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Article

Variable-Order Fractional Linear Systems with Distributed Delays—Existence, Uniqueness and Integral Representation of the Solutions

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Abstract: In this work, we study a general class of retarded linear systems with distributed delays and variable-order fractional derivatives of Caputo type. We propose an approach consisting of finding an associated one-parameter family of constant-order fractional systems, which is "almost" equivalent to the considered variable-order system in an appropriate sense. This approach allows us to replace the study of the initial problem (IP) for variable-order fractional systems with the study of an IP for these one-parameter families of constant-order fractional systems. We prove that the initial problem for the variable-order fractional system with a discontinuous initial function possesses a unique continuous solution on the half-axis when the function describing the variable order of differentiation is locally bounded, Lebesgue integrable and has an appropriate decomposition similar to the Lebesgue decomposition of functions with bounded variation. The obtained results lead to the existence and uniqueness of a fundamental matrix for the studied variable-order fractional homogeneous system. As an application of the obtained results, we establish an integral representation of the solutions of the studied IP.

Keywords: distributed delay; variable-order fractional derivatives; linear fractional system

MSC: 34A08; 34A12; 34D20

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1. Introduction

It is practically established that many real-world phenomena, especially the physical processes, appear to manifest variable fractional order behaviors in respect to time. The mathematical models based on an equation/system with variable-order (VO) fractional derivatives seem to be a more adequate description of these types of phenomena. The VO fractional operators can be considered as a natural analytical generalization of the constant order (CO) operators, which explains the high number of articles devoted to the development of the concept of variable and distributed order fractional operators as well as to the study of their important characteristics in terms of practical applicability (physical realization, ...), memory behaviors of the defining kernels and so on.

For the necessary mathematical (technical) details such as linearity and time invariance of the operator, the operator initialization, operational transforms and so on, see [1–3]. However, the definitions, useful for physical applications, have some conceptual mathematical problems. We note that the theory of VO fractional equations and systems with or without delay is, generally speaking, substantially more complicated, at least technically but not only. This is based on the obstacle that, in general (without special additional conditions), the VO fractional differential operator is not left inverse of the corresponding VO integral operator. This fact is established in [4,5], which to the best of our knowledge are the first deep studies devoted to these conceptual mathematical problems. Historically, as far we know, the first works devoted to finding a pair of kernels, via which it is possible to define the VO fractional differential operator and VO fractional integral operator, where the VO differential operator is left inverse of the VO fractional integral operator, are [6] (Sonin pair kernels) and [7] (Scarpi pair kernels). For more detailed information about the Sonin kernels and Scarpi's approach, see the works [8,9], respectively. For comprehensive information concerning the development of this theme as well as for different approaches by the formulation of various kinds of VO fractional integrals and derivatives, we recommend the remarkable works [10–16] and the references therein.

As mentioned in the works cited above, the inverse property of the VO differential operator, however, recovers in the constant-order (CO) differentiation theory, and hence from this point of view our approach has a natural reason. The main idea of our approach is to replace the study of the VO fractional systems with the study of the one-parameter family of CO fractional systems. This idea leads to some technical difficulties, but these difficulties can be managed.

The basic aim of the present work is to study a class linear system with distributed delays and VO fractional derivatives. The definition of the VO Caputo-type fractional derivative used in this article is introduced in several works such as [10,14,16]. Our work was partially inspired by [17,18].

This paper is organized as follows. Section 2 presents the necessary definitions of CO- and VO-type fractional integrals, the problem statement and the needed auxiliary definitions and facts for our exposition. Section 3 contains an introduction to our approach, which consists of finding an associated one-parameter family of CO fractional systems, which is "almost" equivalent to the VO system in an appropriate sense and can replace the study of the initial problem (IP) for the VO fractional system with the study of IP for the one-parameter family of CO fractional systems. Using this approach, we prove theorems for the existence and uniqueness of a continuous solution of the considered IP for class linear systems with distributed delays and VO fractional derivatives of Caputo type and also for the prolongation of the solutions on the half-axis. Section 4 is devoted to establishing the existence and uniqueness of fundamental matrix of the studied homogeneous system, and, as an application of the obtained results, we establish an integral representation of the solutions of the studied IP. Section 5 contains conclusion and comments as well as some ideas for future research.

2. Preliminaries and Problem Statement

In the present section, to avoid possible misunderstandings, we recall the necessary definitions of CO and VO fractional integrals and the needed auxiliary definitions and facts used in our exposition throughout this paper. The problem statement and the related assumptions and conditions are also presented. For all details concerning the constant order CO fractional derivatives and other properties, we refer to [19,20]. Concerning the case of variable-order (VO) fractional derivatives and their properties, we refer to the remarkable works and surveys of [8,10,12,13].

Let $a \in \mathbb{R}$, $q \in (0,1)$ be arbitrary. Throughout the following, we say that for a class of functions $g : \mathbb{R} \to \mathbb{R}$, some property holds locally if it is fulfilled on every compact subinterval $[b,c] \subset \mathbb{R}$. The following notation will be used for the real linear spaces: $L_1^{loc}(\mathbb{R},\mathbb{R})$ consisting of all locally Lebesgue integrable functions and the subspaces $BV^{loc}(\mathbb{R},\mathbb{R}) \subset BL_1^{loc}(\mathbb{R},\mathbb{R}) \subset L_1^{loc}(\mathbb{R},\mathbb{R})$ of all functions which are locally bounded or have locally bounded variation, respectively. The left-sided fractional integral operators of constant order $q \in (0,1)$, with $a \in \mathbb{R}$ for any $g \in L_1^{loc}(\mathbb{R},\mathbb{R})$, we define via the relation

$$(I_{a+}^{q}g)(t) = \frac{1}{\Gamma(q)} \int_{a}^{t} (t-s)^{q-1} g(s)\mathrm{d}s$$

and the corresponding left side Riemann–Liouville and Caputo derivatives via the equalities

$$^{RL}D^{q}_{a+}g(t) = \frac{\mathrm{d}}{\mathrm{d}t}(I^{1-q}_{a+}g(t)) \quad \text{and} \quad {}_{C}D^{q}_{a+}g(t) = ({}_{RL}D^{q}_{a+}[g(s) - g(a)])(t)$$

for any $t > a$. For an arbitrary $q(t) \in BL^{loc}_{1}(J, (0, 1))$, $J \in [a, \infty)$, we define the VO left-sided fractional integral with the lower limit $a \in \mathbb{R}$ via

$$\mathfrak{I}^{q(t)}_{a+}g(t) = \frac{1}{\Gamma(q(t))}\int_{a}^{t}(t-\tau)^{q(t)-1}g(\tau)\mathrm{d}\tau$$

and the VO left-sided Riemann–Liouville-type fractional derivatives via

$$^{RL}\mathfrak{D}^{q(t)}_{a+}g(t) = \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}(t-\tau)^{-q(t)}g(\tau)\mathrm{d}\tau$$

and the Caputo-type VO derivative (see [10,16]) via the equality

$$^{C}\mathfrak{D}^{q(t)}_{a+}g(t) =^{RL} D^{q(t)}_{a+}(g(t) - g(a)) = \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}(t-\tau)^{-q(t)}(g(t) - g(a))\mathrm{d}\tau.$$

When $g \in AC(\mathbb{R}, \mathbb{R})$ (i.e., g is absolutely continuous), then

$$(^{C}\mathfrak{D}^{q(t)}_{a})g(t) = \frac{1}{\Gamma(1-q(t))}\int_{a}^{t}(t-s)^{-q(t)}g'(s)\mathrm{d}s.$$

Consider the retarded fractional linear system with Caputo-type variable-order (VO) derivatives and distributed delays in the following general form:

$$^{C}\mathfrak{D}^{q(t)}_{a+}X(t) = \int_{-h}^{0}[\mathrm{d}_{\theta}U(t,\theta)]X(t+\theta) + \mathfrak{F}(t), \tag{1}$$

where $h > 0$, $J = [a, \infty)$, $J^{0} = (a, \infty)$, $a \in \mathbb{R}$, $X(t) = col(x_1(t), \ldots, x_n(t))\colon J \to \mathbb{R}^n$ (the notation col mean vector-column), $\mathfrak{F}(t) = col(f_1(t), \ldots, f_n(t)) \in BL^{loc}_{1}(J, \mathbb{R}^n)$, $^{C}\mathfrak{D}^{q(t)}_{a+}X(t) = \mathrm{diag}(\mathfrak{D}^{q(t)}_{a+}x_1(t), \ldots, \mathfrak{D}^{q(t)}_{a+}x_n(t))$, where $\mathfrak{D}^{q(t)}_{a+}x_k(t)$, $k \in \langle n \rangle = \{1, 2, \ldots, n\}$, $X_t(\theta) = X(t+\theta)$, $t \in J$, $\theta \in [-h, 0]$ denotes the left side VO derivative of Caputo type $U(t,\theta) = \sum\limits_{i\in\langle m\rangle_0} U^i(t,\theta)$, $U^i(t,\theta) = \{u^i_{kj}(t,\theta)\}^n_{k,j=1} : J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $\langle m\rangle_0 = \langle m\rangle \bigcup \{0\}$.

The corresponding homogenous system of system (1) described in detail has the form:

$$^{C}\mathfrak{D}^{q(t)}_{a+}x_k(t) = \sum_{i\in\langle m\rangle_0}\left(\sum_{j\in\langle n\rangle}\int_{-h}^{0}x_j(t+\theta)\mathrm{d}_{\theta}u^i_{kj}(t,\theta)\right). \tag{2}$$

The following notations will also be used: $J_{a-h} = [a-h, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty)$, $\mathbf{0} \in \mathbb{R}^n$, and $I, \Theta \in \mathbb{R}^{n\times n}$ are the zero vector, the identity and the zero matrices. For $Y(t,\theta)$: $J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $Y(t,\theta) = \{y^i_j(t,\theta)\}^n_{i,j=1}$, $|Y(t,\theta)| = \sum\limits_{k,j=1}^{n}|y^j_k(t,\theta)|$, and when $y^j_k(t,\cdot) \in BV^{loc}(\mathbb{R}, \mathbb{R})$, for any fixed $t \in J$ and $k, j \in \langle n\rangle$, then $Var_{[a,b]}Y(t,\cdot) = \{Var_{[a,b]}y^j_k(t,\cdot)\}^n_{k,j=1}$, $[a, b] \subset \mathbb{R}$ is arbitrary, and we denote $Y(t,\cdot) \in BV^{loc}_{\theta}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$. $C(\mathbb{R}, \mathbb{R}^n)$ denotes the real linear space of all continuous vector-functions and $BC(J, \mathbb{R}^n) \subset C(J, \mathbb{R}^n)$.

The spaces of initial functions $\Phi = col(\phi_1, \ldots, \phi_n) : [-h, 0] \to \mathbb{R}^n$ are denoted as follows: $\mathbf{PC} = PC([-h, 0], \mathbb{R}^n)$ (piecewise continuous); $\mathbf{PC}^* = \mathbf{PC} \cap BV([-h, 0], \mathbb{R}^n)$ (piecewise continuous with bounded variation), $\mathbf{C} = C([-h, 0], \mathbb{R}^n)$ (continuous) and $\mathbf{AC} = AC([-h, 0], \mathbb{R}^n)$ (absolutely continuous). All these real linear spaces endowed with the sup-norm $||\Phi|| = \sum_{k\in\langle n\rangle} \sup_{s\in[-h,0]} |\phi_k(s)| < \infty$ are Banach spaces. S^{Φ} denotes the set of all jump points for any $\Phi \in \mathbf{PC}$; in addition, we will assume that they are right continuous at $t \in S^{\Phi}$.

For an arbitrary $\Phi \in \mathbf{PC}$, we introduce the following initial condition for system (1) or (2):

$$X(t) = \Phi(t-a), \quad t \in [a-h, a] \ (X_a(\theta) = \Phi(\theta), \theta \in [-h, 0]), \quad h \in \mathbb{R}_+. \tag{3}$$

Definition 1 ([21–23])**.** *We say that for the kernels $U^i : J \times \mathbb{R} \to \mathbb{R}^{n\times n}$, the hypothesis $(\mathfrak{S})$ holds if the following conditions are fulfilled for any $i \in \langle m\rangle_0$:*

$(\mathfrak{S}1)$ The functions $(t, \theta) \to U^i(t, \theta)$ are measurable in $(t, \theta) \in J \times \mathbb{R}$ and normalized so that $U^i(t, \theta) = 0$ for $\theta \geq 0$, $U^i(t, \theta) = U^i(t, -\sigma_i)$, $\sigma_i \in [0, h]$ for $\theta \leq -\sigma^i$, $\theta \leq -\sigma^i$, $0 = \sigma^0 < \sigma^1 < \ldots < \sigma^m = h$ and $t \in J$.

$(\mathfrak{S}2)$ For any fixed $t \in J$, the kernels $U^i(t, \theta)$ are left continuous in θ on $(-\sigma_i, 0)$ and the kernels $U^i(t, \cdot) \in BV^{loc}_{\partial}(J \times \mathbb{R}, \mathbb{R}^{n\times n})$ uniformly in $t \in J$ and

$$\left|Var_{[-h,0]} U^i(t, \cdot)\right|, \sup_{\theta\in[-h,0]} \partial_\theta |U^i(t, \theta)| \in BL^{loc}_1(J, \mathbb{R}_+).$$

$(\mathfrak{S}3)$ For any fixed $t \in J$, the Lebesgue decomposition of the kernels $U^i(t, \theta)$ has the form:

$$U^i(t, \theta) = U^i_{st}(t, \theta) + U^i_{ac}(t, \theta) + U^i_s(t, \theta),$$

$$U^i_{st}(t, \theta) = \{a^i_{kj}(t) H(\theta + \sigma^i(t))\}^n_{k,j=1}, \ \sigma^i(t) \in C(J, [0, h]), \ \sigma^0(t) \equiv 0,$$

$$A^i(t) = \{a^i_{kj}(t)\}^n_{k,j=1} \in BL^{loc}_1(J, \mathbb{R}^n), \ U^i_{ac}(t, \cdot) \in AC([-h, 0], \mathbb{R}^{n\times n}),$$

$$U^i_s(t, \cdot) \in C([-h, 0], \mathbb{R}^{n\times n}),$$

where $k, j \in \langle n\rangle$, and $H(t)$ is the Heaviside function.

$(\mathfrak{S}4)$ The sets $S^i_{\Phi} = \{t \in J | \ t - \sigma^i(t) \in S_{\Phi}\}$ do not have limit points, and for any $t_* \in J$ the relations $\lim_{t\to t_*} \int_{-h}^{0} |U^i(t, \theta) - U^i(t_*, \theta)| \mathrm{d}\theta = 0$ hold.

$(\mathfrak{S}5)$ The function $q(t) \in BL^{loc}_1(J, (0, 1))$ and has the representation $q(t) = q_s(t) + q_c(t)$, where $q_s(t) \in PC(J, (0, 1))$ is a right continuous step function, and $q_c(t) \in C(J, (0, 1))$ with

$$q^m = \inf_{t\in J} q(t), \quad q^M = \sup_{t\in J} q(t), \quad [q^m, q^M] \subset (0, 1).$$

The set $S^{q_s} = \{t_k \in J^0 | q_s(t_k + 0) - q_s(t_k - 0) \neq 0, k \in \mathbb{N}\}$ of all first kind jump points of $q_s(t)$ is at most countable and does not have limit points in j.

Remark 1. *Note that the condition $(\mathfrak{S}5)$ is inspired from physical aspects connected to rheology (the stress and/or strain leads to hysteresis-type reactions) but also leads to some mathematical advantages. So, from a practical point of view, we consider the case when the set S^{q_s} is most countable and does not have limit points in J.*

Theorem 1 ([24], Krasnosel'skii's fixed point theorem)**.** *Let $(\mathfrak{B}, \| \cdot \|)$ be a Banach space, $H \subseteq \mathfrak{B}$ be a nonempty, closed and convex subset of $\mathfrak{B}$ and the maps $\mathbf{T}, \mathbf{K} : H \to \mathfrak{B}$ satisfy the following conditions:*

(i) The operator $\mathbf{T}$ is contraction with constant $\gamma \in (0, 1)$;

(ii) The operator $\mathbf{K}$ is continuous, and the set $\mathbf{K}(H)$ is contained in a compact set;

(iii) *For any $x, y \in H$, we have that $\mathbf{T}x + \mathbf{K}y \in H$.*

Then, there exists $z \in H$ with $\mathbf{T}z + \mathbf{K}z = z$.

3. Main Results

In this section, we study the existence and the uniqueness of continuous solutions of IP (1), (3) without the additional assumption of them being absolutely continuous, which essentially simplifies the problem at least technically. The reason for this is the possible application in economics taking into account that the diagram of the stock courses is more similar to the Weierstrass function than to a smooth function. We propose an approach to convert the IP with VO to a system of integral equations which is more convenient for study, considering the fact that, in general, the fractional integral of VO is not the left inverse operator of the VO differential operator and vice versa.

Let $a = t_0$, $a - h = t_0 - a = t_{-1}$, $\Delta_k = (t_{k-1}, t_k]$, $\Delta_k^0 = (t_{k-1}, t_k)$, $\overline{\Delta}_k = [t_{k-1}, t_k]$, $k \in \mathbb{N}_0$. Then, for any $t \in J$, we have $q_s(t) = \sum_{k=1}^{\infty} q_s^k \mathbf{1}_{\Delta_k}$, where $q_s(t) = q_s^k$, $t \in \Delta_k$, and $\mathbf{1}_{\Delta_k}$ is the indicator function of Δ_k (note that for any $t \in J$, the sum is finite). The next lemma is an immediate generalization of Theorem 4 in [16] and establishes a simple but important statement.

Lemma 1. *Let the condition ($\mathfrak{S}5$) hold.*

Then, for any function $g(t) \in L^1_{loc}(J, \mathbb{R})$ for which $g(t) = o((t-a)^{q^M + \varepsilon})$, we have that ${}^{C}\mathfrak{D}^{q(t)}_{a+} g(t) = 0$ at $t = a$.

Proof. Let $g(t) \in L^1_{loc}(J, \mathbb{R})$ be arbitrary. Then, for $t, s \in [a, a+\delta]$, $s \le t$, we have

$$\begin{aligned} {}^{C}\mathfrak{D}^{q(t)}_{a+} g(t) &= {}^{RL}\mathfrak{D}^{q(t)}_{a+} (g(t) - g(a)) = {}^{RL}\mathfrak{D}^{q(t)}_{a+} g(t) \\ &= \frac{1}{\Gamma(1-q(t))} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-q(t)} g(s)\mathrm{d}s. \end{aligned} \tag{4}$$

It is well known that the Γ function on the real line has a minimum at $z_{min} \approx +1.46$ with $\Gamma(z_{min}) \approx +0.8856$, and hence we have that

$$0 < \frac{1}{\Gamma(1-q^M)} \le \lim_{t \to a+0} \frac{1}{\Gamma(1-q(t))} \le \frac{1}{\Gamma(z_{min})} \le \frac{1}{0.8856} < \infty.$$

Since $g(t) = o((t-a)^{q^M + \varepsilon})$, there exist constants $c > 0$ and $\delta \in (0, \min(1, t_1))$, such that for some $\varepsilon > 0$, we have that $|g(t)| \le c(t-a)^{\varepsilon + q^M}$ for any $t \in [a, a+\delta]$. Then, from (4), for any $t, s \in [a, a+\delta]$, $s \le t$, taking into account that $\inf\limits_{t \in [a, a+\delta]} (1 - q(t)) > 0$, it follows that

$$\begin{aligned} \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-s)^{-q(t)} g(s)\mathrm{d}s|_{t=a} \right| &\le \lim_{t \to a+0} \frac{\int_a^t (t-s)^{-q(t)} |g(s)|\mathrm{d}s}{(t-a)} \\ &\le \lim_{t \to a+0} \frac{c\int_a^t (t-s)^{-q(t)} (t-a)^{\varepsilon + q^M}\mathrm{d}s}{(t-a)} \\ &\le \lim_{t \to a+0} \frac{c(t-a)^{\varepsilon + q^M} \int_a^t (t-s)^{-q(t)}\mathrm{d}s}{(t-a)} \\ &\le \lim_{t \to a+0} \frac{c(t-a)^{\varepsilon + q^M} (t-a)^{1-q(t)}}{(1-q(t))(t-a)} = \lim_{t \to a+0} \frac{c(t-a)^{\varepsilon}}{(1-q(t))} = 0, \end{aligned}$$

which completes the proof. □

Definition 2. *The vector function $X(t) = col(x_1(t), \ldots, x_n(t))$ is a solution of IP (1), (3) or IP (2), (3) in $[a,b]$, $b > a$ arbitrary, if $X|_{[a,b]} \in C([a,b], \mathbb{R}^n)$, $(X|_J \in C(J, \mathbb{R}^n))$ satisfies system (1) or (2), respectively, for all $t \in (a,b](t > a)$, as well as the initial condition (3).*

Let us assume that $X(t)$ is a solution of IP (1), (3) for any $t > a$ and denote

$$Z(t) = \Gamma(1 - q^M)(X(t) - X(a)).$$

The aim of our considerations below is to find (establish) an one-parameter family of CO fractional systems which the function $Z(t)$ satisfies for any $t > a$ as well as the corresponding initial conditions for $Z(t)$, induced from the initial condition (3) which the solution $X(t)$ satisfies.

Then, from (3) it follows that for $t \in [a-h, a]$ $Z(t)$ satisfies the initial condition

$$Z(t) = \Gamma(1 - q^M)(X(t) - X(a)) = \Gamma(1 - q^M)(\Phi(t-a) - \Phi(0)) = Z_0(t-a), \tag{5}$$

and hence $Z(a) = Z_0(0) = \mathbf{0}$.

Consider the function $Y^t(s) = \omega(t-s)Z(s)$, where

$$\omega(t-s) = (t-s)^{q^M - q(t)} = (t-s)^{q^M - q_s(t) - q_c(t)}$$

for $t \in J$ and $s \in [a-h, t]$. In addition we define: $\omega(t-s) = 0$ when $t \leq s$ and $\omega_k(t-s)$ mean that $t \in \Delta_k$ for any $k \in \mathbb{N}_0$. Then, from (3) it follows that for any fixed $t \in J^0$ and $s \in [a-h, a] = \overline{\Delta}_0$ we have that $Y^t(s) = \omega(t-s)Z(s) = \omega(t-s)Z_0(s-a)$ and hence it must satisfy the following initial condition for any fixed $t \in J$ and $s \in [a-h, a]$

$$Y^t(s) = \omega(t-s)Z(s) = \omega(t-s)(X(s) - X(a)) = \omega(t-s)Z_0(s-a) \quad \text{and} \quad Y(t,a) = \mathbf{0}. \tag{6}$$

Then, applying Lemma 1 we have

$$\begin{aligned} \Gamma(1-\alpha(t))\mathfrak{D}^{q(t)}_{a+}G(t) &= \frac{1}{\Gamma(1-q^M)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{-q(t)}(G(s) - G(a))\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-q^M)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{-q^M}(t-a)^{q^M - q(t)}\Gamma(1-q^M)(g(s) - g(a))\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-q^M)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{-q^M}Y'(a)\tau =_{RL} D^{q^M}_{a+}Y^t(s)|_{s=t} =_C D^{q^M}_{a+}Y^t(s)|_{s=t}. \end{aligned} \tag{7}$$

From (1), (7) for an arbitrary fixed $t > a$ with $s \in [a, t]$ we obtain

$$
\begin{aligned}
&{}_{RL}D_{a+}^{q^M} Y^t(s)|_{s=t} = {}_C D_{a+}^{q^M} Y^t(s)|_{s=t} = \Gamma(1-q(t))\left(\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X(t+\theta)\mathrm{d}t + \mathfrak{F}(t)\right) \\
&\overset{\eta=t+\theta}{=} \Gamma(1-q(t))\left(\int_{t-h}^{t}[\mathrm{d}_s U(t,\eta-t)](X(\eta)-X(a)+X(a))\mathrm{d}t + \mathfrak{F}(t)\right) \\
&= \Gamma(1-q(t))\int_{t-h}^{t}[\mathrm{d}_s U(t,\eta-t)](X(\eta)-X(a))\mathrm{d}t \\
&+ \Gamma(1-q(t))\int_{t-h}^{t}[\mathrm{d}_s U(t,s-t)]X(a)\mathrm{d}t + \Gamma(1-q(t))\mathfrak{F}t) \\
&= \Gamma(1-q(t))\int_{t-h}^{t}[\mathrm{d}_s U(t,\eta-t)](X(\eta)-X(a))\mathrm{d}t \\
&- \Gamma(1-q(t))X(a)U(t,-h) + \Gamma(1-q(t))\mathfrak{F}(t) \\
&= \Gamma(1-q(t))\int_{-h}^{0}[\mathrm{d}_s U(t,\theta)](X(t+\theta)-X(a))\mathrm{d}t + \Gamma(1-q(t))\overline{\mathfrak{F}}(t),
\end{aligned} \tag{8}
$$

where $\overline{\mathfrak{F}}(t) = \mathfrak{F}(t) - \Phi(0)U(t,-h)$, and hence $\overline{\mathfrak{F}}(t) \in BL_1^{loc}(J,\mathbb{R}^n)$.

Using (8) we introduce for any fixed $t > a$, arbitrary $\mathfrak{F}(t) \in BL_1^{loc}(J,\mathbb{R}^n)$ and each $s \in (a,t]$ the following auxiliary system

$$
\begin{aligned}
{}^C D_{a+}^{q^M} Y^t(s) &= {}^C D_{a+}^{q^M}[\omega(t-s)Z(s)](s) \\
&= \Gamma(1-q(s))\int_{-h}^{0}[\mathrm{d}_\theta U(s,\theta)](Z(s+\theta))\mathrm{d}t + \Gamma(1-q(s))\overline{\mathfrak{F}}(s).
\end{aligned} \tag{9}
$$

Since according (6) we have $Y^t(a) = \mathbf{0}$ for any $t \in J$, then applying the operator $I_{a+}^{q^M}$ (note that q^M is a constant order) to both sides of (9) for any fixed $t > a$ and $s \in [a,t]$ we obtain

$$
\begin{aligned}
\omega(t-s)Z(s) = \Gamma^{-1}(q^M)&\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)](Z(s+\theta))\mathrm{d}\eta \\
&+ \Gamma^{-1}(q^M)\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\overline{\mathfrak{F}}(\eta)\mathrm{d}\eta.
\end{aligned} \tag{10}
$$

Taking into account (5) we can rewrite (10) in a more convenient form, in view of application of the Krasnosel'skii's fixed point theorem MK [24] as follow

$$
\begin{aligned}
Z(s) = &Z(s)(1-\omega(t-s)) + \Gamma^{-1}(q^M)\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)](Z(\eta+\theta))\mathrm{d}\eta \\
&+ \Gamma^{-1}(q^M)\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\overline{\mathfrak{F}}(\eta)\mathrm{d}\eta = Z(s)(1-\omega(t-s)) + \Im_1(s) + \Im_2(s),
\end{aligned} \tag{11}
$$

where by $\Im_1(s)$ and $\Im_2(s)$ are denoted the second and the third addends in the right side of (11). We will establish that $\lim_{s\to t-0}\Im_i(s) = \Im_i(t)$. Let $t \in \Delta_k$ for some $k \in \mathbb{N}$ and then we have

$$
{}^{C}\mathfrak{D}_{a+}^{q(t)}g(t) = {}^{RL}\mathfrak{D}_{a+}^{q(t)}(g(t)-g(a)) = \frac{1}{\Gamma(1-\alpha(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-q(t)}(g(\tau)-g(a))\mathrm{d}\tau,
$$

$$
{}^{RL}\mathfrak{D}_{a+}^{q(t)}g(t) = \frac{1}{\Gamma(1-\alpha(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-q(t)}g(\tau)\mathrm{d}\tau,
$$

$$
|\mathfrak{J}_1(s)| = \left|\Gamma^{-1}(q^M)\right|\left|\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\int_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)](Z(\eta+\theta))\mathrm{d}\eta\right| \tag{12}
$$

$$
\leq \frac{\sup_{\eta\in[a,t]}\Gamma(1-q(\eta))}{\Gamma(q^M)}\int_a^s (s-\eta)^{q^M-1}\left|\int_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)](s-\eta-\theta)^{-q^M+q(t)}Y^s(\eta+\theta)\right|\mathrm{d}\eta
$$

$$
\leq \frac{U_{k+1}\Gamma(1-q^M)}{\Gamma^2(z_{min})}\left|\int_a^s (s-\eta)^{q(t)-1}\sup_{\theta\in[-h,0]} Y^s(\eta+\theta)\mathrm{d}\eta\right| < \infty,
$$

where $U_k = \max\Big(\sup_{t\in[a,t_k]}\mathrm{Var}_{\theta\in[-h,0]}|U(t,\theta)|, \sup_{t\in[a,t_k]}\Big|\sup_{\theta\in[-h,0]}\partial_\theta|U(t,\theta)|\Big|\Big) < \infty$ by virtue of conditions ($\mathfrak{S}2$) and ($\mathfrak{S}3$). Then, we conclude that $\lim\limits_{s\to t-0}|\mathfrak{I}_s(s)| < \infty$ implies $\lim\limits_{s\to t-0}\mathfrak{I}_1(s) = \mathfrak{J}_1(t)$. The relation $\lim\limits_{s\to t-0}\mathfrak{I}_s(s) = \mathfrak{I}_2(t)$ obviously holds.

Definition 3. *The vector function $Z(s) \in C([a,b],\mathbb{R}^n)(C(J,\mathbb{R}^n))$ is a solution of IP (11), (5) or IP (9), (5) in the interval $(a,b](J^0)$, where $b > a$ be arbitrary, if for any fixed $t \in (a,b](t \in J^0)$ and for $s \in (a,b](s \in (a,t])$, the function $Z(s)$ satisfies system (11) or (9), respectively, for any $t \in (a,b](t \in J^0)$ as well as the initial condition (5).*

Remark 2. *Taking into account that the order of differentiation of system (8) is constant, then according to Lemma 3.3 in [25], each solution in J of IP (10), (8) in the sense of Definition 3 is a solution in J of IP (12), (8) and vice versa.*

So, with our consideration above, we proved the following statement.

Proposition 1. *For any solution $X(t)$ of IP (1), (3) with interval of existence J^0, the function $Z(s) = \Gamma(1-q^M)(X(s)-X(a))$ satisfies system (11) for every fixed $t > a$ and any $s \in (a,t]$ with the initial condition (5) too and vice versa.*

Let $j \in \langle l_k\rangle_0$, $l_k \in \mathbb{N}$ for any $k \in \mathbb{N}$. Thus, without loss of generality, we can assume for simplicity in our exposition below that $|\overline{\Delta}_k| < 1$ for $k \in \mathbb{N}$ and $|\overline{\Delta}_0| = h$

Consider the real linear space $\mathbf{B} = \{G : J \to \mathbb{R}^n | G(s) \in C(J,\mathbb{R}^n)\}$ and for an arbitrary fixed $t \in J^0$ the real Banach spaces

$$
\mathbf{B}^t = \{G_t(s) : [a,t] \to \mathbb{R}^n | G_t(s) = G(s)|_{[a,t]}, G(s) \in \mathbf{B}, \|G_t\|_t = \sup_{s\in[a,t]}|G(s)|\} \tag{13}
$$

and for any fixed $t \in J^0$ define the nonempty, closed and convex subsets $\mathbf{H}_0^t$ as follows:

$$
\mathbf{H}_0^t \subset \mathbf{B}^t = \{G_t(s) \in \mathbf{B}^t | G(a) = 0\}, \tag{14}
$$

which endowed with the metric function $d_t^{\Phi}(G_t,\overline{G}_t) = \|G_t - \overline{G}_t\|_t = \sup\limits_{s\in[a,t]}|G_t(s) - \overline{G}_t(s)|$ for arbitrary $G_t(s), \overline{G}_t(s) \in \mathbf{H}_0^t$, are complete metric spaces. It is clear that for any $t, \overline{t} \in J$ with $t \leq \overline{t}$ for any $G_{\overline{t}}(s) \in \mathbf{H}_0^{\overline{t}}$, the restriction $G_{\overline{t}}(s)|_{[a-h,t]} \in \mathbf{H}_0^t$.

For any $t \in \Delta_1$ and every $G_t(s) \in \mathbf{H}_0^t$ and $s \in (a, t]$, the operator $(\mathfrak{R}G_t)(s)$ is defined via the equality:

$$
\begin{aligned}
(\mathbf{R}G_t)(s) = {} & G_t(s)(1-\omega(t_1 - s)) \\
& + \Gamma^{-1}(q^M) \int_a^s (s-\eta)^{q^M - 1}\Gamma(1-q(\eta)) \int_{-h}^{0} [\mathrm{d}_\theta U(\eta,\theta)](G_t(\eta+\theta))\mathrm{d}\eta \\
& + \Gamma^{-1}(q^M) \int_a^s (s-\eta)^{q^M - 1}\Gamma(1-q(\eta))\overline{\mathfrak{F}}(\eta)\mathrm{d}\eta,
\end{aligned}
\tag{15}
$$

where the initial condition $(\mathbf{R}G_t)(s) = Z_0(s-a)$ for $s \in [a-h, a]$ (i.e., $(\mathbf{R}G_t)(s)$ satisfies the initial condition (5)).

Theorem 2. *Let the following conditions be fulfilled:*

1. *The hypothesis* $(\mathfrak{S})$ *hold.*
2. *The function* $\overline{\mathfrak{F}}(t) \in BL_1^{loc}(J, \mathbb{R}^n)$*.*

Then, for any $\Phi \in \mathbf{PC}$ *IP* (10), (5) *has a unique solution* $Z_1^*(s) \in \mathbf{H}_0^{t_1}$ *with a* Δ_1 *interval of existence.*

Proof. Existence: Let $t \in \Delta_1$ be arbitrary, and for any $G_t(s) \in \mathbf{H}_0^t$ using (14), we split the operator $(\mathbf{R}G_t)(s)$ in two parts $(\mathbf{R}G_t)(s) = (\mathbf{T}G_t)(s) + (\mathbf{K}G_t)(s)$. The operators $\mathbf{T}$ and $\mathbf{K}$ are defined as follows:

$$
(\mathbf{T}G_t)(s) = G_t(s)(1-\omega(t_1-s)) + \Gamma^{-1}(q^M)\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\overline{\mathfrak{F}}(\eta)\mathrm{d}\eta, \tag{16}
$$

$$
(\mathbf{K}G_t)(s) = \Gamma^{-1}(q^M)\int_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\int_{-h}^{0}[\mathrm{d}_\theta U(\eta,\theta)](G_t(\eta+\theta))\mathrm{d}\eta, \tag{17}
$$

and $(\mathbf{K}G_t)(s)$ satisfies the initial condition (5) for any $t \in \Delta_1$, i.e.,

$$
(\mathbf{K}G_t)(s) = (\Phi(s-a) - \Phi(0)) = Z_0(s-a) \quad \text{for} \quad s \in [a-h, a].
$$

Since both integrands in the right sides of (16) and (17) by virtue of Conditions 1 and 2 of the Theorem are at least locally bounded and Lebesgue integrable functions, then both integrals in the right sides of (16) and (17) are continuous functions. So, since $(TG_t)(a) = 0$, $(KG_t)(a) = 0$, and hence $(RG_t)(a) = 0$ for any $G_t(s) \in \mathbf{H}_0^t$, we have that $(\mathbf{R}G_t)(a) \in \mathbf{H}_0^t$, and hence for any $t \in \Delta_1$, $\mathbf{TH}_\Phi^t \subseteq \mathbf{H}_\Phi^t$, $\mathbf{KH}_\Phi^t \subseteq \mathbf{H}_\Phi^t$ and $\mathbf{RH}_\Phi^t \subseteq \mathbf{H}_\Phi^t$ are fulfilled.

The main obstacle is that Theorem 1 (Krasnosel'skii's fixed point theorem) cannot be applied directly since for $s = t$ the obtained system (11) is degenerate in the sense that the operator $\mathbf{T}$, which is defined pointwise, is not a contraction for $s = t$. To overcome this obstacle, we carried out a continuous prolongation of the fixed point of the operator $\mathbf{R}$ existing for all $s < t$ so that the prolonged fixed point (as a continuous function) also satisfies system (11) at $s = t$. Let $t \in \Delta_1$ be arbitrary and define the sequence $t_j = t - \dfrac{1}{j}$, $j \in \mathbb{N}$. We will prove that $\mathbf{TH}_0^{t_j} \subseteq \mathbf{H}_0^{t_j}$ is a contraction map for any $j \in \mathbb{N}$. Indeed, for an arbitrary $G_{t_j}(s), \overline{G}_{t_j}(s) \in \mathbf{H}_0^{t_j}$ and denoting $\gamma_j = 1 - \omega\left(\dfrac{1}{j}\right) < 1$, $j \in \mathbb{N}$, we have that

$$
|(\mathbf{T}G_{t_j})(s) - (\mathbf{T}\overline{G}_{t_j})(s)| \le \sup_{s\in[a,t_j]} (1-\omega(t-s))|G_{t_j}(s) - \overline{G}_{t_j}(s)| \le \gamma_j |G_{t_j}(s) - \overline{G}_{t_j}(s)|
$$

and hence

$$
\|\mathbf{T}G_{t_j} - \mathbf{T}\overline{G}_{t_j}\| \le \gamma_i \|G_{t_j} - \overline{G}_{t_j}\|_{t_j}. \tag{18}
$$

For an arbitrary $G_{t_j}(s), \overline{G}_{t_j}(s) \in \mathbf{H}_0^{t_j}$ from (17), it follows that

$$\begin{aligned}
&\left|(\mathbf{K}G_{t_j})(s) - (\mathbf{K}\overline{G}_{t_j})(s)\right| \le \\
&\le \Gamma^{-1}(q^M)\int\limits_a^s (s-\eta)^{q^M-1}\Gamma(1-q(\eta))\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]\Big(G_{t_j}(\eta+\theta) - \overline{G}_{t_j}(\eta+\theta)\Big)\mathrm{d}\eta \\
&\le \frac{\Gamma(1-q^M)}{\Gamma(q^M)}\int\limits_a^s (s-\eta)^{q^M-1}\left|\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]\Big(G_{t_j}(\eta+\theta) - \overline{G}_{t_j}(\eta+\theta)\Big)\right|\mathrm{d}\eta \\
&\le \frac{U_1\Gamma(1-q^M)}{\Gamma(q^M)}\|G_{t_j} - \overline{G}_{t_j}\|_{t_j}\int\limits_a^s (t-\eta)^{q^M-1}\mathrm{d}\eta \le \frac{U_1\Gamma(1-q^M)(s-a)^{q^M}}{q^M\Gamma(q^M)}\|G_{t_j} - \overline{G}_{t_j}\|_{t_j} \\
&\le \frac{U_1\Gamma(1-q^M)(t_1-a)^{q^M}}{\Gamma(1+q^M)}\|G_{t_j} - \overline{G}_{t_j}\|_{t_j}
\end{aligned}$$

and hence

$$\|\mathbf{K}G_{t_j} - \mathbf{K}\overline{G}_{t_j}\| \le \frac{U_1|\overline{\Delta}_1|^{q^M}}{\Gamma(1+q^M)}\|G_{t_j} - \overline{G}_{t_j}\|_{t_k} \le \frac{U_1}{\Gamma(1+q^M)}\|G_{t_j} - \overline{G}_{t_j}\|_{t_j}. \tag{19}$$

Thus, we proved that the operator $\mathbf{K}\mathbf{H}_0^{t_j} \subseteq \mathbf{H}_0^{t_j}$ is continuous. For an arbitrary fixed $r > 0$ and $G_{t_j}^0(s) \in \mathbf{H}_0^{t_j}$, consider the ball $B(r, G_{t_k}^0(s)) = \{G_{t_j}(s) \in \mathbf{H}_0^{t_j}, \|G_{t_j} - G_{t_j}^0\|_{t_j} \le r\} \subset \mathbf{H}_0^{t_j}$. For any $G_{t_k}(s) \in B(r, G_{t_k}^0(s))$ from (17) and (19), we obtain the estimation

$$\begin{aligned}
\|\mathbf{K}G_{t_j}\|_{t_j} &\le \|\mathbf{K}G_{t_j}^0\|_{t_j} + \|\mathbf{K}G_{t_j} - \mathbf{K}G_{t_j}^0\|_{t_j} \le \frac{U_1}{\Gamma(1+q^M)}\Big(\|\mathbf{K}G_{t_j}^0\|_{t_j} + \|G_{t_j} - G_{t_j}^0\|_{t_j}\Big) \\
&\le \frac{U_1}{\Gamma(1+q^M)}\Big(\|\mathbf{K}G_{t_j}^0\|_{t_j} + r\Big)
\end{aligned}$$

which implies that the set $\mathbf{K}\Big(B(r, G_{t_j}^0(s))\Big)$ is uniformly bounded.

To apply Theorem 1, it must also be proved that the set $\mathbf{K}\Big(B(r, G_{t_j}^0(s))\Big)$ is equicontinuous.

Let $s_1, s_2 \in [a, t_k]$ be arbitrary, and for definiteness assume that $s_1 < s_2$. For any $G_{t_k}(s) \in B(r, G_{t_k}^0(s))$ from (17) and (19), it follows that

$$\begin{aligned}
&\left|(\mathbf{K}G_{t_j})(s_2) - (\mathbf{K}\overline{G}_{t_j})(s_1)\right| \le \\
&\le \frac{\Gamma(1-q^M)}{\Gamma(q^M)}\left|\int\limits_a^{s_1}\Big((s_2-\eta)^{q^M-1} - (s_1-\eta)^{q^M-1}\Big)\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]G_{t_j}(\eta+\theta)\mathrm{d}\eta\right| \\
&+ \frac{\Gamma(1-q^M)}{\Gamma(q^M)}\left|\int\limits_{s_1}^{s_2} I_{q^M-1}(s_2-\eta)\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]G_{t_j}(\eta+\theta)\mathrm{d}\eta\right| \\
&\le \frac{\Gamma(1-q^M)}{\Gamma(q^M)}\int\limits_a^{s_1}\Big((s_2-\eta)^{q^M-1} - (s_1-\eta)^{q^M-1}\Big)\left|\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]G_{t_j}(\eta+\theta)\right|\mathrm{d}\eta \\
&+ \frac{\Gamma(1-q^M)}{\Gamma(q^M)}\int\limits_{s_1}^{s_2}(s_2-\eta)^{q^M-1}\left|\int\limits_{-h}^0 [\mathrm{d}_\theta U(\eta,\theta)]G_{t_j}(\eta+\theta)\right|\mathrm{d}\eta \\
&\le \frac{U_1\Gamma(1-q^M)}{\Gamma(q^M)}\|G_{t_j}\|_{t_j}\left(\int\limits_a^{s_1}\Big((s_1-\eta)^{q^M-1} - (s_2-\eta)^{q^M-1}\Big)\mathrm{d}\eta + \int\limits_{s_1}^{s_2}(s_2-\eta)^{q^M-1}\mathrm{d}\eta\right)
\end{aligned}$$

and hence we obtain the estimation

$$\begin{aligned}&\Big|(\mathbf{K}G_{t_j})(s_2)-(\mathbf{K}\overline{G}_{t_j})(s_1)\Big|\\&\leq\frac{\Gamma(1-q^M)}{\Gamma(1+q^M)}\Big(\|G^0_{t_j}\|_{t_{t_j}}+r\Big)\Big(2(s_2-s_1)^{q^M}-(s_2-\eta)^{q^M-1}+\Big|(s_1-a)^{q^M}-(s_2-a)^{q^M}\Big|\Big).\end{aligned}\tag{20}$$

Denoting $\tilde{C}=\frac{\Gamma(1-q^M)}{\Gamma(1+q^M)}\Big(\|G^0_{t_j}\|_{t_{t_j}}+r\Big)$ and taking into account that $(s-a)^{q^M}$ is uniformly continuous at $s\in[a,t]$, then for each $\varepsilon>0$ there exists $\delta\in(0,\varepsilon)$, such that if $|s_2-s_1|<\delta$, we have that $\Big|(s_1-a)^{q^M}-(s_2-a)^{q^M}\Big|<\varepsilon$. Then, from (20), for $|s_2-s_1|<\delta$, it follows that

$$\Big|(\mathbf{K}G_{t_j})(s_2)-(\mathbf{K}\overline{G}_{t_j})(s_1)\Big|\leq\tilde{C}(2\delta+\varepsilon)<3\varepsilon\tilde{C}$$

and hence the set $\mathbf{K}\Big(B(r,G^0_{t_j}(s))\Big)$ is equicontinuous for any $j\in\mathbb{N}$. Thus, the Arzela–Ascoli theorem implies that the closure of the set $\mathbf{K}\Big(B(r,G^0_{t_j}(s))\Big)$ is compact. Thus, according to Theorem 1, it follows that for any $j\in\mathbb{N}$, there exists at least one $G^*_{t_j}(s)\in\mathbf{H}_0^{t_j}$, such that for any $s\in[a,t_j)$, we obtain that $G^*_{t_j}(s)=(\mathbf{R}G^*_{t_j})(s)$. Then, for $j\to\infty$, taking into account that $G^*_{t_j}(s)\in\mathbf{H}_0^{t_j}$ for any $j\in\mathbb{N}$ are continuous functions and (11), we obtain that there exists $G^*_t(s)=(\mathbf{R}G^*_t)(s)$ for any $s\in[a,t]$ and $G^*_t(t)=(\mathbf{R}G^*_t)(t)$. Since $t\in\Delta_1$ is arbitrary then passing to the limit $t\to t_1-0$, we obtain that $G^*_{t_1}(t_1)=(\mathbf{R}G^*_{t_1})(t_1)$ and $G^*_{t_1}(s)=(\mathbf{R}G^*_{t_1})(s)$ for any $s\in[a,t_1)$. Then, the function $Z^*_1(s)=G^*_{t_1}(s)$ is a solution of IP (10), (5) for any $s\in\Delta_1$. Note that according to (10), we have that $\lim_{t\to a+0}Z^*_1(t)=\mathbf{0}$, and hence $Z_1(s)\in C(\overline{\Delta}_1,\mathbb{R}^n)$.

Uniqueness: Assume the contrary, that IP (11), (5) has two different solutions $Z^*_i(s)\in C(\overline{\Delta}_1,\mathbb{R}^n)$ and denote $W(s)=Z^*_1(s)-Z^*_2(s)$. Then, from (11), for any $s\in\overline{\Delta}_1$ it follows that

$$\begin{aligned}&|Z^*_1(s)-Z^*_2(s)|=|W(s)|\leq|(\mathbf{T}W)(s)+(\mathbf{K}W)(t)|\leq|(\mathbf{T}W)(s)|+|(\mathbf{K}W)(t)|\\&\leq\omega(t_1-s)|W(s)|+\frac{\Gamma(1-q^M)}{\Gamma(q^M)}\int_a^s(s-\eta)^{q^M-1}\left|\int_{-h}^0[\partial_\theta U(\eta,\theta)W(\eta+\theta)\mathrm{d}\theta\right|\mathrm{d}\eta\\&\leq q|W(s)|+\frac{\Gamma(1-q^M)hU_1}{\Gamma(q^M)}\int_a^s(s-\eta)^{q^M-1}\sup_{\xi\in[a,\eta]}|W(\xi)|\mathrm{d}\eta.\end{aligned}\tag{21}$$

Let $s_*\in\Delta_1$ be arbitrary, such that $Z^*_1(s_*)-Z^*_2(s_*)=W(s_*)\neq\mathbf{0}$, and then from (20), we obtain

$$\sup_{s\in[a,s_*]}|W(s)|\leq\frac{\Gamma(1-q^M)hU_1}{\Gamma(q^M)(1-\omega(t_1-a))}\int_a^{s_*}(s_*-\eta)^{\alpha-1}\sup_{\xi\in[a,\eta]}|W(\xi)|\mathrm{d}\eta,$$

and denote $a(s)\equiv0$ and $g=\frac{\Gamma(1-q^M)hU_1}{\Gamma(q^M)(1-\omega(t_1-a))}$ for any $s\in[a,s_*]$. Then, Corollary 2 in [26] implies that $\sup_{s\in[a,s_*]}|W(s)|\leq a(t)E_\alpha(g\Gamma(1-q^M)s_*^{q^M})\equiv0$, which contradicts our assumption. □

Theorem 3. *Let the conditions of Theorem 2 hold.*

Then, for any $\Phi\in\mathbf{PC}$ IP (11), (5) has a unique solution $Z^*(t)\in C(J,\mathbb{R}^n)$.

Proof. Assume the contrary, that there exists a maximal solution $Z^*_{Max}(t)\in C([a,b_{Max}],\mathbb{R}^n)$ of IP (11), (5) with the interval of its existence $[a,b_{Max}]$ (which is closed from the right), and then we have that $b_{Max} < \infty$. We consider the most difficult case when $b_{Max} = t_k$ for some $k \in \mathbb{N}$. Then, let $b \in \Delta_{k+1}$ be arbitrary and define the Banach space

$$\mathbf{B}^b_{k+1} = \{G_b(s) : [t_k, b] \to \mathbb{R}^n | G_b(s) = G(s)|_{[t_k,b]}, G(s) \in \mathbf{B}, \|G_b(s)\|_b = \sup_{s\in[t_k,b]} |G(s)|\},$$

and the nonempty, closed and convex subsets set

$$\mathbf{H}^b_{k+1} \subset \mathbf{B}^b_{k+1} = \{G_b(s) \in \mathbf{B}^b_{k+1} | G_b(t_k) = Z^*_{Max}(t)\}.$$

As above, $\mathbf{B}^b_{k+1}$ endowed with the metric function

$$d^b_{k+1}(G_b, \overline{G}_b) = \|G_b - \overline{G}_b\|_b = \sup_{s\in[t_k,b]} |G(s) - \overline{G}(s)|$$

for any $G_b(s), \overline{G}_b(s) \in \mathbf{H}^b_{k+1}$ is a complete metric space.

Consider the system (11) for $t = t_k$, $s \in (t_k, b]$ and introduce the initial condition:

$$Z(s) = Z^*_{Max}(t) \quad \text{for} \quad s \in [t_k - h, t_k]. \tag{22}$$

In the same way that the estimations (18) and (19) are established, for arbitrary $G_b(s), \overline{G}_b(s) \in \mathbf{H}^b_{k+1}$ we obtain that

$$\begin{aligned} \|\mathbf{T}G_b - \mathbf{T}\overline{G}_b\|_b &\le \sup_{s\in[t_k,b]} (1 - \omega(t_{k+1} - s))\|G_b - \overline{G}_b\|_b \\ &\le (1 - (t_{k+1} - b)^{q^M - q(t)})\|G_b - \overline{G}_b\|_b, \end{aligned} \tag{23}$$

$$\|\mathbf{K}G_b - \mathbf{K}\overline{G}_b\|_b \le \frac{U_1(b - t_k)^{q^M}}{\Gamma(1 + q^M)} \|G_b - \overline{G}_b\|_b, \tag{24}$$

and from (23) and (24), we obtain that

$$\|\mathbf{R}G_b - \mathbf{R}\overline{G}_b\|_b \le \left(\frac{U_1(b - t_k)^{q^M}}{\Gamma(1 + q^M)} + \left(1 - (t_{k+1} - b)^{q^M - q^m}\right)\right) \|G_b - \overline{G}_b\|_b. \tag{25}$$

We will choose $b \in \Delta_{k+1}$, such that

$$\frac{U_1(b - t_k)^{q^M}}{\Gamma(1 + q^M)} + 1 - (t_{k+1} - b)^{q^M - q^m} < 1. \tag{26}$$

After elementary calculations, we obtain that (26) is equivalent to the following inequality

$$\frac{U_b}{\Gamma(1 + q^M)} < \frac{(t_{k+1} - b)^{q^M - q^m}}{(b - t_k)^{q^M}}, \tag{27}$$

which is obviously fulfilled when $b - t_k$ is sufficiently small (note that if $b_1 > b_2$, then $U_{b_1} \ge U_{b_2}$, and since $q^M - q^m > 0$, $(t_{k+1} - b_1)^{q^M - q^m} \ge (t_{k+1} - b_2)^{q^M - q^m}$). Thus, choosing $b = b^*$, for which the inequality (27) holds, we obtain that the operator $\mathbf{R}$ is a contraction map and has a unique fixed point $G_{b^*}(s) \in \mathbf{H}^{b^*}_{t_{k+1}}$, and hence we obtain a prolongation of the maximal solution $Z^*_{Max}(t)$, which is impossible. Then, we conclude that the interval of existence of the maximal solution $Z^*_{Max}(t)$ must be open from the right, i.e., $[a, b_{Max})$.

Let us assume that $b_{Max} < \infty$. Then, taking into account (11) and (12), we can conclude that passing to the limit when $s \to b_{Max} - 0$, we obtain that system (10) is fulfilled for $s = b_{Max}$ too, and hence we obtain a prolongation of the maximal solution $Z^*_{Max}(t)$, which contradicts our assumption that $Z^*_{Max}(t)$ is the maximal solution of IP (11), (5). Thus, it follows that $b_{Max} = \infty$, which completes the proof of the theorem. □

4. Fundamental Matrices and Integral Representation

In this section, we will apply the results from the former to prove the existence and uniqueness of the fundamental matrix of the studied homogeneous system (2), and then using the fundamental matrices we will obtain integral representation of the solutions of the systems (2) and (1).

Let $s \in J$ be an arbitrary fixed point and denote $J_s = [a, \infty)$. Consider the matrix-valued function $t \to \mathfrak{N}(t,s) = \{n_{kj}(t,s)\}_{k,j=1}^{n}$, and let $\mathfrak{N}^j(t,s) = col(n_{1j}(t,s), \dots, n_{nj}(t,s))$ be its j-th column. From Theorems 2 and 3, it follows that $\mathfrak{N}(t,s)$ for any $j \in \langle n \rangle$ is the unique solution of IP (2), (3) for $t \in J_s^0 = [s, \infty)$ and the initial function $\Phi^j(t,s) = I^j$ when $t = s$ and $\Phi^j(t,s) = \mathbf{0}$ for $t < s$. Thus, we call the matrix $C(t,s)$ the fundamental matrix (FM) of system (2). Analogically, let $s \in [a-h, a]$ be an arbitrary fixed point and consider the matrix-valued function $t \to \mathfrak{M}(t,s) = \{m_{kj}(t,s)\}_{k,j=1}^{n}$ with its j-th column $\mathfrak{M}^j(t,s) = col(m_{1j}(t,s), \dots, m_{nj}(t,s))$ for $j \in \langle n \rangle$. As above, by virtue of Theorems 2 and 3, we have that $\mathfrak{M}^j(t,s)$ for any $j \in \langle n \rangle$ is the unique solution of IP (2), (3) in the interval $t \in J^0$ for the initial function $\Psi^j(t,s) = I^j$ when $a - h \le s \le t \le a$ and $\Psi^j(t,s) = \mathbf{0}$ for $t < s$ and is called the extended fundamental matrix (EFM) of (2). It is clear that $\mathfrak{N}(t,a) = \mathfrak{M}(t,a)$ if and only if $s = a$.

Remark 3. *Since the columns $\mathfrak{N}^j(t,s)$ and $\mathfrak{M}^j(t,s)$ satisfy system (9), where the differentiation is of constant order, then according to Theorem 6 in [27], it follows that $\mathfrak{N}^j(t,\cdot) \in AC(J_s, \mathbb{R}^n)$ for any fixed $s \in J^0$ and $\mathfrak{N}^j(\cdot,s)$ for any fixed $t \in J_s^0$ are continuous when $s \neq t$, have the first kind jump for $s = t$, and at the jump points both limits $\mathfrak{N}^j(t,t-0)$ and $\mathfrak{N}^j(t,t+0)$ exist inclusively when $s = t = a$. Theorem 7 in [27] implies that for any fixed $s \in [a-h,a]$, we have that $\mathfrak{M}^j(t,\cdot) \in AC(J, \mathbb{R}^n)$ for any fixed $t \in [a-h, \infty)$ are continuous when $s \neq t$ has the first kind jump for $s = t$, and at the jump points both limits $\mathfrak{M}^j(t,t-0)$ and $\mathfrak{M}^j(t,t+0)$ exist inclusively when $s = t = a$.*

Let $\Phi \in \mathbf{PC}^*$ be arbitrary and define the function

$$X^{\mathfrak{M}}(t) = \int_{a-h}^{a} \mathfrak{M}(t,\tau) \mathrm{d}_\tau \overline{\Phi}(\tau), \tag{28}$$

where $\Phi(t) \equiv \overline{\Phi}(t)$ for $s \in (a-h, a]$ and $\overline{\Phi}(a-h) = \mathbf{0}$.

Theorem 4. *Let the conditions of Theorem 2 hold.*

Then, for an arbitrary $\Phi \in \mathbf{PC}^*$, the function $X^{\mathfrak{M}}(t)$ defined via (28) is the unique solution of IP (2), (3) for $t \in J^0$.

Proof. The proof uses the same ideas as in [24,25], but we will present it because of some technical differences in the case of variable order differentiation. First, integrating by parts the equality (28), we have

$$X^{\mathfrak{M}}(t) = \int_{a-h}^{a} \mathfrak{M}(t,\tau)\, \mathrm{d}_\tau \overline{\Phi}(\tau) = \mathfrak{M}(t,a)\overline{\Phi}(a) - \int_{a-h}^{a} \overline{\Phi}(\tau) \mathrm{d}_\tau \mathfrak{M}(t,\tau). \tag{29}$$

Then, from (29) (see [28], page 229, point 5) and the hypothesis $(\mathfrak{S})$, it follows that

$$\mathrm{Var}_{\tau \in [a-h,a]} |\mathfrak{M}(t,\cdot)| < \infty$$

and $\partial_\tau \mathfrak{M}(\cdot,\tau) \in BL_1^{loc}([a-h,a],\mathbb{R}^n)$. Thus, taking into account Lemma 1 and Remark 3 and applying the Fubini theorem for the left side of (2), we obtain that

$$\begin{aligned}
({}^C\mathfrak{D}_a^{q(t)} X^{\mathfrak{M}})(t) &= \frac{1}{\Gamma(1-q(t))}\int_a^t (t-s)^{-q(t)} \frac{\mathrm{d}}{\mathrm{d}s} X^{\mathfrak{M}}(s)\mathrm{d}s \\
&= \frac{1}{\Gamma(1-q(t))}\int_a^t (t-s)^{-q(t)} \frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{a-h}^a \mathfrak{M}(s,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau)\right)\mathrm{d}s \\
&= \frac{1}{\Gamma(1-q(t))}\int_a^t (t-s)^{-q(t)} \left(\int_{a-h}^a \partial_s\mathfrak{M}(s,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau)\right)\mathrm{d}s \\
&= \int_{a-h}^a \left(\frac{1}{\Gamma(1-q(t))}\int_a^t (t-s)^{-q(t)}\partial_s\mathfrak{M}(s,\tau)\mathrm{d}s\right)\mathrm{d}_\tau\overline{\Phi}(\tau) \\
&= \int_{a-h}^a ({}^C\mathfrak{D}_a^{q(t)}\mathfrak{M}(t,\tau))\mathrm{d}_\tau\overline{\Phi}(\tau).
\end{aligned} \tag{30}$$

For the right side of (2), applying again the Fubini theorem, we have

$$\begin{aligned}
\int_{-h}^0 [\mathrm{d}_\theta U(t,\theta)]X(t+\theta) &= \int_{-h}^0 [\mathrm{d}_\theta U(t,\theta)]X^{\mathfrak{M}}(t+\theta) \\
&= \int_{-h}^0 [\mathrm{d}_\theta U(t,\theta)]\left(\int_{a-h}^a \mathfrak{M}(t+\theta,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau)\right) \\
&= \int_{a-h}^a \left(\int_{-h}^0 [\mathrm{d}_\theta U(t,\theta)]\mathfrak{M}(t+\theta,\tau)\right)\mathrm{d}_\tau\overline{\Phi}(\tau)
\end{aligned} \tag{31}$$

and hence from (30) and (31), it follows that

$$\int_{a-h}^a \left({}^C\mathfrak{D}_a^{q(t)}\mathfrak{M}(t,\tau) - \int_{-h}^0 [\mathrm{d}_\theta U(t,\theta)]\mathfrak{M}(t+\theta,\tau)\right)\mathrm{d}_\tau\overline{\Phi}(\tau) = \mathbf{0}$$

which ultimately holds since $\mathfrak{M}(t,\tau)$ is the EFM for system (2).

We will establish that the function $X^{\mathfrak{M}}(t)$ defined via (28) satisfies the initial condition (3). Indeed, let $t \in [a-h,a]$ be arbitrary. Then, from (27), it follows that

$$\begin{aligned}
X^{\mathfrak{M}}(t) &= \int_{a-h}^a \mathfrak{M}(t,\tau)\,\mathrm{d}_\tau\overline{\Phi}(\tau) = \int_{a-h}^a \mathfrak{M}(t,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau) + \int_t^a \mathfrak{M}(t,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau) \\
&= \int_{a-h}^a \mathbf{I}\mathrm{d}_\tau\overline{\Phi}(\tau) + \int_t^a \Theta(t,\tau)\mathrm{d}_\tau\overline{\Phi}(\tau) = \overline{\Phi}(t) - \overline{\Phi}(a-h) = \Phi(t).
\end{aligned}$$

and hence $X^{\mathfrak{M}}(t)$ is the unique solution of the IP (2), (3) for $t \in J^0$.

Consider for an arbitrary function $\mathfrak{F}(t) \in BL_1^{loc}(J,\mathbb{R}^n)$, the auxiliary system for

$$({}^C\mathfrak{D}_a^{q(t)} T^*)(t) = \mathfrak{F}(t) \quad \text{and} \quad T^*(a) = \mathfrak{F}(a), \tag{32}$$

where $T^*(t) = \mathfrak{F}(a) + \int_a^t T(\eta)\mathrm{d}\eta$ and then by virtue of Theorems 2 and 3 IP (32), possesses a unique solution. □

Theorem 5. *Let the conditions of Theorem 2 hold.*

Then, for $t \in J^0$, the function $X_{T^*}(t)$ defined via the equality

$$X_{T^*}(t) = \int_a^t \mathfrak{N}(t,s)T^*(s)\mathrm{d}s = \int_a^t \mathfrak{P}(t,s)\mathrm{d}s \tag{33}$$

is the unique solution of IP (1), (3) with the initial function $\Phi(t) \equiv \mathbf{0}$.

Proof. Substituting the function $X_{T^*}(t)$ in the left side of system (1) and applying the Fubini theorem for any $t \in J^0$ and formula (2.1.40) in [20], we have

$$\begin{aligned}
({}^C\mathfrak{D}_a^{q(t)} X_{T^*})(t) &= \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\eta)^{-q(t)} X_{T^*}(\eta)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\eta)^{-q(t)} \left(\int_a^\eta \mathfrak{P}(t,s)\mathrm{d}s\right)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\int_s^t (t-\eta)^{-q(t)}\mathfrak{P}(\eta,s)\mathrm{d}s\right)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\int_s^t (t-\eta)^{-q(t)}\mathfrak{P}(\eta,s)\mathrm{d}\eta\right)\mathrm{d}s \\
&= \frac{1}{\Gamma(1-q(t))}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t W(t,s)\mathrm{d}s \\
&= \frac{1}{\Gamma(1-q(t))}\int_a^t \partial_t W(t,s)\mathrm{d}s + \frac{1}{\Gamma(1-q(t))}\lim_{s\to t-0} W(t,s),
\end{aligned} \tag{34}$$

where $W(t,s) = \int_a^t (t-\eta)^{-q(t)}\mathfrak{P}(\eta,s)\mathrm{d}\eta$.

For the second addend in the right side of (34), we have

$$\begin{aligned}
\frac{1}{\Gamma(1-q(t))}\lim_{s\to t-0} W(t,s) &= \frac{1}{\Gamma(1-q(t))}\lim_{s\to t-0}\int_a^t (t-\eta)^{-q(t)}\mathfrak{P}(\eta,s)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-q(t))}\int_a^t (t-\eta)^{-q(t)}\lim_{s\to \eta-0}\mathfrak{P}(\eta,s)\mathrm{d}\eta \\
&= \frac{1}{\Gamma(1-q(t))}\int_a^t (t-\eta)^{-q(t)}\lim_{s\to \eta-0}\mathfrak{N}(\eta,s)T^*(s)\mathrm{d}\eta \\
&= ({}^C\mathfrak{D}_a^{q(t)} T^*)(t)
\end{aligned} \tag{35}$$

and then from (34) and (35), it follows for $t \in J^0$ that

$$({}^C\mathfrak{D}_a^{q(t)} X_{T^*})(t) = \int_a^t T^*(s)({}^C\mathfrak{D}_a^{q(t)}\mathfrak{N}(t,s))(t)\mathrm{d}s + ({}^C\mathfrak{D}_a^{q(t)} T^*)(t). \tag{36}$$

Substituting the function $X_{T^*}(t)$ in the right side of system (1) and applying again the Fubini theorem, we obtain for $t \in J^0$ that

$$\begin{aligned}
&\int\limits_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X(t+\theta)+\mathfrak{F}(t)=\int\limits_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]X_{T^*}(t+\theta)+\mathfrak{F}(t)\\
&=\int\limits_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]\left(\int_a^{t+\theta}\mathfrak{N}(t+\theta,s)T^*(s)\mathrm{d}s\right)+\mathfrak{F}(t)\\
&=\int\limits_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]\left(\int_a^{t}\mathfrak{N}(t+\theta,s)T^*(s)\mathrm{d}s\right)+\mathfrak{F}(t)\\
&=\int\limits_{a}^{t}T^*(s)\left(\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]\mathfrak{N}(t+\theta,s)\right)\mathrm{d}s+\mathfrak{F}(t),
\end{aligned} \tag{37}$$

taking into account that since $\mathfrak{N}(t,s)$ is the FM, then $\int_t^{t+\theta}\mathfrak{N}(t+\theta,s)T^*(s)\mathrm{d}s=\mathbf{0}$. Then, from (36) and (37), it follows that

$$\int_a^t T^*(s)({}^{C}\mathfrak{D}_a^{q(t)}\mathfrak{D}(t,s))(t)\mathrm{d}s+({}^{C}\mathfrak{D}_a^{q(t)}T^*)(t)=\int\limits_a^t T^*(s)\left(\int_{-h}^{0}[\mathrm{d}_\theta U(t,\theta)]\mathfrak{N}(t+\theta,s)\right)\mathrm{d}s+\mathfrak{F}(t),$$

the equality of which ultimately holds for $t\in J^0$, since $\mathfrak{N}(t,s)$ is the FM and $T^*(t)$ is the unique solution of IP (32). □

Corollary 1. *Let the conditions of Theorem 2 hold.*

Then, for any $\Phi\in\mathbf{PC}^*$, IP (1), (3) possess a unique solution with the following integral representation:

$$X(t)=X^{\mathfrak{M}}(t)+X_{T^*}(t)=\int\limits_{a-h}^{a}\mathfrak{M}(t,s)\mathrm{d}_s\overline{\Phi}(s)+\int\limits_a^t\mathfrak{N}(t,s)T^*(s)\mathrm{d}s. \tag{38}$$

Proof. The statement follows immediately from the superposition principle and Theorems 4 and 5. □

Example 1. *It is well known that in all applications, even in the cases with integer order derivatives, the Lebesgue decomposition of the kernels with bounded variation does not include a singular term (see [23]). Consider the following system*

$${}^{C}\mathfrak{D}_{a+}^{q(t)}X(t)=A(t)X(t-\sigma(t))+B(t)\int_{-h}^{0}\theta X(t+\theta)\mathrm{d}\theta+\mathfrak{F}(t), \tag{39}$$

where the kernel of system (39) has the form

$$U(t,\theta)=U_{st}(t,\theta)+U_{ac}(t,\theta)=A(t)H(\theta+\sigma(t))+B(t)\frac{\theta^2}{2},$$

i.e., substituting $U(t,\theta)=A(t)H(\theta+\sigma(t))+B(t)\frac{\theta^2}{2}$ in (1), we will obtain system (39). It is simple to check that if $\sigma(t)\in C(J,[0,h])$, $A(t),B(t)\in BL_1^{loc}(J,\mathbb{R}^{n\times n})$ and $\mathfrak{F}\in BL_1^{loc}(J,\mathbb{R}^n)$, then all conditions of the hypothesis $\mathfrak{S}$ hold. Thus, all conditions of Theorem 2 hold, and applying Corollary 1, we obtain that for any $\Phi\in\mathbf{PC}^*$ IP (39), (3) has a unique solution $X_\Phi(t)$, which possesses the integral representation (38):

$$X_\Phi(t)=X^{\mathfrak{M}}(t)+X_{T^*}(t)=\int\limits_{a-h}^{a}\mathfrak{M}(t,s)\mathrm{d}\overline{\Phi}(s)+\int\limits_a^t\mathfrak{N}(t,s)T^*(s)\mathrm{d}s.$$

where $\mathfrak{N}(t,s)$, $\mathfrak{M}(t,s)$ *are the fundamental matrices of the corresponding homogeneous system of* (39) *(i.e.,* $\mathfrak{F}(t) \equiv 0$*), and* $T^*(s)$ *is the unique solution of IP* (32)*.*

5. Conclusions and Comments

As already mentioned, the main obstacle in the study of fractional equations or systems including a VO integral and/or differential fractional operator is that, in general (without special additional conditions), the VO fractional differential operator is not left inverse of the corresponding VO integral operator. So, generally speaking, it is not possible to directly transform fractional equations or systems including a VO fractional operator into an equivalent Volterra-type integral equation or system with eventually a low singularity, which is the standard approach in the CO case.

As far as we know, VO delayed equations are studied for the existence and uniqueness of the solution only in the partial cases when the VO of differentiation $q(t)$ is either a step function (with finitely many jumps of the first kind [18]) or a continuous function, mainly via numerical methods [16]. These facts were the main reason for us to introduce another approach which considers that the general case with VO of differentiation $q(t)$ is a piecewise continuous function. Our approach consists of finding an associated one-parameter family of CO fractional systems which is "almost" equivalent to the VO system in the appropriate sense and can replace the study of the initial problem (IP) for the VO fractional system with the study of an IP for the one-parameter family of CO fractional systems. Using this approach, we prove theorems for existence and uniqueness of a continuous solution of the considered initial problem (IP) for a class linear systems with distributed delays and VO fractional derivatives of Caputo type as well as concerning the prolongation of the solutions on the half-axis.

In addition, we prove the existence and uniqueness of a fundamental matrix and an extended fundamental matrix of the studied homogeneous system. The presence of fundamental matrices allows us to establish integral representations of the solutions of the studied IP.

Notably, the obtained integral representation as well as and the presence of fundamental matrices can be used as main tools in future research of the stability behaviors of the studied systems.

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