion algebra.

| * | 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ |
| $e_1$ | $e_1$ | -1 | $e_3$ | $-e_2$ | $e_5$ | $-e_4$ | $-e_7$ | $e_6$ |
| $e_2$ | $e_2$ | $-e_3$ | -1 | $e_1$ | $e_6$ | $e_7$ | $-e_4$ | $-e_5$ |
| $e_3$ | $e_3$ | $e_2$ | $-e_1$ | -1 | $e_7$ | $-e_6$ | $e_5$ | $-e_4$ |
| $e_4$ | $e_4$ | $-e_5$ | $-e_6$ | $-e_7$ | -1 | $e_1$ | $e_2$ | $e_3$ |
| $e_5$ | $e_5$ | $e_4$ | $-e_7$ | $e_6$ | $-e_1$ | -1 | $-e_3$ | $e_2$ |
| $e_6$ | $e_6$ | $e_7$ | $e_4$ | $-e_5$ | $-e_2$ | $e_3$ | -1 | $-e_1$ |
| $e_7$ | $e_7$ | $-e_6$ | $e_5$ | $e_4$ | $-e_3$ | $-e_2$ | $e_1$ | -1 |

**Figure 1.** Multiplication rules in octonion algebra.

The norm of octonion algebra is defined by $|o| = \sqrt{o\bar{o}} = \sqrt{\bar{o}o}$ and $|o|^2 = \sum_{r=0}^{7} o_r^2$. It satisfies $|\epsilon\varepsilon|{=}|\epsilon||\varepsilon|$ for all $\epsilon, \varepsilon \in \mathbb{O}$.

For any octonion algebra, $o \in \mathbb{O}$ can be represented as

$$o = \jmath + \imath e_4,$$

where $\jmath = o_0 + o_1e_1 + o_2e_2 + o_3e_3$ and $\imath = o_4 + o_5e_1 + o_6e_2 + o_7e_3$ are quaternion algebras. That is to say, any octonion algebra can be composed of the sum of two quaternion algebras.

**Property 1** ([25]). *Let $\jmath, \imath \in \mathbb{H}$. Then, any octonion algebra satisfies the following properties:*

$$(1)\ e_4\jmath = \bar{\jmath}e_4;\ \ (2)\ e_4(\jmath e_4) = -\bar{\jmath};\ \ (3)\ (\jmath e_4)e_4 = -\jmath,$$

$$(4)\ \jmath(\imath e_4) = (\imath\jmath)e_4;\ \ (5)\ (\jmath e_4)\imath = (\jmath\bar{\imath})e_4;\ \ (6)\ (\jmath e_4)(\imath e_4) = -\bar{\imath}\jmath.$$

**Property 2** ([25]). *For any octonion algebras $\jmath + \imath e_4, \jmath, \imath \in \mathbb{H}$ in the quaternionic form, then the following formulas are right:*

$$\overline{\jmath + \imath e_4} = \bar{\jmath} - \imath e_4,$$

$$|\jmath + \imath e_4|^2 = |\jmath|^2 + |\imath|^2.$$

According to octonion algebra, we provide the definition of an octonion-valued function. For any octonion-valued function, $x(\mathbf{n})$ is defined by

$$x(\mathbf{n}) = x_0(\mathbf{n}) + x_1(\mathbf{n})e_1 + \cdots + x_7(\mathbf{n})e_7 = \tilde{x}(\mathbf{n}) + \hat{x}(\mathbf{n})e_4, \tag{1}$$

where $\tilde{x}(\mathbf{n}) = x_0(\mathbf{n}) + x_1(\mathbf{n})e_1 + x_2(\mathbf{n})e_2 + x_3(\mathbf{n})e_3$ and $\hat{x}(\mathbf{n}) = x_4(\mathbf{n}) + x_5(\mathbf{n})e_1 + x_6(\mathbf{n})e_2 + x_7(\mathbf{n})e_3$ are quaternion valued functions. $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{R}^3$.

For $1 \le p < \infty$, the norm for the 3-D octonion-valued signal $x(\mathbf{n})$ is given by

$$\| x \|_p^p = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} |x(\mathbf{n})|^p.$$

If $p = 2$, then the norm for the 3-D octonion-valued signal $x(\mathbf{n})$ is

$$\| x \|_2^2 = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} |x(\mathbf{n})|^2.$$

*2.2. Discrete Linear Canonical Transform*

Next, we present the definition of the 3-D DLCT.

**Definition 1** ([9]). *Let $A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be a matrix parameter satisfying $\det(A_k) = 1$ $(k = 1, 2, 3)$. For any function $x : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$, the 3-D DLCT is defined by*

$$\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1, m_1) D_{A_2}^{e_1}(n_2, m_2) D_{A_3}^{e_1}(n_3, m_3), \tag{2}$$

*where the discrete linear canonical transform kernel signal is*

$$D_{A_k}^{e_1}(n_k, m_k) = \frac{1}{\sqrt{N_k}} e^{e_1\left(\frac{a_k}{2b_k} n_k^2 \Delta s_k^2 - \frac{2\pi}{N_k} n_k m_k + \frac{d_k}{2b_k} m_k^2 \Delta y_k^2 - \frac{\pi}{2}\right)}, \tag{3}$$

*where $\Delta s_k$ is the periodic sampling interval in the space domain $s_k$ and $\Delta y_k$ is the periodic sampling interval in the DLCT domain $y_k$.*

*2.3. Discrete Quaternion Linear Canonical Transform*

In paper [23], the authors proposed the DQLCT and exploited knowledge about the DQLCT. The DQLCT expands the research scope of the discrete quaternion Fourier transform, and provides a way to solve problems about quaternion non-stationary finite-length signals. Due to the non-commutativity and non-associative properties of quaternion, there are three kinds of DQLCT. The authors of [23] proposed the two-sided DQLCT.

**Definition 2** ([23])**.** *Let* $A_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ *be a matrix parameter satisfying* $\det(A_k) = 1$ *($k = 1, 2$). The DQLCT of a function* $x : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$ *is defined by*

$$\mathcal{L}_{A_1,A_2}\{x\}(\mathbf{m}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} K^{e_1}_{L,A_1}(n_1, m_1)x(\mathbf{n})K^{e_2}_{L,A_2}(n_2, m_2), \tag{4}$$

*where the discrete quaternion linear canonical transform kernel signal is*

$$K^{e_k}_{L,A_k}(n_k, m_k) = \frac{1}{\sqrt{N_k}} e^{e_k\left(\frac{a_k}{2b_k}n_k^2\Delta s_k^2 - \frac{2\pi}{N_k}n_k m_k + \frac{d_k}{2b_k}m_k^2\Delta y_k^2\right)}, \tag{5}$$

*where $\Delta s_k$ is the periodic sampling interval in the space domain $s_k$ and $\Delta y_k$ is the periodic sampling interval in the DQLCT domain $y_k$.*

The inverse transform of the DQLCT is displayed by

$$x(\mathbf{n}) = \sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1} K^{e_1}_{L,A_1^{-1}}(m_1, n_1)\mathcal{L}_{A_1,A_2}\{x\}(\mathbf{m})K^{e_2}_{L,A_2^{-1}}(m_2, n_2), \tag{6}$$

where $A_k^{-1} = \begin{bmatrix} d_k & -b_k \\ -c_k & a_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ are inverse matrices.

*2.4. Discrete Octonion Fourier Transform*

In paper [29], the DOFT of 3-D octonion finite-length signals was given. The DOFT is a very good tool for studying octonion finite-length signals. In the following description, a 3-D octonion finite-length function is equivalent to a 3-D octonion finite-length signal.

**Definition 3.** *Let a 3-D octonion-valued function $x(\boldsymbol{n})$ be a 3-D finite-length signal and $x : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$. The DOFT is defined as follows:*

$$F_o(x)(\mathbf{m}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) e^{-e_1\frac{2\pi n_1 m_1}{N_1}} e^{-e_2\frac{2\pi n_2 m_2}{N_2}} e^{-e_4\frac{2\pi n_3 m_3}{N_3}}, \tag{7}$$

*where $\boldsymbol{n} = (n_1, n_2, n_3) \in [N_1] \times [N_2] \times [N_3]$, $\boldsymbol{m} = (m_1, m_2, m_3) \in [N_1] \times [N_2] \times [N_3]$, $[N_k] = \{0, 1, \cdots, N_k - 1\}$ and $(k = 1, 2, 3)$.*

The inverse transform of the DOFT is presented by the following formula [29]:

$$x(\mathbf{n}) = \frac{1}{N_1N_2N_3}\sum_{m_1=0}^{N_1-1}\sum_{m_2=0}^{N_2-1}\sum_{m_3=0}^{N_3-1} F_o(x)(\mathbf{m})\mathbf{e}^{e_4\frac{2\pi n_3 m_3}{N_3}}\mathbf{e}^{e_2\frac{2\pi n_2 m_2}{N_2}}\mathbf{e}^{e_1\frac{2\pi n_1 m_1}{N_1}}. \tag{8}$$

## 3. Discrete Octonion Linear Canonical Transform

In this section, based on the DOFT, we extend the 2-D DQLCT to the 3-D discrete octonion linear canonical transform domain. A new transform, the discrete octonion linear canonical transform, is proposed. We can use DOCLCT to solve the problem of non-stationary 3-D octonion finite-length signals.

**Definition 4.** *The DOCLCT of an octonion-valued signal* $x : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$ *is defined by*

$$\mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D^{e_1}_{A_1}(n_1,m_1) D^{e_2}_{A_2}(n_2,m_2) D^{e_4}_{A_3}(n_3,m_3), \tag{9}$$

*where the kernel signals of the DOCLCT are*

$$D^{e_1}_{A_1}(n_1,m_1) = \frac{1}{\sqrt{N_1}} \boldsymbol{e}^{e_1\left(\frac{a_1}{2b_1}n_1^2\Delta s_1^2 - \frac{2\pi}{N_1}n_1m_1 + \frac{d_1}{2b_1}m_1^2\Delta y_1^2 - \frac{\pi}{2}\right)}, \tag{10}$$

$$D^{e_2}_{A_2}(n_2,m_2) = \frac{1}{\sqrt{N_2}} \boldsymbol{e}^{e_2\left(\frac{a_2}{2b_2}n_2^2\Delta s_2^2 - \frac{2\pi}{N_2}n_2m_2 + \frac{d_2}{2b_2}m_2^2\Delta y_2^2 - \frac{\pi}{2}\right)}, \tag{11}$$

*and*

$$D^{e_4}_{A_3}(n_3,m_3) = \frac{1}{\sqrt{N_3}} \boldsymbol{e}^{e_4\left(\frac{a_3}{2b_3}n_3^2\Delta s_3^2 - \frac{2\pi}{N_3}n_3m_3 + \frac{d_3}{2b_3}m_3^2\Delta y_3^2 - \frac{\pi}{2}\right)}. \tag{12}$$

When $A_k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $(k = 1,2,3)$, the DOCLCT reduces to the DOFT.

*Properties of the DOCLCT*

Next, we present several properties of the DOCLCT.

**Theorem 1** (Linearity). *For any octonion-valued signals* $x, y : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$, $\natural, \sharp \in \mathbb{R}$. *Then, the linearity of the DOCLCT is*

$$\mathcal{D}^o_{A_1,A_2,A_3}\{\natural x + \sharp y\}(\mathbf{m}) = \natural \mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}) + \sharp \mathcal{D}^o_{A_1,A_2,A_3}\{y\}(\mathbf{m}). \tag{13}$$

**Proof.** This proof step can be directly obtained by the definition of the DOCLCT. □

**Theorem 2** (Scaling). *For an octonion-valued signal* $x : [N_1] \times [N_2] \times [N_3] \to \mathbb{O}$, $\mathbf{t} = (t_1, t_2, t_3) \neq 0 \in [N_1] \times [N_2] \times [N_3]$. *Then, the scaling of the DOCLCT is*

$$\mathcal{D}^o_{A_1,A_2,A_3}\{x(\mathbf{tn})\}(\mathbf{m}) = \mathcal{D}^o_{\Omega_1,\Omega_2,\Omega_3}\{x\}\left(\tfrac{\mathbf{m}}{\mathbf{t}}\right), \tag{14}$$

*where* $\Omega_k = \begin{bmatrix} \frac{a_k}{t_k^2} & b_k \\ c_k & t_k^2 d_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ *and* $(k = 1,2,3)$.

**Proof.** According to the definition of the DOCLCT, we have

$$\mathcal{D}^o_{A_1,A_2,A_3}\{x(\mathbf{tn})\}(\mathbf{m}) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{tn}) D^{e_1}_{A_1}(n_1,m_1) D^{e_2}_{A_2}(n_2,m_2) D^{e_4}_{A_3}(n_3,m_3). \tag{15}$$

Let $\mathbf{u} = \mathbf{tn} = (u_1, u_2, u_3)$. Then, the above formula becomes

$$\begin{aligned}
&\mathcal{D}^o_{A_1,A_2,A_3}\{x(\mathbf{tn})\}(\mathbf{m}) \\
&= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{u}) \frac{1}{\sqrt{N_1}} \mathbf{e}^{e_1\left(\frac{a_1}{2b_1}\left(\frac{u_1}{t_1}\right)^2\Delta s_1^2 - \frac{2\pi}{N_1}\frac{u_1}{t_1}m_1 + \frac{d_1}{2b_1}m_1^2\Delta y_1^2 - \frac{\pi}{2}\right)} \\
&\times \frac{1}{\sqrt{N_2}} \mathbf{e}^{e_2\left(\frac{a_2}{2b_2}\left(\frac{u_2}{t_2}\right)^2\Delta s_2^2 - \frac{2\pi}{N_2}\frac{u_2}{t_2}m_2 + \frac{d_2}{2b_2}m_2^2\Delta y_2^2 - \frac{\pi}{2}\right)} \frac{1}{\sqrt{N_3}} \mathbf{e}^{e_4\left(\frac{a_3}{2b_3}\left(\frac{u_3}{t_3}\right)^2\Delta s_3^2 - \frac{2\pi}{N_3}\frac{u_3}{t_3}m_3 + \frac{d_3}{2b_3}m_3^2\Delta y_3^2 - \frac{\pi}{2}\right)} \\
&= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{u}) D^{e_1}_{\Omega_1}\left(u_1, \frac{m_1}{t_1}\right) D^{e_2}_{\Omega_2}\left(u_2, \frac{m_2}{t_2}\right) D^{e_4}_{\Omega_3}\left(u_3, \frac{m_3}{t_3}\right) \\
&= \mathcal{D}^o_{\Omega_1,\Omega_2,\Omega_3}\{x(\mathbf{u})\}\left(\tfrac{\mathbf{m}}{\mathbf{t}}\right).
\end{aligned}$$

□

**Theorem 3.** *[Boundedness] Assume that* $x_e(\boldsymbol{n}) = \frac{x(n_1,n_2,n_3)+x(n_1,n_2,-n_3)}{2}$ *is the even part and* $x_o(\boldsymbol{n}) = \frac{x(n_1,n_2,n_3)-x(n_1,n_2,-n_3)}{2}$ *is the odd part of a function* $x(\boldsymbol{n})$ *in the third variable* $n_3$*, respectively. Then,*

$$\begin{aligned}\left|\mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m})\right| = &\left[\frac{1}{N_3}\left(\left|\mathcal{D}^o_{A_1,A_2}\{\tilde{x}_e\}(\mathbf{m})\right|^2 + \left|\mathcal{D}^o_{A_1,A_2}\{\hat{x}_o\}(\mathbf{m})\right|^2\right.\right.\\ &\left.\left.+\left|\mathcal{D}^o_{A_1,A_2}\{\hat{x}_e\}(\mathbf{m})\right|^2 + \left|\mathcal{D}^o_{A_1,A_2}\{\tilde{x}_o\}(\mathbf{m})\right|^2\right)\right]^{\frac{1}{2}}.\end{aligned} \tag{16}$$

**Proof.** According to the Formula (1), the DOCLCT of the function $x(\mathbf{n})$ becomes

$$\begin{aligned}&\mathcal{D}^o_{A_1,A_2,A_3}\{x(\mathbf{n})\}(\mathbf{m})\\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}(\tilde{x}+\hat{x}e_4)(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)D^{e_4}_{A_3}(n_3,m_3)\\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\tilde{x}(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)D^{e_4}_{A_3}(n_3,m_3)\\ &+ \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\hat{x}(\mathbf{n})D^{-e_1}_{A_1}(n_1,m_1)D^{-e_2}_{A_2}(n_2,m_2)D^{e_4}_{A_3}(n_3,m_3)e_4.\end{aligned} \tag{17}$$

From the Euler formula, then

$$\begin{aligned}&\mathcal{D}^o_{A_1,A_2,A_3}\{x(\mathbf{n})\}(\mathbf{m})\\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\tilde{x}_e(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}\cos\alpha_3\\ &- \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\hat{x}_o(\mathbf{n})D^{-e_1}_{A_1}(n_1,m_1)D^{-e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}\sin\alpha_3\\ &+ \left(\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\tilde{x}_o(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}\sin\alpha_3\right.\\ &\left.+ \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\hat{x}_e(\mathbf{n})D^{-e_1}_{A_1}(n_1,m_1)D^{-e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}\cos\alpha_3\right)e_4,\end{aligned} \tag{18}$$

where $\alpha_3 = \frac{a_3}{2b_3}n_3^2\Delta s_3^2 - \frac{2\pi}{N_3}n_3m_3 + \frac{d_3}{2b_3}m_3^2\Delta y_3^2 - \frac{\pi}{2}$.

Hence, we have the result. □

**Theorem 4.** *(Plancherel theorem) The Plancherel theorem of the DOCLCT is*

$$\| x \|_2^2 = N_1N_2N_3\| \mathcal{D}^o_{A_1,A_2,A_3}\{x\} \|_2^2. \tag{19}$$

**Proof.** According to Theorem 3, then

$$\begin{aligned}\|\mathcal{D}^o_{A_1,A_2,A_3}\{x\}\|_2^2 &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\left|\mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m})\right|^2\\ &= \frac{1}{N_3}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\left(\left|\mathcal{D}^o_{A_1,A_2}\{\tilde{x}_e\}(\mathbf{m})\right|^2 + \left|\mathcal{D}^o_{A_1,A_2}\{\hat{x}_o\}(\mathbf{m})\right|^2\right.\\ &\left.+\left|\mathcal{D}^o_{A_1,A_2}\{\hat{x}_e\}(\mathbf{m})\right|^2 + \left|\mathcal{D}^o_{A_1,A_2}\{\tilde{x}_o\}(\mathbf{m})\right|^2\right)\\ &= \frac{1}{N_3}\left(\|\mathcal{D}^o_{A_1,A_2}\{\tilde{x}_e\} \|_2^2 + \| \mathcal{D}^o_{A_1,A_2}\{\hat{x}_o\} \|_2^2\right.\\ &\left.+ \|\mathcal{D}^o_{A_1,A_2}\{\hat{x}_e\} \|_2^2 + \| \mathcal{D}^o_{A_1,A_2}\{\tilde{x}_o\} \|_2^2\right).\end{aligned} \tag{20}$$

By the Plancherel theorem for the DQLCT [23], we have

$$\|\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}\|_2^2 = \frac{1}{N_1N_2N_3}\left(\|\widetilde{x}_e\|_2^2 + \|\hat{x}_o\|_2^2 + \|\hat{x}_e\|_2^2 + \|\widetilde{x}_o\|_2^2\right). \tag{21}$$

In addition, based on the formula

$$\| x \|_2^2 = \|\widetilde{x}_e\|_2^2 + \|\widetilde{x}_o\|_2^2 + \|\hat{x}_e\|_2^2 + \|\hat{x}_o\|_2^2, \tag{22}$$

then we have

$$\|\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}\|_2^2 = \frac{1}{N_1N_2N_3}\| x \|_2^2,$$

that is to say

$$N_1N_2N_3\|\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}\|_2^2 = \| x \|_2^2.$$

□

**Theorem 5** (Inversion transform)**.** *The inversion transform of the DOCLCT is obtained as follows:*

$$x(\mathbf{n}) = \sum_{u_1=0}^{N_1-1}\sum_{u_2=0}^{N_2-1}\sum_{u_3=0}^{N_3-1}\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}(\mathbf{m})D^{e_4}_{A_3^{-1}}(u_3,m_3)D^{e_2}_{A_2^{-1}}(u_2,m_2)D^{e_1}_{A_1^{-1}}(u_1,m_1), \tag{23}$$

*where* $A_k^{-1} = \begin{bmatrix} d_k & -b_k \\ -c_k & a_k \end{bmatrix} \in \mathbb{R}^{2\times 2}$ *and* $(k = 1,2,3)$*.*

**Proof.** According to the definition of the DOCLCT, we have

$$\begin{aligned}
&\sum_{u_1=0}^{N_1-1}\sum_{u_2=0}^{N_2-1}\sum_{u_3=0}^{N_3-1}\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}(\mathbf{m})D^{e_4}_{A_3^{-1}}(u_3,m_3)D^{e_2}_{A_2^{-1}}(u_2,m_2)D^{e_1}_{A_1^{-1}}(u_1,m_1)\\
&=\sum_{u_1=0}^{N_1-1}\sum_{u_2=0}^{N_2-1}\sum_{u_3=0}^{N_3-1}\left(\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}x(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)D^{e_4}_{A_3}(n_3,m_3)\right)\\
&\times D^{e_4}_{A_3^{-1}}(u_3,m_3)D^{e_2}_{A_2^{-1}}(u_2,m_2)D^{e_1}_{A_1^{-1}}(u_1,m_1).
\end{aligned} \tag{24}$$

In addition, by the kernel signals of the DOCLCT, the following formulas hold:

$$\begin{aligned}
&\sum_{u_3=0}^{N_3-1}\sum_{n_3=0}^{N_3-1}D^{e_4}_{A_3}(n_3,m_3)D^{e_4}_{A_3^{-1}}(u_3,m_3)\\
&=\frac{1}{N_3}\sum_{u_3=0}^{N_3-1}\sum_{n_3=0}^{N_2-1}\mathbf{e}^{e_4\left(\frac{a_3}{2b_3}(n_3^2-u_3^2)\Delta s_3^2-\frac{2\pi}{N_3}(n_3-u_3)m_3\right)}\\
&=\begin{cases}1, & n_3 = u_3\\ 0, & n_3 \neq u_3\end{cases}
\end{aligned} \tag{25}$$

By these three formulas (25), the inverse transform of the DOCLCT can be established. □

The following lemma shows that the DOCLCT can be disassembled by the Euler formula.

**Lemma 1.** *The DOCLCT can be expressed in another formula:*

$$\begin{aligned}
&\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\\
&=\Re_{eee}+\Re_{oee}e_1+\Re_{eoe}e_2+\Re_{ooe}e_3+\Re_{eeo}e_4+\Re_{oeo}e_5+\Re_{eoo}e_6+\Re_{ooo}e_7,
\end{aligned} \tag{26}$$

*where*

$$\Re_{eee}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}x_{eee}(\mathbf{n})\cos\alpha_1\cos\alpha_2\cos\alpha_3,$$

$$\Re_{oee}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{oee}(\mathbf{n}) \sin\alpha_1 \cos\alpha_2 \cos\alpha_3,$$

$$\Re_{eoe}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{eoe}(\mathbf{n}) \cos\alpha_1 \sin\alpha_2 \cos\alpha_3,$$

$$\Re_{ooe}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{ooe}(\mathbf{n}) \sin\alpha_1 \sin\alpha_2 \cos\alpha_3,$$

$$\Re_{eeo}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{eeo}(\mathbf{n}) \cos\alpha_1 \cos\alpha_2 \sin\alpha_3,$$

$$\Re_{oeo}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{oeo}(\mathbf{n}) \sin\alpha_1 \cos\alpha_2 \sin\alpha_3,$$

$$\Re_{eoo}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{eoo}(\mathbf{n}) \cos\alpha_1 \sin\alpha_2 \sin\alpha_3,$$

$$\Re_{ooo}(\mathbf{m}) = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \sum_{n_3=0}^{N_3-1} x_{ooo}(\mathbf{n}) \sin\alpha_1 \sin\alpha_2 \sin\alpha_3,$$

*and* $\alpha_k = \frac{a_k}{2b_k} n_k^2 \Delta s_k^2 - \frac{2\pi}{N_k} n_k m_k + \frac{d_k}{2b_k} m_k^2 \Delta y_k^2 - \frac{\pi}{2}$, $(k = 1, 2, 3)$*. The subscripts e and o indicate a function is either even (e) or odd (o) for an appropriate variable, i.e.,* $x_{ooe}(\boldsymbol{n})$ *is odd for* $n_1$ *and* $n_2$*, and even for* $n_3$*.*

**Proof.** The kernel function of the DOCLCT can be expanded as follows:

$$\begin{aligned}
&\left(D_{A_1}^{e_1}(n_1, m_1) D_{A_2}^{e_2}(n_2, m_2)\right) D_{A_3}^{e_4}(n_3, m_3) \\
&= \frac{1}{\sqrt{N_1N_2N_3}} (\mathbf{e}^{e_1\alpha_1}\mathbf{e}^{e_2\alpha_2})\mathbf{e}^{e_4\alpha_3} \\
&= \frac{1}{\sqrt{N_1N_2N_3}} ((\cos\alpha_1 + e_1 \sin\alpha_1)(\cos\alpha_2 + e_2 \sin\alpha_2))(\cos\alpha_3 + e_4 \sin\alpha_3) \\
&= \frac{1}{\sqrt{N_1N_2N_3}} (\cos\alpha_1 \cos\alpha_2 \cos\alpha_3 + \sin\alpha_1 \cos\alpha_2 \cos\alpha_3 e_1 \\
&+ \cos\alpha_1 \sin\alpha_2 \cos\alpha_3 e_2 + \sin\alpha_1 \sin\alpha_2 \cos\alpha_3 e_3 + \cos\alpha_1 \cos\alpha_2 \sin\alpha_3 e_4 \\
&+ \sin\alpha_1 \cos\alpha_2 \sin\alpha_3 e_5 + \cos\alpha_1 \sin\alpha_2 \sin\alpha_3 e_6 + \sin\alpha_1 \sin\alpha_2 \sin\alpha_3 e_7).
\end{aligned} \tag{27}$$

By the definition of the DOCLCT, we have the result. □

This lemma eliminates the obstacle caused by the non-commutative and non-associative properties of the DOCLCT.

Next, we give the shift transform of the DOCLCT. There are three forms of the shift function based on three variables, $x^{T_1}(n_1, n_2, n_3) = x(n_1 - l_1, n_2, n_3)$, $x^{T_2}(n_1, n_2, n_3) = x(n_1, n_2 - l_2, n_3)$, and $x^{T_3}(n_1, n_2, n_3) = x(n_1, n_2, n_3 - l_3)$. These three shift functions are independent of each other and are not affected by other remaining variables.

**Theorem 6** (Shift transform of the DOCLCT)**.** *Let* $\wp^{T_1}$, $\wp^{T_2}$ *and* $\wp^{T_3}$ *denote the DOCLCT of the three shift functions* $x(n_1 - l_1, n_2, n_3)$, $x(n_1, n_2 - l_2, n_3)$ *and* $x(n_1, n_2, n_3 - l_3)$*, respectively. Then,*

$$\wp^{T_1}(\mathbf{m}) = \cos\gamma_1 \mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}') - \sin\gamma_1 \Phi_1(\mathbf{m}'), \tag{28}$$

$$\wp^{T_1}(\mathbf{m}) = \cos\gamma_1 \mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}'') - \sin\gamma_1 \Phi_2(\mathbf{m}''), \tag{29}$$

$$\wp^{T_1}(\mathbf{m}) = \cos\gamma_1 \mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}''') - \sin\gamma_1 \Phi_3(\mathbf{m}'''), \tag{30}$$

*where* $\boldsymbol{m}' = (m_1', m_2, m_3)$*;* $\boldsymbol{m}'' = (m_1, m_2', m_3)$*;* $\boldsymbol{m}''' = (m_1, m_2, m_3')$*;* $m_k' = m_k - \frac{N_k a_k}{2\pi b_k} \Delta s_k^2 l_k$*;* $\gamma_k = \frac{a_k}{2b_k} l_k^2 \Delta s_k^2 - \frac{2\pi}{N_k} l_k m_k - \frac{N_k a_k d_k}{4\pi b_k^2} \Delta y_k^2 \Delta s_k^2 l_k \left(\frac{N_k a_k}{2\pi b_k} \Delta s_k^2 l_k + 2m_k\right)$, $(k = 1, 2, 3)$*; and* $\Phi_1 = \Re_{see} - \Re_{cee} e_1 + \Re_{soe} e_2 - \Re_{coe} e_3 + \Re_{seo} e_4 - \Re_{ceo} e_5 + \Re_{soo} e_6 - \Re_{coo} e_7$*,* $\Phi_2 = \Re_{ese} + \Re_{ose} e_1 - \Re_{ece} e_2 -$

$\Re_{oce}e_3 + \Re_{eso}e_4 + \Re_{oso}e_5 - \Re_{eco}e_6 - \Re_{oco}e_7$, $\Phi_3 = \Re_{ees} + \Re_{oes}e_1 + \Re_{eos}e_2 + \Re_{oos}e_3 - \Re_{eec}e_4 - \Re_{oec}e_5 - \Re_{eoc}e_6 - \Re_{ooc}e_7$.

**Proof.** We present the proof process for the DOCLCT of the function $x^{T_1}$, and the other two types are obtained according to the same steps.

According to Lemma 1, we have

$$\begin{aligned}&\wp^{T_1} = \mathcal{D}^o_{A_1,A_2,A_3}\{x^{T_1}\}(\mathbf{m})\\&= \Re^{T_1}_{eee} + \Re^{T_1}_{oee}e_1 + \Re^{T_1}_{eoe}e_2 + \Re^{T_1}_{ooe}e_3 + \Re^{T_1}_{eeo}e_4 + \Re^{T_1}_{oeo}e_5 + \Re^{T_1}_{eoo}e_6 + \Re^{T_1}_{ooo}e_7.\end{aligned} \tag{31}$$

Then, we can compute the formula:

$$\begin{aligned}\Re^{T_1}_{eee}(\mathbf{m}) &= \frac{1}{\sqrt{N_1N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x^{T_1}_{eee}(\mathbf{n})\cos\alpha_1\cos\alpha_2\cos\alpha_3\\&= \frac{1}{\sqrt{N_1N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x_{eee}(n_1-l_1,n_2,n_3)\cos\alpha_1\cos\alpha_2\cos\alpha_3.\end{aligned}$$

Let $h_1 = n_1 - l_1, h_2 = n_2, h_3 = n_3$ ($\mathbf{h} = (h_1,h_2,h_3) \in \mathbb{R}^3$), hence

$$\begin{aligned}\Re^{T_1}_{eee}(\mathbf{m}) &= \frac{1}{\sqrt{N_1N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x_{eee}(n_1-l_1,n_2,n_3)\cos\alpha_1\cos\alpha_2\cos\alpha_3\\&= \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eee}(\mathbf{h})\cos(\beta_1+\gamma_1)\cos\beta_2\cos\beta_3\\&= \frac{\cos(\gamma_1)}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eee}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3\\&\quad-\frac{\sin(\gamma_1)}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eee}(\mathbf{h})\sin\beta_1\cos\beta_2\cos\beta_3,\end{aligned} \tag{32}$$

where $\beta_1 = \frac{a_1}{2b_1}h_1^2\Delta s_1^2 - \frac{2\pi}{N_1}h_1\left(m_1 - \frac{N_1a_1}{2\pi b_1}\Delta s_1^2 l_1\right) + \frac{d_1}{2b_1}\left(m_1 - \frac{N_1a_1}{2\pi b_1}\Delta s_1^2 l_1\right)^2\Delta y_1^2 - \frac{\pi}{2}$, $\gamma_1 = \frac{a_1}{2b_1}l_1^2\Delta s_1^2 - \frac{2\pi}{N_1}l_1m_1 - \frac{N_1a_1d_1}{4\pi b_1^2}\Delta y_1^2\Delta s_1^2 l_1\left(\frac{N_1a_1}{2\pi b_1}\Delta s_1^2 l_1 + 2m_1\right)$, and $\beta_k = \frac{a_k}{2b_k}n_k^2\Delta s_k^2 - \frac{2\pi}{N_k}n_km_k + \frac{d_k}{2b_k}m_k^2\Delta y_k^2 - \frac{\pi}{2}$, $(k = 2,3)$.

Let $\mathbf{m}' = (m_1', m_2, m_3)$, $m_1' = m_1 - \frac{N_1a_1}{2\pi b_1}\Delta s_1^2 l_1$ and

$$\Re_{see}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eee}(\mathbf{h})\sin\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{cee}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{oee}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{soe}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eoe}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{coe}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{ooe}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{seo}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eeo}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{ceo}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{oeo}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{soo}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}}\sum_{h_1=0}^{N_1-1}\sum_{h_2=0}^{N_2-1}\sum_{h_3=0}^{N_3-1} x_{eoo}(\mathbf{h})\cos\beta_1\cos\beta_2\cos\beta_3,$$

$$\Re_{coo}(\mathbf{m}') = \frac{1}{\sqrt{N_1N_2N_3}} \sum_{h_1=0}^{N_1-1} \sum_{h_2=0}^{N_2-1} \sum_{h_3=0}^{N_3-1} x_{ooo}(\mathbf{h}) \cos\beta_1 \cos\beta_2 \cos\beta_3.$$

Then, Formula (32) becomes

$$\Re_{eee}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{eee}(\mathbf{m}') - \sin\gamma_1 \Re_{see}(\mathbf{m}'),$$

In addition, we can prove other formulas, such as

$$\begin{aligned} \Re_{oee}^{T_1}(\mathbf{m}) &= \frac{1}{\sqrt{N_1N_2N_3}} \sum_{h_1=0}^{N_1-1} \sum_{h_2=0}^{N_2-1} \sum_{h_3=0}^{N_3-1} x_{oee}(\mathbf{h}) \sin(\beta_1+\gamma_1) \cos\beta_2 \cos\beta_3 \\ &= \frac{\cos(\gamma_1)}{\sqrt{N_1N_2N_3}} \sum_{h_1=0}^{N_1-1} \sum_{h_2=0}^{N_2-1} \sum_{h_3=0}^{N_3-1} x_{oee}(\mathbf{h}) \sin\beta_1 \cos\beta_2 \cos\beta_3 \\ &+ \frac{\sin(\gamma_1)}{\sqrt{N_1N_2N_3}} \sum_{h_1=0}^{N_1-1} \sum_{h_2=0}^{N_2-1} \sum_{h_3=0}^{N_3-1} x_{oee}(\mathbf{h}) \cos\beta_1 \cos\beta_2 \cos\beta_3, \end{aligned}$$

By continuing in this way, we have

$$\Re_{eee}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{eee}(\mathbf{m}') - \sin\gamma_1 \Re_{see}(\mathbf{m}'),$$

$$\Re_{oee}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{oee}(\mathbf{m}') + \sin\gamma_1 \Re_{cee}(\mathbf{m}'),$$

$$\Re_{eoe}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{eoe}(\mathbf{m}') - \sin\gamma_1 \Re_{soe}(\mathbf{m}'),$$

$$\Re_{ooe}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{ooe}(\mathbf{m}') + \sin\gamma_1 \Re_{coe}(\mathbf{m}'),$$

$$\Re_{eeo}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{eeo}(\mathbf{m}') - \sin\gamma_1 \Re_{seo}(\mathbf{m}'),$$

$$\Re_{oeo}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{oeo}(\mathbf{m}') + \sin\gamma_1 \Re_{ceo}(\mathbf{m}'),$$

$$\Re_{eoo}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{eoo}(\mathbf{m}') - \sin\gamma_1 \Re_{soo}(\mathbf{m}'),$$

$$\Re_{oo}^{T_1}(\mathbf{m}) = \cos\gamma_1 \Re_{ooo}(\mathbf{m}') + \sin\gamma_1 \Re_{coo}(\mathbf{m}').$$

According to Lemma 1, the first conclusion can be inferred:

$$\wp^{T_1}(\mathbf{m}) = \cos\gamma_1 \mathcal{D}^o_{A_1,A_2,A_3}\{x\}(\mathbf{m}') - \sin\gamma_1 \Phi_1(\mathbf{m}'). \tag{33}$$

where $\Phi_1 = \Re_{see} - \Re_{cee}e_1 + \Re_{soe}e_2 - \Re_{coe}e_3 + \Re_{seo}e_4 - \Re_{ceo}e_5 + \Re_{soo}e_6 - \Re_{coo}e_7$.

Using the same steps, the other two formulas, (29) and (30), can be obtained. □

## 4. Convolution Theorem of the DOCLCT

In this section, we first give the definition of the convolution operator associated with the DOCLCT. Then, we obtain the convolution theorem of the DOCLCT by the convolution operator.

**Definition 5.** *Suppose the real functions $x(\mathbf{n})$ and $f(\mathbf{n})$ are given; the convolution operator $\circledast_\wedge$ is defined by*

$$\begin{aligned} (x \circledast_\wedge f)(\mathbf{n}) &= \sum_{u_1=0}^{U_1-1} \sum_{u_2=0}^{U_2-1} \sum_{u_3=0}^{U_3-1} \frac{1}{\sqrt{U_1}} \frac{1}{\sqrt{U_2}} \frac{1}{\sqrt{U_3}} x(\mathbf{u}) f(\mathbf{n}-\mathbf{u}) \\ &\times e^{-e_1 \frac{a_1}{b_1} u_1(n_1-u_1)\Delta s_1^2} e^{-e_1 \frac{a_2}{b_2} u_2(n_2-u_2)\Delta s_2^2} e^{-e_1 \frac{a_3}{b_3} u_3(n_3-u_3)\Delta s_3^2}, \end{aligned} \tag{34}$$

*where $U_k$ denotes the intervals $[0,1,2,\cdots,U_k-1]$, $(k=1,2,3)$.*

**Lemma 2.** *The convolution theorem for the 3-D DLCT is obtained:*

$$\begin{aligned}\mathcal{D}_{A_1,A_2,A_3}\{x \circledast_{\wedge} f\}(\mathbf{m}) &= \mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\mathcal{D}_{A_1,A_2,A_3}\{f\}(\mathbf{m}) \\ &\times \boldsymbol{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})}.\end{aligned} \tag{35}$$

**Proof.** According to the definition of the 3-D DLCT and the convolution operator, then

$$\begin{aligned}&\mathcal{D}_{A_1,A_2,A_3}\{x \circledast_{\wedge} f\}(\mathbf{m}) \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1}\Bigg(\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}\sum_{u_3=0}^{U_3-1}\frac{1}{\sqrt{U_1}}\frac{1}{\sqrt{U_2}}\frac{1}{\sqrt{U_3}}x(\mathbf{u})f(\mathbf{n}-\mathbf{u}) \\ &\times \mathrm{e}^{-e_1\frac{a_1}{b_1}u_1(n_1-u_1)\Delta s_1^2}\mathrm{e}^{-e_1\frac{a_2}{b_2}u_2(n_2-u_2)\Delta s_2^2}\mathrm{e}^{-e_1\frac{a_3}{b_3}u_3(n_3-u_3)\Delta s_3^2}\Bigg) \\ &\times D_{A_1}^{e_1}(n_1,m_1)D_{A_2}^{e_1}(n_2,m_2)D_{A_3}^{e_1}(n_3,m_3).\end{aligned} \tag{36}$$

Let $\mathbf{r}=\mathbf{n}-\mathbf{u}=(r_1,r_2,r_3)$. Then, the above formula becomes

$$\begin{aligned}&\mathcal{D}_{A_1,A_2,A_3}\{x \circledast_{\wedge} f\}(\mathbf{m}) \\ &= \sum_{r_1=-u_1}^{N_1-U_1-1}\sum_{r_2=-u_2}^{N_2-U_2-1}\sum_{r_3=-u_3}^{N_3-U_3-1}\Bigg(\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}\sum_{u_3=0}^{U_3-1}\frac{1}{\sqrt{U_1}}\frac{1}{\sqrt{U_2}}\frac{1}{\sqrt{U_3}}x(\mathbf{u})f(\mathbf{r}) \\ &\times \mathrm{e}^{-e_1\frac{a_1}{b_1}u_1r_1\Delta s_1^2}\mathrm{e}^{-e_1\frac{a_2}{b_2}u_2r_2\Delta s_2^2}\mathrm{e}^{-e_1\frac{a_3}{b_3}u_3r_3\Delta s_3^2}\Bigg) \\ &\times \frac{1}{\sqrt{N_1}}\mathbf{e}^{e_1(\frac{a_1}{2b_1}(r_1+u_1)^2\Delta s_1^2-\frac{2\pi}{N_1}(r_1+u_1)m_1+\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{\pi}{2})} \\ &\times \frac{1}{\sqrt{N_2}}\mathbf{e}^{e_1(\frac{a_2}{2b_2}(r_2+u_2)^2\Delta s_2^2-\frac{2\pi}{N_2}(r_2+u_2)m_2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{\pi}{2})} \\ &\times \frac{1}{\sqrt{N_3}}\mathbf{e}^{e_1(\frac{a_3}{2b_3}(r_3+u_3)^2\Delta s_3^2-\frac{2\pi}{N_3}(r_3+u_3)m_3+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{\pi}{2})}.\end{aligned} \tag{37}$$

Since $\mathrm{e}^{-e_1\frac{a_1}{b_1}u_1r_1\Delta s_1^2}\mathrm{e}^{e_1\frac{a_1}{b_1}u_1r_1\Delta s_1^2}=1$, $\mathrm{e}^{-e_1\frac{a_2}{b_2}u_2r_2\Delta s_2^2}\mathrm{e}^{e_1\frac{a_2}{b_2}u_2r_2\Delta s_2^2}=1$ , $\mathrm{e}^{-e_1\frac{a_3}{b_3}u_3r_3\Delta s_3^2}\mathrm{e}^{e_1\frac{a_3}{b_3}u_3r_3\Delta s_3^2}=1$. Then, we have

$$\begin{aligned}&\mathcal{D}_{A_1,A_2,A_3}\{x \circledast_{\wedge} f\}(\mathbf{m}) \\ &= \Bigg(\sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}\sum_{u_3=0}^{U_3-1}\frac{1}{\sqrt{U_1U_2U_3}}x(\mathbf{u})\mathbf{e}^{e_1(\frac{a_1}{2b_1}u_1^2\Delta s_1^2-\frac{2\pi}{N_1}u_1m_1+\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{\pi}{2})} \\ &\times \mathbf{e}^{e_1(\frac{a_2}{2b_2}u_2^2\Delta s_2^2-\frac{2\pi}{N_2}u_2m_2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{\pi}{2})}\mathbf{e}^{e_1(\frac{a_3}{2b_3}u_3^2\Delta s_3^2-\frac{2\pi}{N_3}u_3m_3+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{\pi}{2})} \\ &\times \sum_{r_1=-u_1}^{N_1-U_1-1}\sum_{r_2=-u_2}^{N_2-U_2-1}\sum_{r_3=-u_3}^{N_3-U_3-1}\frac{f(\mathbf{r})\mathbf{e}^{e_1(\frac{a_1}{2b_1}r_1^2\Delta s_1^2-\frac{2\pi}{N_1}r_1m_1+\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{\pi}{2})}}{\sqrt{N_1N_2N_3}} \\ &\times \mathbf{e}^{e_1(\frac{a_2}{2b_2}r_2^2\Delta s_2^2-\frac{2\pi}{N_2}r_2m_2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{\pi}{2})}\mathbf{e}^{e_1(\frac{a_3}{2b_3}r_3^2\Delta s_3^2-\frac{2\pi}{N_3}r_3m_3+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{\pi}{2})} \\ &\times \mathbf{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})}.\end{aligned} \tag{38}$$

Based on the definition of the 3-D DLCT, we can obtain the proof. □

Next, we obtain the relation between the DOCLCT and the 3-D DLCT. Then, we derive a convolution theorem for the DOCLCT.

**Lemma 3.** *The relation between the DOCLCT and the 3-D DLCT is given as follows:*

$$\begin{aligned}&\mathcal{D}^{o}_{A_1,A_2,A_3}\{x\}(\mathbf{m}) \\ &= \tfrac{1}{4}\big\{\big(\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})-\mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big)(1-e_3) \\ &+\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho)-\mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\big)(1+e_3)\big\} \\ &+\tfrac{1}{4}\big\{\big(\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})+\mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big)(1+e_3) \\ &-\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho)+\mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\big)(1-e_3)\big\}\cdot e_5,\end{aligned} \tag{39}$$

*where* $B_k = \begin{bmatrix} a_k & -b_k \\ -c_k & d_k \end{bmatrix}$, $(k = 1,2,3)$, $\tau = (m_1, -m_2, m_3)$, $\varsigma = (m_1, m_2, -m_3)$ *and* $\varrho = (m_1, -m_2, -m_3)$.

**Proof.** According to Definition 1, then

$$\begin{aligned}\mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau) &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)\frac{1}{\sqrt{N_2}}\mathrm{e}^{e_1\frac{a_2}{-2b_2}n_2^2\Delta s_2^2+\frac{2\pi}{N_2}n_2m_2+\frac{d_2}{-2b_2}m_2^2\Delta y_2^2-\frac{\pi}{2}} \\ &\times D_{A_3}^{e_1}(n_3,m_3) \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)\frac{1}{\sqrt{N_2}}\mathrm{e}^{-e_1(\frac{a_2}{2b_2}n_2^2\Delta s_2^2-\frac{2\pi}{N_2}n_2m_2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{\pi}{2})} \\ &\times D_{A_3}^{e_1}(n_3,m_3) \\ &= -\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)\frac{1}{\sqrt{N_2}}\mathrm{e}^{-e_1\alpha_2}D_{A_3}^{e_1}(n_3,m_3).\end{aligned} \tag{40}$$

The third equation is based on this fact: $\mathrm{e}^{-e_1\frac{\pi}{2}} = -\mathrm{e}^{e_1\frac{\pi}{2}}$.
According to the same method, we can obtain the following two equations:

$$\mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma) = -\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1) D_{A_2}^{e_1}(n_2,m_2)\frac{1}{\sqrt{N_3}}\mathrm{e}^{-e_1\alpha_3}, \tag{41}$$

$$\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)\frac{1}{\sqrt{N_2}}\mathrm{e}^{-e_1\alpha_2}\frac{1}{\sqrt{N_3}}\mathrm{e}^{-e_1\alpha_3}, \tag{42}$$

where $\alpha_k = \frac{a_k}{2b_k}n_k^2\Delta s_k^2 - \frac{2\pi}{N_k}n_k m_k + \frac{d_k}{2b_k}m_k^2\Delta y_k^2 - \frac{\pi}{2}, (k = 2,3)$. By the sine and cosine functions, we write the second kernel function $D_{A_2}^{e_1}(n_2, m_2)$ of the 3-D DLCT in the cosine and sine forms, respectively.

$$\begin{aligned}&\frac{1}{2}\big(\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m}) - \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\big) \\ &= \frac{1}{\sqrt{N_2}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)\cos\alpha_2 D_{A_3}^{e_1}(n_3,m_3),\end{aligned} \tag{43}$$

$$\begin{aligned}&\frac{1}{2}\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho) + \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big) \\ &= \frac{1}{\sqrt{N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)(-e_1\sin\alpha_2)\mathrm{e}^{-e_1\alpha_3}.\end{aligned} \tag{44}$$

In Formula (44), if $e_1\sin\alpha_2$ becomes $e_2\sin\alpha_2$, this multiplies Formula (44) from the right by $e_3$ and according to Figure 1. Hence,

$$\begin{aligned}&\frac{1}{2}\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho) + \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big)e_3 \\ &= \frac{1}{\sqrt{N_2N_3}}\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1)e_2\sin\alpha_2\mathrm{e}^{e_1\alpha_3}.\end{aligned} \tag{45}$$

Adding Formulas (43) and (45), then

$$\begin{aligned}&\tfrac{1}{2}\big(\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m}) - \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\big) \\ &+\tfrac{1}{2}\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho) + \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big)e_3 \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n}) D_{A_1}^{e_1}(n_1,m_1) D_{A_2}^{e_2}(n_2,m_2) D_{A_3}^{e_1}(n_3,m_3).\end{aligned} \tag{46}$$

For the convenience of calculation, assume that

$$\begin{aligned}\Theta_{A_1,A_2,A_3}\{x\}(\mathbf{m}) &= \tfrac{1}{2}\big(\mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m}) - \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\big) \\ &+\tfrac{1}{2}\big(\mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho) + \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\big)e_3 \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)D^{e_1}_{A_3}(n_3,m_3).\end{aligned} \tag{47}$$

By the same steps, we can write $D^{e_1}_{A_3}(n_3,m_3)$ in the form

$$\begin{aligned}&\frac{1}{2}\big(\Theta_{A_1,A_2,A_3}\{x\}(\mathbf{m}) - \Theta_{A_1,A_2,B_3}\{x\}(\varsigma)\big) \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}\cos\alpha_3,\end{aligned} \tag{48}$$

$$\begin{aligned}&-\frac{1}{2}\big(\Theta_{A_1,A_2,A_3}\{x\}(\mathbf{m}) + \Theta_{A_1,A_2,B_3}\{x\}(\varsigma)\big)e_5 \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)\frac{1}{\sqrt{N_3}}e_4\sin\alpha_3,\end{aligned} \tag{49}$$

Adding the above two formulas, the following formula is obtained:

$$\begin{aligned}&\tfrac{1}{2}\big(\Theta_{A_1,A_2,A_3}\{x\}(\mathbf{m}) - \Theta_{A_1,A_2,B_3}\{x\}(\varsigma)\big) \\ &-\tfrac{1}{2}\big(\Theta_{A_1,A_2,A_3}\{x\}(\mathbf{m}) + \Theta_{A_1,A_2,B_3}\{x\}(\varsigma)\big)e_5 \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}\sum_{n_3=0}^{N_3-1} x(\mathbf{n})D^{e_1}_{A_1}(n_1,m_1)D^{e_2}_{A_2}(n_2,m_2)D^{e_3}_{A_3}(n_3,m_3).\end{aligned} \tag{50}$$

Finally, the result can be proven. □

**Theorem 7.** *The convolution theorem for the DOCLCT is obtained as follows:*

$$\begin{aligned}&\mathcal{D}^{o}_{A_1,A_2,A_3}\{x \circledast_{\wedge} f\}(\mathbf{m}) \\ &= \tfrac{1}{4}\big\{\big(\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\Pi_{A_1,A_2,A_3}\{f\}(\mathbf{m}) - \Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\Pi_{A_1,A_2,B_3}\{f\}(\varsigma)\big)(1-e_3) \\ &+\big(\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\Pi_{A_1,B_2,B_3}\{f\}(\varrho) - \Xi_{A_1,B_2,A_3}\{x\}(\tau)\Pi_{A_1,B_2,A_3}\{f\}(\tau)\big)(1+e_3)\big\} \\ &+\tfrac{1}{4}\big\{\big(\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\Pi_{A_1,A_2,A_3}\{f\}(\mathbf{m}) + \Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\Pi_{A_1,A_2,B_3}\{f\}(\varsigma)\big)(1+e_3) \\ &-\big(\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\Pi_{A_1,B_2,B_3}\{f\}(\varrho) + \Xi_{A_1,B_2,A_3}\{x\}(\tau)\Pi_{A_1,B_2,A_3}\{f\}(\tau)\big)(1-e_3)\big\}\cdot e_5,\end{aligned} \tag{51}$$

*where*

$$\begin{aligned}\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\Pi_{A_1,A_2,A_3}\{f\}(\mathbf{m}) &= \mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\mathcal{D}_{A_1,A_2,A_3}\{f\}(\mathbf{m}) \\ &\times \boldsymbol{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},\end{aligned} \tag{52}$$

$$\begin{aligned}\Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\Pi_{A_1,A_2,B_3}\{f\}(\varsigma) &= \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\mathcal{D}_{A_1,A_2,B_3}\{f\}(\varsigma) \\ &\times \boldsymbol{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},\end{aligned} \tag{53}$$

$$\begin{aligned}\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\Pi_{A_1,B_2,B_3}\{f\}(\varrho) &= \mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho)\mathcal{D}_{A_1,B_2,B_3}\{f\}(\varrho) \\ &\times \boldsymbol{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},\end{aligned} \tag{54}$$

$$\begin{aligned}\Xi_{A_1,B_2,A_3}\{x\}(\tau)\Pi_{A_1,B_2,A_3}\{f\}(\tau) &= \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\mathcal{D}_{A_1,B_2,A_3}\{f\}(\tau) \\ &\times \boldsymbol{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})}.\end{aligned} \tag{55}$$

**Proof.** Using Lemma 3, then we have

$$\begin{aligned}
&\mathcal{D}^{o}_{A_1,A_2,A_3}\{x \divideontimes_{\wedge} f\}(\mathbf{m}) \\
&= \tfrac{1}{4}\{\big(\mathcal{D}_{A_1,A_2,A_3}\{x \divideontimes_{\wedge} f\}(\mathbf{m}) - \mathcal{D}_{A_1,A_2,B_3}\{x \divideontimes_{\wedge} f\}(\varsigma)\big)(1-e_3) \\
&+\big(\mathcal{D}_{A_1,B_2,B_3}\{x \divideontimes_{\wedge} f\}(\varrho) - \mathcal{D}_{A_1,B_2,A_3}\{x \divideontimes_{\wedge} f\}(\tau)\big)(1+e_3)\} \\
&+\tfrac{1}{4}\{\big(\mathcal{D}_{A_1,A_2,A_3}\{x \divideontimes_{\wedge} f\}(\mathbf{m}) + \mathcal{D}_{A_1,A_2,B_3}\{x \divideontimes_{\wedge} f\}(\varsigma)\big)(1+e_3) \\
&-\big(\mathcal{D}_{A_1,B_2,B_3}\{x \divideontimes_{\wedge} f\}(\varrho) + \mathcal{D}_{A_1,B_2,A_3}\{x \divideontimes_{\wedge} f\}(\tau)\big)(1-e_3)\}\cdot e_5.
\end{aligned} \tag{56}$$

According to Lemma 2, we obtain

$$\begin{aligned}
\mathcal{D}_{A_1,A_2,A_3}\{x \divideontimes_{\wedge} f\}(\mathbf{m}) &= \mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\mathcal{D}_{A_1,A_2,A_3}\{f\}(\mathbf{m}) \\
&\quad\times \mathbf{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},
\end{aligned} \tag{57}$$

$$\begin{aligned}
\mathcal{D}_{A_1,A_2,B_3}\{x \divideontimes_{\wedge} f\}(\varsigma) &= \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\mathcal{D}_{A_1,A_2,B_3}\{f\}(\varsigma) \\
&\quad\times \mathbf{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},
\end{aligned} \tag{58}$$

$$\begin{aligned}
\mathcal{D}_{A_1,B_2,B_3}\{x \divideontimes_{\wedge} f\}(\varrho) &= \mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho)\mathcal{D}_{A_1,B_2,B_3}\{f\}(\varrho) \\
&\quad\times \mathbf{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})},
\end{aligned} \tag{59}$$

$$\begin{aligned}
\mathcal{D}_{A_1,B_2,A_3}\{x \divideontimes_{\wedge} f\}(\tau) &= \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\mathcal{D}_{A_1,B_2,A_3}\{f\}(\tau) \\
&\quad\times \mathbf{e}^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2-\frac{3\pi}{2})}.
\end{aligned} \tag{60}$$

Hence, the result can be obtained. □

## 5. Correlation Theorem of the DOCLCT

In this section, the correlation operator is presented; then, we obtain the correlation theorem of the DOCLCT.

**Definition 6.** *Given the real functions $x(\boldsymbol{n})$ and $f(\boldsymbol{n})$, the correlation operator $\ltimes$ is defined by*

$$\begin{aligned}
(x \ltimes f)(\mathbf{n}) &= \sum_{u_1=0}^{U_1-1}\sum_{u_2=0}^{U_2-1}\sum_{u_3=0}^{U_3-1}\frac{1}{\sqrt{U_1}}\frac{1}{\sqrt{U_2}}\frac{1}{\sqrt{U_3}}x(\mathbf{u})\overline{f(\mathbf{u}-\mathbf{n})} \\
&\quad\times e^{-e_1\frac{a_1}{b_1}u_1(n_1-u_1)\Delta s_1^2}\mathbf{e}^{-e_1\frac{a_2}{b_2}u_2(n_2-u_2)\Delta s_2^2}\mathbf{e}^{-e_1\frac{a_3}{b_3}u_3(n_3-u_3)\Delta s_3^2} \\
&= x(\mathbf{n}) \divideontimes_{\wedge} \overline{f(-\mathbf{n})}.
\end{aligned} \tag{61}$$

**Lemma 4.** *The correlation theorem for the 3-D DLCT is obtained as follows:*

$$\begin{aligned}
\mathcal{D}_{A_1,A_2,A_3}\{x \ltimes f\}(\mathbf{m}) &= \mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\overline{\mathcal{D}_{B_1,B_2,B_3}\{f\}(\mathbf{m})} \\
&\quad\times e^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2)}\cdot e_1.
\end{aligned} \tag{62}$$

**Proof.** The proof process is similar to Lemma 2, so it is omitted. □

According to Lemma 4, we obtain the following theorem:

**Theorem 8.** *The correlation theorem for the DOCLCT is obtained as follows:*

$$
\begin{aligned}
&\mathcal{D}^{o}_{A_1,A_2,A_3}\{x \ltimes f\}(\mathbf{m}) \\
&= \tfrac{1}{4}\Big\{\Big(\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\overline{\Pi_{B_1,B_2,B_3}\{f\}(\mathbf{m})} - \Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\overline{\Pi_{B_1,B_2,A_3}\{f\}(\varsigma)}\Big)(1-e_3) \\
&+\Big(\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\overline{\Pi_{B_1,A_2,A_3}\{f\}(\varrho)} - \Xi_{A_1,B_2,A_3}\{x\}(\tau)\overline{\Pi_{B_1,A_2,B_3}\{f\}(\tau)}\Big)(1+e_3)\Big\} \\
&+\tfrac{1}{4}\Big\{\Big(\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\overline{\Pi_{B_1,B_2,B_3}\{f\}(\mathbf{m})} + \Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\overline{\Pi_{B_1,B_2,A_3}\{f\}(\varsigma)}\Big)(1+e_3) \\
&-\Big(\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\overline{\Pi_{B_1,A_2,A_3}\{f\}(\varrho)} + \Xi_{A_1,B_2,A_3}\{x\}(\tau)\overline{\Pi_{B_1,A_2,B_3}\{f\}(\tau)}\Big)(1-e_3)\Big\}\cdot e_5,
\end{aligned} \tag{63}
$$

*where*

$$
\begin{aligned}
\Xi_{A_1,A_2,A_3}\{x\}(\mathbf{m})\overline{\Pi_{B_1,B_2,B_3}\{f\}(\mathbf{m})} &= \mathcal{D}_{A_1,A_2,A_3}\{x\}(\mathbf{m})\overline{\mathcal{D}_{B_1,B_2,B_3}\{f\}(\mathbf{m})} \\
&\quad\times e^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2)}\cdot e_1,
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
\Xi_{A_1,A_2,B_3}\{x\}(\varsigma)\overline{\Pi_{B_1,B_2,A_3}\{f\}(\varsigma)} &= \mathcal{D}_{A_1,A_2,B_3}\{x\}(\varsigma)\overline{\mathcal{D}_{B_1,B_2,A_3}\{f\}(\varsigma)} \\
&\quad\times e^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2)}\cdot e_1,
\end{aligned} \tag{65}
$$

$$
\begin{aligned}
\Xi_{A_1,B_2,B_3}\{x\}(\varrho)\overline{\Pi_{B_1,A_2,A_3}\{f\}(\varrho)} &= \mathcal{D}_{A_1,B_2,B_3}\{x\}(\varrho)\overline{\mathcal{D}_{B_1,A_2,A_3}\{f\}(\varrho)} \\
&\quad\times e^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2-\frac{d_2}{2b_2}m_2^2\Delta y_2^2-\frac{d_3}{2b_3}m_3^2\Delta y_3^2)}\cdot e_1,
\end{aligned} \tag{66}
$$

$$
\begin{aligned}
\Xi_{A_1,B_2,A_3}\{x\}(\tau)\overline{\Pi_{B_1,A_2,B_3}\{f\}(\tau)} &= \mathcal{D}_{A_1,B_2,A_3}\{x\}(\tau)\overline{\mathcal{D}_{B_1,A_2,B_3}\{f\}(\tau)} \\
&\quad\times e^{-e_1(\frac{d_1}{2b_1}m_1^2\Delta y_1^2+\frac{d_2}{2b_2}m_2^2\Delta y_2^2+\frac{d_3}{2b_3}m_3^2\Delta y_3^2)}\cdot e_1.
\end{aligned} \tag{67}
$$

**Proof.** The proof process is similar to Theorem 7, so it is omitted. □

## 6. Conclusions

In the present work, the results presented show that the DLCT can be generalized to the case of octonion algebras. We proposed the DOCLCT and studied some basic properties associated with the DOCLCT. Then, according to a new convolution operate, we obtained the convolution theorem of the DOCLCT by the relationship between the DOCLCT and the 3-D DLCT. Finally, the correlation theorem of the DOCLCT was exploited. The properties of the DOCLCT show that they can be used for the analysis of the convolution theorem. The most important contribution of this article is that it provides basic tools for the time–frequency analysis of non-stationary 3-D octonion finite-length signals. The presented results form the foundation of octonion-based signals and system theory.

For applications, the proposed convolution theorem can be used to solve integral equations with special kernels [37]. We can also discuss the design of multiplicative filters with the convolution theorem of the DOCLCT.

We can use the convolution theorem of the DOCLCT in the analysis of some 3-D linear time-invariant systems described in [28]. This hypercomplex generalization of the DOCLCT provides an excellent tool for the analysis of 3-D discrete linear time-invariant systems and 3-D discrete data.

The authors of [38] show how some interesting properties of 1-D complex Gabor filters are extended to 2-D by quaternionic Gabor filters. In parallel, they introduce the corresponding quaternionic Gabor filter-based approach to disparity estimation and texture segmentation. Thus, if one is interested in the development of Gabor filters with values in octonion algebras it is possible to define octonion Gabor filters based on the DOCLCT and to introduce the local octonion phase. This is a theoretical necessity to develop and analyze the DOCLCT.

The DOCLCT is a new concept which allows for the processing of a few gray-scale or color images, as well as images with their local information as one octonion 8-D image in the spectral domain. This concept generalizes the traditional complex and quaternion 2-D DLCTs and can be effectively used for parallel processing up to eight gray-scale images or two color images [39,40]. Surely, the methods presented here are only a first step in using hypercomplex methods in image processing. Detailed work in this area remains in the plan for further action, as well as the further development of this theory.

As a theoretical basis of the frequency-domain definitions of high-dimensional analytic signals, the DOCLCT can be applied in the domain of analytic signals. Other potential applications can be found in noise analysis and image processing, such as the methods presented in [41,42].

Future research will be concerned with extensions of the applications sketched in this paper. Thus, we hope to open a door for future research on high-dimensional signal processing using representations in hypercomplex algebras.

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