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# On a Faster Iterative Method for Solving Fractional Delay Differential Equations in Banach Spaces

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**Abstract:** In this paper, we consider a faster iterative method for approximating the fixed points of generalized $\alpha$-nonexpansive mappings. We prove several weak and strong convergence theorems of the considered method in mild conditions within the control parameters. In order to validate our findings, we present some nontrivial examples of the considered mappings. Furthermore, we show that the class of mappings considered is more general than some nonexpansive-type mappings. Also, we show numerically that the method studied in our article is more efficient than several existing methods. Lastly, we use our main results to approximate the solution of a delay fractional differential equation in the Caputo sense. Our results generalize and improve many well-known existing results.

**Keywords:** fixed point; iterative method; fractional delay differential equation; strong convergence

**MSC:** 05C07; 05C09; 05C31; 05C76; 05C99

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## 1. Introduction

In this paper, let $\mathcal{M}$, $\mathcal{G}$ and $\mathbb{N}$ denote a Banach space, a nonempty subset of $\mathcal{M}$ and the set of all natural numbers, respectively. The mapping $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ is said to have a fixed point $u \in \mathcal{G}$ if $\mathcal{L}u = u$. The set of all fixed points of $\mathcal{L}$ is denoted by $F(\mathcal{L})$. If there exists a constant $l \in [0,1)$, such that $\|\mathcal{L}u - \mathcal{L}v\| \leq l\|u - v\|$ for any $u, v \in \mathcal{G}$ then $\mathcal{L}$ is called a contraction, and it is called nonexpansive if $\|\mathcal{L}u - \mathcal{L}v\| \leq \|u - v\|$ for all $u, v \in \mathcal{G}$. It is called quasi-nonexpansive if $F(\mathcal{L}) \neq \emptyset$ and $\|\mathcal{L}u - u^*\| \leq \|u - u^*\|$ for all $u \in \mathcal{G}$ and $u^* \in F(\mathcal{L})$.

The numerous applications of nonexpansive mappings to the solutions of problems in applied science and engineering have drawn the attention of many authors to study their basic concepts and their generalizations.

A prominent generalization of nonexpansive mappings was given in 2008 by Suzuki [1]. The author introduced a nonexpansive-type mapping known as generalized nonexpansive mappings or mappings satisfying the condition $(C)$.

**Definition 1.** *A mapping $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ is said to satisfy condition $(C)$ if*

$$\frac{1}{2}\|u - \mathcal{L}u\| \leq \|u - v\| \;\Rightarrow\; \|\mathcal{L}u - \mathcal{L}v\| \leq \|u - v\|, \tag{1}$$

*for all $u, v \in \mathcal{G}$.*

Another generalization of the class of mappings fulfilling (1) was given in 2017 by Pant and Shukla [2]. This class of mappings is known as generalized $\alpha$–nonexpansive mappings.

**Definition 2.** *A mapping $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ is called generalized $\alpha$-nonexpansive if a constant $\alpha \in [0,1)$ exists and for each $u, v \in \mathcal{G}$ the following holds:*

$$\frac{1}{2}\|u - \mathcal{L}u\| \le \|u - v\| \;\Rightarrow\; \|\mathcal{L}u - \mathcal{G}v\| \le \alpha\|\mathcal{L}u - v\| + \alpha\|\mathcal{L}v - u\| + (1 - 2\alpha)\|u - v\|. \quad (2)$$

In recent years, many authors have studied this class of mappings; see, e.g., [3–8].

One of the widely used approaches for approximating the fixed points of operators that can be nonlinear is the use of iterative methods. Many iterative methods have been introduced in the past few years. Some well-known iterative methods in the literature are given in [9–13].

The following iterative method is known as the Mann [14] iterative method:

$$\begin{cases} u_0 \in \mathcal{G}, \\ u_{m+1} = (1 - \alpha_m)u_m + \alpha_m \mathcal{L}u_m, \end{cases} \quad m \in \mathbb{N}, \quad (3)$$

where the sequence $\{\alpha_m\} \in (0,1)$.

The Ishikawa iterative method was introduced in 1974 by Ishikawa [15] to approximate the fixed points of nonexpansive mappings as follows:

$$\begin{cases} u_0 \in \mathcal{G}, \\ v_m = (1 - \beta_m)u_m + \beta_m \mathcal{L}u_m, \\ u_{m+1} = (1 - \alpha_m)u_m + \alpha_m \mathcal{L}u_m. \end{cases} \quad m \in \mathbb{N}, \quad (4)$$

where the sequences $\{\alpha_m\}$, $\{\beta_m\} \in (0,1)$.

In 2000, Noor [16] proposed the following three-steps iterative method, which incorporates the Mann and the two-step man (Ishikawa) iterative methods:

$$\begin{cases} u_0 \in \mathcal{G}, \\ w_m = (1 - \gamma_m)u_m + \gamma_m \mathcal{L}u_m, \\ v_m = (1 - \beta_m)u_m + \beta_m \mathcal{L}w_m, \\ u_{m+1} = (1 - \alpha_m)u_m + \alpha_m \mathcal{L}v_m. \end{cases} \quad m \in \mathbb{N}, \quad (5)$$

where the sequences $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\} \in (0,1)$.

A modification of the Ishikawa iterative method was given in 2007 by Agarwal et al. [17]. This method is known as the $S$ iterative method, and it is defined as follows:

$$\begin{cases} u_0 \in \mathcal{G}, \\ v_m = (1 - \beta_m)u_m + \beta_m \mathcal{L}u_m, \\ u_{m+1} = (1 - \alpha_m)\mathcal{L}u_m + \alpha_m \mathcal{L}v_m, \end{cases} \quad m \in \mathbb{N}, \quad (6)$$

where the sequences $\{\alpha_m\}$, $\{\beta_m\} \in (0,1)$. For contraction mappings, the authors proved that (6) has a better rate of convergence than the Mann iterative method.

The Picard–Mann iterative method was introduced in 2013 by Khan [18] as follows:

$$\begin{cases} u_0 \in \mathcal{G}, \\ v_m = (1 - \beta_m)u_m + \alpha_m \mathcal{L}u_m, \\ u_{m+1} = \mathcal{L}v_m, \end{cases} \quad m \in \mathbb{N}, \quad (7)$$

where the sequence $\{\alpha_m\} \in (0,1)$.

In 2014, Abbas and Nazir [19] provided the following three-steps iterative method:

$$\begin{cases} u_0 \in \mathcal{G}, \\ w_m = (1-\gamma_m)u_m + \gamma_m \mathcal{L}u_m, \\ v_m = (1-\beta_m)\mathcal{L}u_m + \beta_m \mathcal{L}w_m, \\ u_{m+1} = (1-\alpha_m)h_m + p_m \mathcal{L}u_m, \end{cases} \quad m \in \mathbb{N}, \tag{8}$$

where the sequences $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\} \in (0,1)$. The authors showed that their method has a better rate of convergence than the $S$ iterative method.

Recently, an efficient and faster iterative method for finding the fixed points of contraction mappings was introduce by Okeke [20]; the method, which is called the Picard–Ishikawa iterative method, is as follows:

$$\begin{cases} u_0 \in \mathcal{G}, \\ w_m = (1-\beta_m)u_m + \beta_m \mathcal{L}u_m, \\ v_m = (1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m, \\ u_{m+1} = \mathcal{L}v_m, \end{cases} \quad m \in \mathbb{N}, \tag{9}$$

where the sequences $\{\alpha_m\}, \{\beta_m\} \in (0,1)$. The author proved that (9) has a better convergence rate than most of the iterative methods defined above for contraction mappings. Now, it is natural to ask the following question:

*Is it possible to approximate the fixed points of a class of mappings, such as generalized $\alpha$-nonexpansive mappings, that is more general than those studied with the iterative methods (3)–(8)?*

One of our aims in this article is to give an affirmative answer to the above question.

On the other hand, fractional calculus is utilized in different aspects of mathematics as a result of its several applications in modeling various physical phenomena in science and engineering. The notion of fractional calculus emanated from the fact $D^{\alpha}(f(x))$, where $\alpha$ is a non-integer. Over time, various researchers, such as Euler, Riemann–Liouville, Leibniz, Wallis and Bernoulli have made a number of contributions in this research direction. Fractional calculus can be applied to several fields of sciences. For instance, fluid mechanics [21], dynamic of viscoelastic materials [22], propagation of spherical flames [23], viscoelastic materials [24] and electromagnetism [25].

In the real world, differential equations (DEs) are utilized to model many physical problems. Most of these problems are more complex and cannot be modeled by using the classical DEs [26–28]. In order to model these complex problems, many authors have used a new approach known as fractional differential equations (FDEs). FDEs are widely used in the mathematical modeling of real-life physical problems, and this is as a result of their several applications in real-world science and engineering problems, such as solid mechanics, economics, oscillation of earthquakes, continuum and statistical mechanics, anomalous transport, rough substrates, dynamics of interfaces between soft nanoparticles and solid mechanics, fluid-dynamic traffic models and bio–engineering and colored noise (see [29] and the references therein).

Delay differential equations (DDEs) have been applied in various aspects of science and engineering. DDEs are suitable for physical systems, depend on past data and simplify the ordinary differential equation. FDDs have been applied in different areas of mathematical modelings, such as epidemiology, population dynamics, physiology, immunology and neural networks [29].

It is important to note that there is no precise approach for finding an analytical or exact solution for every FDDE. Different techniques have been developed for solving these problems.

Motivated by the ongoing research in these directions, we prove several weak and strong convergence theorems of the Picard–Ishikawa method (9) for fixed points of generalized $\alpha$-nonexpansive mappings in mild conditions within the control parameters. We present some nontrivial examples of generalized $\alpha$–nonexpansive mappings. We further show that the class of generalized $\alpha$–nonexpansive mappings is more general than some nonexpansive-type mappings. The efficiency of the studied method over several existing methods is tested using numerical examples. Lastly, we use our main results to solve a delay fractional differential equation in the Caputo sense. Our results generalize and improve many well-known existing results.

The article is organized as follows: In Section 2, some useful definitions and lemmas are recalled. The convergence results of the considered method and numerical examples that validate our findings are presented in Section 3. In Section 4, we consider the application of the considered algorithm in approximating the solution of a delay differential equation in the Caputo sense.

## 2. Preliminaries

The following definitions, lemmas and propositions will be used in obtaining our main results:

**Definition 3** ([30]). *The Opial condition is said to be satisfied by a Banach space $\mathcal{M}$ if, for any sequence $\{u_m\}$ in $\mathcal{M}$, $\{u_m\} \rightharpoonup u \in \mathcal{M}$ implies*

$$\limsup_{m\to\infty} \|u_m - u\| < \limsup_{m\to\infty} \|u_m - v\|, \forall\, v \in \mathcal{M} \text{ with } v \neq u.$$

**Definition 4** ([30]). *If $\mathcal{M}$ is a Banach space, such that for each $\epsilon \in (0,2]$ there exists $\delta > 0$ with $u, v \in \mathcal{M}$ satisfying $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| > \epsilon$ such that we have $\|\frac{u+v}{2}\| < 1 - \delta$, then we say that $\mathcal{M}$ is uniformly convex.*

**Lemma 1** ([30]). *Let $\{\mu_m\}$ be any sequence that satisfies $0 < u \leq \mu_m \leq v < 1$ for all $m \geq 1$ and $\{u_m\}$ and let $\{v_m\}$ be any sequences in a uniformly convex Banach space $\mathcal{M}$, such that*

$$\begin{aligned} \limsup_{m\to\infty} \|u_m\| &\leq w, \\ \limsup_{m\to\infty} \|v_m\| &\leq w \text{ and} \\ \limsup_{m\to\infty} \|\mu_m u_m + (1-\mu_m)v_m\| &= w \end{aligned}$$

*hold for some $w \geq 0$. Then, $\lim_{m\to\infty} \|u_m - v_m\| = 0$.*

Let $\mathcal{M}$ be a Banach space and $\mathcal{G}$ be a nonempty closed convex subset of $\mathcal{M}$. Let $\{u_m\}$ be a bounded sequence in $\mathcal{M}$. For $u \in \mathcal{M}$, we set

$$r(u, \{u_m\}) = \limsup_{m\to\infty} \|u_m - u\|.$$

The asymptotic radius of $\{u_m\}$ relative to $\mathcal{G}$ is defined by

$$r(\mathcal{G}, \{u_m\}) = \inf\{r(u, \{u_m\}) : u \in \mathcal{G}\}.$$

The asymptotic center of $\{u_m\}$ relative to $\mathcal{G}$ is given as:

$$A(\mathcal{G}, \{u_m\}) = \{u \in \mathcal{G} : r(u, \{u_m\}) = r(\mathcal{G}, \{u_m\})\}.$$

It is well known that $A(\mathcal{G}, \{u_m\})$ consists of exactly one point in a uniformly convex Banach space.

Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be a nonlinear operator, such that $F(\mathcal{L}) \neq \emptyset$. Then, $I - \mathcal{L}$ is called demiclosed at zero if for any $u_m \in \mathcal{G}$ the following implication holds:

$$u_m \rightharpoonup u \text{ and } (I - \mathcal{L})u_m \to 0 \implies u \in F(\mathcal{L}).$$

**Lemma 2** ([31])**.** *Let* $\{_m\}$ *and* $\{\sigma_m\}$ *be sequences in* $[0, \infty)$*, such that:*

$$_{m+1} \leq (1 - \lambda_m)_m + \lambda_m \sigma_m,$$

*where* $\lambda_m \in (0, 1)$ *for all* $m \in \mathbb{N}$, $\sum_{m=0}^{\infty} \lambda_m = \infty$ *and* $\sigma_m \geq 0$ *for all* $m \in \mathbb{N}$*; then,*

$$0 \leq \limsup_{m \to \infty} {}_m \leq \limsup_{m \to \infty} \sigma_m.$$

**Definition 5** ([32])**.** *The condition* $(I)$ *is said to be satisfied by a self-mapping* $\mathcal{L}$ *defined on* $\mathcal{G}$ *if a nondecreasing function* $g : [0, \infty) \to [0, \infty)$ *exists with* $g(0) = 0$*, and if* $g(s) > 0$ *for all* $s > 0$*, such that* $\|u - \mathcal{L}u\| \geq g(d(u, F(\mathcal{L}))))$ *for all* $u \in \mathcal{G}$*, where* $d(u, F(\mathcal{L})) = \inf_{u^* \in F(\mathcal{L})} \|u - u^*\|$*.*

**Proposition 1** ([2])**.** *Let* $\mathcal{M}$ *be a Banach space and* $\mathcal{G}$ *be a nonempty subset of* $\mathcal{M}$ *if* $\mathcal{L}$ *is any self map. Then:*

*(i)* *If* $\mathcal{L}$ *fulfills condition (C) then* $\mathcal{L}$ *satisfies* (2)*.*
*(ii)* *If* $\mathcal{L}$ *satisfies* (2)*, such that* $F(\mathcal{L}) \neq \emptyset$*, then* $\mathcal{L}$ *is quasi-nonexpansive.*
*(iii)* *If* $\mathcal{L}$ *satisfies* (2) *then* $F(\mathcal{L})$ *is closed. Moreover, if* $\mathcal{M}$ *is strictly convex and* $\mathcal{G}$ *is convex then* $F(\mathcal{L})$ *is also convex.*
*(iv)* *If* $\mathcal{L}$ *satisfies* (2) *then we obtain*

$$\|u - \mathcal{L}v\| \leq \left( \frac{3 + \alpha}{1 - \alpha} \right) \|u - \mathcal{L}u\| + \|u - v\|, \ \forall \ u, v \in \mathcal{G}. \tag{10}$$

**Definition 6** ([33])**.** *The Caputo fractional order derivation of a continuous function F in the closed interval* $[a, b]$ *is defined by*

$$({}^c\mathfrak{D}^q_{a+}F)(s) = \frac{1}{\Gamma(k - q)} \int_a^s (s - t)^{k-q-1} F^{(k)}(t) dt, \tag{11}$$

*where* $k = [q] + 1$*.*

## 3. Weak and Strong Convergence Theorems

In this part of the article, several weak and strong convergence theorems will be stated and proved using the Picard–Ishikawa iterative method (9) for mappings satisfying (2). Furthermore, we provide some novel numerical examples. One of the provided examples will be used to compare the computational efficiency of (9) with some well-known iterative methods in the literature.

**Theorem 1.** *Let* $\mathcal{G}$ *be a closed, convex and nonempty subset of a Banach space* $\mathcal{M}$*. Let* $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ *be a generalized* $\alpha$*-Reich–Suzuki nonexpansive mapping. If* $\{u_m\}$ *is the sequence defined by* (9) *then* $\lim_{m \to \infty} \|u_m - u^*\|$ *exists for all* $u^* \in F(\mathcal{L})$*.*

**Proof.** Assume that $u^* \in F(\mathcal{L})$; then, by Proposition 1(ii) and (9) we obtain

$$\begin{aligned} \|w_m - u^*\| &= \|(1 - \beta_m)u_m + \beta_m \mathcal{L}u_m - u^*\| \\ &\leq (1 - \beta_m)\|u_m - u^*\| + \beta_m \|\mathcal{L}u_m - u^*\| \\ &\leq (1 - \gamma_m)\|u_m - u^*\| + \gamma_m \|u_m - u^*\| \\ &\leq \|u_m - u^*\|. \end{aligned} \tag{12}$$

Again, by (12) we obtain

$$
\begin{aligned}
\|v_m - u^*\| &= \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m - u^*\| \\
&\le (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|\mathcal{L}w_m - u^*\| \\
&\le (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|w_m - u^*\| \\
&\le \|u_m - u^*\|. \qquad (13)
\end{aligned}
$$

From (13), we obtain

$$
\begin{aligned}
\|u_{m+1} - u^*\| &= \|\mathcal{L}v_m - u^*\| \\
&\le \|v_m - u^*\| \\
&\le \|u_m - u^*\|, \qquad (14)
\end{aligned}
$$

which means that the sequence $\{\|u_m - u^*\|\}$ is bounded and decreasing. Therefore, $\lim_{m\to\infty} \|u_m - u^*\|$ exists for all $u^* \in F(\mathcal{L})$. □

**Theorem 2.** *If $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ are defined as demonstrated in Theorem 1 then $F(\mathcal{L}) \neq \emptyset$ if and only if the sequence $\{u_m\}$ is bounded and $\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0$.*

**Proof.** We have shown in Theorem 1 that $\{u_m\}$ is a bounded sequence and $\lim_{m\to\infty} \|u_m - u^*\|$ exists for any $u^* \in F(\mathcal{L})$. Setting

$$\lim_{m\to\infty} \|u_m - u^*\| = \ell, \qquad (15)$$

it follows from (12) and (15) that

$$\limsup_{m\to\infty} \|w_m - u^*\| \le \limsup_{m\to\infty} \|u_m - u^*\| = \ell. \qquad (16)$$

By Proposition 1(ii), we obtain

$$\limsup_{m\to\infty} \|\mathcal{L}u_m - u^*\| \le \limsup_{m\to\infty} \|u_m - u^*\| = \ell. \qquad (17)$$

Now, from (9), we obtain

$$
\begin{aligned}
\|u_{m+1} - u^*\| &= \|\mathcal{L}u_m - u^*\| \\
&\le \|v_m - u^*\| \\
&= \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m - u^*\| \\
&\le (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|\mathcal{L}w_m - u^*\| \\
&\le (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|w_m - u^*\| \\
&= \|u_m - u^*\| - \alpha_m\|u_m - u^*\| + \alpha_m\|w_m - u^*\|,
\end{aligned}
$$

which implies that

$$\frac{\|u_{m+1} - u^*\| - \|u_m - u^*\|}{\alpha_m} \le \|w_m - u^*\| - \|u_m - u^*\|$$

Therefore,

$$\|u_{m+1} - u^*\| - \|u_m - u^*\| \le \frac{\|u_{m+1} - u^*\| - \|u_m - u^*\|}{\alpha_m} \le \|w_m - u^*\| - \|u_m - u^*\|,$$

which gives us

$$\|u_{m+1} - u^*\| \le \|w_m - u^*\|. \qquad (18)$$

Therefore,

$$\ell \leq \liminf_{m\to\infty} \|w_m - u^*\|. \tag{19}$$

By (16) and (19), we obtain

$$\begin{aligned} \ell &= \lim_{m\to\infty} \|w_m - u^*\| \\ &= \lim_{m\to\infty} \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}u_m - u^*\| \\ &= \lim_{m\to\infty} \|\alpha_m(\mathcal{L}u_m - u^*) + (1-\alpha_m)(u_m - u^*)\| \end{aligned} \tag{20}$$

Combining Lemmas 1, (15), (17) and (20), we obtain

$$\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0.$$

Let $u^* \in A(\mathcal{G}, \{u_m\})$; then,

$$\begin{aligned} r(\{u_m\}, \mathcal{L}u^*) &= \limsup_{m\to\infty} \|u_m - \mathcal{L}u^*\| \\ &\leq \limsup_{m\to\infty} \left\{ \left(\frac{3+\alpha}{1-\alpha}\right) \|u_m - \mathcal{L}u_m\| + \|\mathcal{L}u_m - \ell\| \right\} \\ &= \limsup_{m\to\infty} \left(\frac{3+\alpha}{1-\alpha}\right) \|u_m - \mathcal{L}u_m\| + \limsup_{m\to\infty} \|\mathcal{L}u_m - u^*\| \\ &\leq \limsup_{m\to\infty} \|u_m - u^*\| \\ &= r(\{u_m\}, u^*). \end{aligned} \tag{21}$$

It follows that $\mathcal{L}u^* \in A(\mathcal{G}, \{u_m\})$. By the uniform convexity of $\mathcal{M}$ it is implied that $A(\mathcal{G}, \{u_m\})$ is a unit set and, thus, one obtains $\mathcal{L}u^* = u^*$. Therefore, $F(\mathcal{L}) \neq \emptyset$. □

**Theorem 3.** *If $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ are the same as in Theorem 1, with $F(\mathcal{L}) \neq \emptyset$, then it is the case that if the Opial property is satisfied by $\mathcal{M}$ then $\{u_m\}$ weakly converges to a point in $F(\mathcal{L})$.*

**Proof.** Since $F(\mathcal{L}) \neq \emptyset$, it follows from Theorem 2 and Theorem 1 that $\lim_{m\to\infty} \|u_m - u^*\|$ exists and $\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0$. Now, we show that $\{u_m\}$ has just one weakly sub-sequential limit in $F(\mathcal{L})$. Assume that $k$ and $h$ are two weak sub-sequential limits of $\{u_{m_j}\}$ and $\{u_{m_k}\}$, respectively. From Theorem 2 and the demicloseness of $(I - \mathcal{L})$ at 0, we know that $(I - \mathcal{L})k = 0$. Thus, $\mathcal{L}k = k$ and, by a similar approach, we obtain $\mathcal{L}h = h$. Next, we prove uniqueness. Assume $k \neq h$; then, by the Opial condition, we obtain

$$\begin{aligned} \lim_{m\to\infty} \|u_m - k\| &= \lim_{m_j\to\infty} \|u_{m_j} - k\| < \lim_{m_j\to\infty} \|u_{m_j} - h\| = \lim_{m\to\infty} \|u_m - h\| \\ &= \lim_{m_k\to\infty} \|u_{m_k} - h\| < \lim_{m_k\to\infty} \|u_{m_k} - k\| = \lim_{m\to\infty} \|u_m - k\|. \end{aligned}$$

This shows contraction; therefore, $k = h$. Thus, $\{u_m\}$ weakly converges to an element in $F(\mathcal{L})$. □

**Theorem 4.** *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be the same as in Theorem 1, with $F(\mathcal{L}) \neq \emptyset$. Then, $\{u_m\}$ converges strongly to an element of $F(\mathcal{L})$ if and only if $\lim_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$, where $d(u_m, F(\mathcal{L})) = \inf\{\|u_m - u^*\| : u^* \in F(\mathcal{L})\}$.*

**Proof.** The necessity is not hard to demonstrate, so we will omit it. Next, we show the converse case. Let $u^*$ be any fixed point of $\mathcal{L}$; then, $\liminf d(u_m, F(\mathcal{L})) = 0$. From Theorem 1, one knows that $\lim_{m\to\infty} \|u_m - u^*\|$ exists for each $u^* \in F(\mathcal{L})$ and this implies that

$\liminf_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$. Next, we demonstrate that $\{u_m\}$ is a Cauchy sequence in $\mathcal{G}$. Due to $\liminf_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$, inasmuch as for any $\wp > 0$ a constant $m_0 \in \mathbb{N}$ exists with

$$d(u_m, F(\mathcal{L})) < \frac{\wp}{2}$$
$$\inf\{\|u_m - u^*\| : u^* \in F(\mathcal{L})\} < \frac{\wp}{2},$$

for all $m \geq m_0$. Therefore, $\inf\{\|u_{m_0} - u^*\| : u^* \in F(\mathcal{L})\} < \frac{\wp}{2}$. Therefore, there exists $u^* \in F(\mathcal{L})$, such that

$$\|u_{m_0} - u^*\| < \frac{\wp}{2}.$$

For $n, m \geq m_0$, we obtain

$$\begin{aligned}
\|u_{m+n} - u_m\| &\leq \|u_{m+n} - u^*\| + \|u_m - u^*\| \\
&\leq \|u_{m_0} - u^*\| + \|u_{m_0} - u^*\| \\
&= 2\|u_{m_0} - u^*\| \\
&< \wp.
\end{aligned}$$

It follows that $\{u_m\}$ is a Cauchy sequence in $\mathcal{G}$. Due to the completeness of $\mathcal{G}$, we obtain $\lim_{m\to\infty} u_m = p$ for some $p \in \mathcal{G}$. Furthermore, $\lim_{m\to\infty} d(u_m, F(\mathcal{L})) = 0$ shows that $p \in F(\mathcal{L})$. □

**Theorem 5.** *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be the same as in Theorem 1 with $F(\mathcal{L}) \neq \emptyset$. Suppose $\mathcal{G}$ is compact; then, $\{u_m\}$ strongly converges to a fixed point, e.g., $u^* \in F(\mathcal{L})$.*

**Proof.** Owing to the hypothesis that $F(\mathcal{L}) \neq \emptyset$, we know from Theorem 2 that $\lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0$. Since $\mathcal{G}$ is compact, one can have a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ with $\lim_{m\to\infty} u_{m_j} \to u^* \in \mathcal{G}$. From Proposition 1, one obtains

$$\|u_{m_j} - \mathcal{L}u^*\| \leq \left(\frac{3+\alpha}{1-\alpha}\right)\|\mathcal{L}u_{m_j} - u_{m_j}\| + \|u_{m_j} - u^*\|.$$

On taking $j \to \infty$, $\mathcal{L}u^* = u^*$, i.e., $u^* \in F(\mathcal{L})$. By Theorem 1, $\lim_{m\to\infty} \|u_m - u^*\|$ exists for any $u^* \in F(\mathcal{L})$, and so the sequence $\{u_m\}$ strongly converges to $u^*$. □

**Theorem 6.** *Let $\mathcal{L}$, $\mathcal{G}$, $\mathcal{M}$ and $\{u_m\}$ be defined as in Theorem 1 with $F(\mathcal{L}) \neq \emptyset$. If $\mathcal{L}$ fulfills condition $(I)$ then $\{u_m\}$ strongly converges to an element of $\mathcal{L}$.*

**Proof.** Due to Theorem 2, one obtains

$$\lim_{s\to\infty} \|\mathcal{L}u_m - u_m\| = 0. \tag{22}$$

From (22) and condition (I) in Definition 5, one obtains

$$\lim_{s\to\infty} g(d(u_m, F(\mathcal{L}))) \leq \lim_{m\to\infty} \|\mathcal{L}u_m - u_m\| = 0, \tag{23}$$

which implies that $\lim_{m\to\infty} g(d(u_m, F(\mathcal{L}))) = 0$. We know that $g$ is a nondecreasing self function defined on $[0,\infty)$ with $g(0) = 0$, $g(s) > 0$ for all $s \in (0,\infty)$; therefore, we obtain

$$\lim_{m\to\infty} d(u_m, F(\mathcal{L})) = 0. \tag{24}$$

By Theorem 4, it follows that $\{u_m\}$ strongly converges to an element of $F(\mathcal{L})$. □

Next, a numerical experiment will be carried out on a mapping that satisfies (2) but does not satisfy condition $(C)$, as follows:

**Example 1.** *Let $\mathcal{M} = \mathbb{R}$ and $\mathcal{G} = [3,6]$. We define $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ by*

$$\mathcal{L}u = \begin{cases} \frac{u+3}{2}, & if \;\; u \in [3,4], \\ 3, & if \;\; u \in (4,6]. \end{cases}$$

*For $\alpha = \frac{1}{3}$, we will show that $\mathcal{L}$ is a mapping satisfying (2) in the following cases:*
***Case (I):*** *Let $u, v \in [3,4]$; then,*

$$\begin{aligned} \alpha|\mathcal{L}u - v| + \alpha|\mathcal{L}v - u| + (1-2\alpha)|u-v| & \\ &= \frac{1}{3}\left|\frac{u+3}{2} - v\right| + \frac{1}{3}\left|\frac{v+3}{2} - u\right| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}\left|\frac{3u}{2} - \frac{3v}{2}\right| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{2}|u-v| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{2}|u-v| = |\mathcal{L}u - \mathcal{L}v| \end{aligned}.$$

***Case (II):*** *Let $u \in [3,4]$ and $v \in (4,6]$; then,*

$$\begin{aligned} \alpha|\mathcal{L}u - v| + \alpha|\mathcal{L}v - u| + (1-2\alpha)|u-v| & \\ &= \frac{1}{3}\left|\frac{u+3}{2} - v\right| + \frac{1}{3}|u-3| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}\left|\frac{u}{2} + v - \frac{9}{2}\right| + \frac{1}{3}|u-v| \\ &\geq \frac{1}{3}\left|\frac{3u}{2} - \frac{9}{2}\right| \\ &\geq \frac{1}{2}|u-3| = |\mathcal{L}u - \mathcal{L}v|. \end{aligned}$$

***Case (III):*** *Let $u, v \in (4,6]$; then,*

$$\alpha|\mathcal{L}u - v| + \alpha|\mathcal{L}v - u| + (1-2\alpha)|u-v| \geq 0 = |\mathcal{L}u - \mathcal{L}v|.$$

*From **Case (I)–(III)**, we see that $\mathcal{L}$ is a generalized $\alpha$-nonexpansive mapping for $\alpha = \frac{1}{3}$.*

Moreover, if we take $u = \frac{39}{10}$ and $v = \frac{29}{7}$, we obtain

$$\frac{1}{2}\|u - \mathcal{L}u\| = \frac{9}{40} \leq \frac{17}{70} = \|u - v\|.$$

On the other hand,

$$\|\mathcal{L}u - \mathcal{L}v\| = \frac{9}{20} > \frac{17}{70} = \|u - v\|.$$

Hence, $\mathcal{L}$ does not fulfill the condition $(C)$.
Next, for $\alpha_m = \beta_m = \gamma_m = \frac{m+5}{m+6}$ for $m \in \mathbb{N}$ and $u_0 = 4$, we obtain the following Tables 1 and 2, Figures 1 and 2.

**Table 1.** Convergence behavior of various iterative methods for Example 1.

| Step | Mann | S | Khan | Picard–Ishikawa |
|---|---|---|---|---|
| 1 | 4.0000000000 | 4.0000000000 | 4.0000000000 | 4.0000000000 |
| 2 | 3.6250000000 | 3.3593750000 | 3.3125000000 | 3.2421875000 |
| 3 | 3.3906250000 | 3.1291503906 | 3.0976562500 | 3.0586547852 |
| 4 | 3.2441406250 | 3.0464134216 | 3.0305175781 | 3.0142054558 |
| 5 | 3.1525878906 | 3.0166798234 | 3.0095367432 | 3.0034403838 |
| 6 | 3.0953674316 | 3.0059943115 | 3.0029802322 | 3.0008332180 |
| 7 | 3.0596046448 | 3.0021542057 | 3.0009313226 | 3.0002017950 |
| 8 | 3.0372529030 | 3.0007741677 | 3.0002910383 | 3.0000488722 |
| 9 | 3.0232830644 | 3.0002782165 | 3.0000909495 | 3.0000118362 |
| 10 | 3.0145519152 | 3.0000999841 | 3.0000284217 | 3.0000028666 |
| 11 | 3.0090949470 | 3.0000359318 | 3.0000088818 | 3.0000006943 |
| 12 | 3.0056843419 | 3.0000129130 | 3.0000027756 | 3.0000001681 |
| 13 | 3.0035527137 | 3.0000046406 | 3.0000008674 | 3.0000000407 |
| 14 | 3.0022204460 | 3.0000016677 | 3.0000002711 | 3.0000000099 |
| 15 | 3.0013877788 | 3.0000005993 | 3.0000000847 | 3.0000000024 |
| 16 | 3.0008673617 | 3.0000002154 | 3.0000000265 | 3.0000000006 |
| 17 | 3.0005421011 | 3.0000000774 | 3.0000000083 | 3.0000000001 |
| 18 | 3.0003388132 | 3.0000000278 | 3.0000000026 | 3.0000000000 |
| 19 | 3.0002117582 | 3.0000000100 | 3.0000000008 | 3.0000000000 |
| 20 | 3.0001323489 | 3.0000000036 | 3.0000000003 | 3.0000000000 |
| 21 | 3.0000827181 | 3.0000000013 | 3.0000000001 | 3.0000000000 |
| 22 | 3.0000516988 | 3.0000000005 | 3.0000000000 | 3.0000000000 |
| 23 | 3.0000323117 | 3.0000000002 | 3.0000000000 | 3.0000000000 |
| 24 | 3.0000201948 | 3.0000000001 | 3.0000000000 | 3.0000000000 |
| 25 | 3.0000126218 | 3.0000000000 | 3.0000000000 | 3.0000000000 |

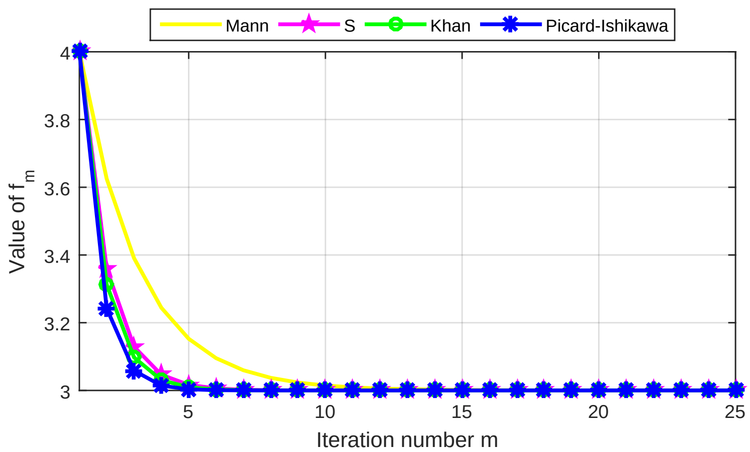


**Figure 1.** Graph corresponding to Table 1.

**Table 2.** Convergence behavior of various iterative methods for Example 1.

| Step | Ishikawa | Noor | Abbas | Picard–Ishikawa |
|---|---|---|---|---|
| 1 | 4.0000000000 | 4.0000000000 | 4.0000000000 | 4.0000000000 |
| 2 | 3.4843750000 | 3.4316406250 | 3.2792968750 | 3.2421875000 |
| 3 | 3.2346191406 | 3.1863136292 | 3.0780067444 | 3.0586547852 |
| 4 | 3.1136436462 | 3.0804205313 | 3.0217870399 | 3.0142054558 |
| 5 | 3.0550461411 | 3.0347127684 | 3.0060850522 | 3.0034403838 |
| 6 | 3.0266629746 | 3.0149834411 | 3.0016995361 | 3.0008332180 |
| 7 | 3.0129148783 | 3.0064674619 | 3.0004746751 | 3.0002017950 |

**Table 2.** *Cont.*

| Step | Ishikawa | Noor | Abbas | Picard–Ishikawa |
|---|---|---|---|---|
| 8 | 3.0062556442 | 3.0027916193 | 3.0001325753 | 3.0000488722 |
| 9 | 3.0030300777 | 3.0012049763 | 3.0000370279 | 3.0000118362 |
| 10 | 3.0014676939 | 3.0005201167 | 3.0000103418 | 3.0000028666 |
| 11 | 3.0007109142 | 3.0002245035 | 3.0000028884 | 3.0000006943 |
| 12 | 3.0003443491 | 3.0000969048 | 3.0000008067 | 3.0000001681 |
| 13 | 3.0001667941 | 3.0000418281 | 3.0000002253 | 3.0000000407 |
| 14 | 3.0000807909 | 3.0000180547 | 3.0000000629 | 3.0000000099 |
| 15 | 3.0000391331 | 3.0000077931 | 3.0000000176 | 3.0000000024 |
| 16 | 3.0000189551 | 3.0000033638 | 3.0000000049 | 3.0000000006 |
| 17 | 3.0000091814 | 3.0000014520 | 3.0000000014 | 3.0000000001 |
| 18 | 3.0000044472 | 3.0000006267 | 3.0000000004 | 3.0000000000 |
| 19 | 3.0000021541 | 3.0000002705 | 3.0000000001 | 3.0000000000 |
| 20 | 3.0000010434 | 3.0000001168 | 3.0000000000 | 3.0000000000 |
| 21 | 3.0000005054 | 3.0000000504 | 3.0000000000 | 3.0000000000 |
| 22 | 3.0000002448 | 3.0000000218 | 3.0000000000 | 3.0000000000 |
| 23 | 3.0000001186 | 3.0000000094 | 3.0000000000 | 3.0000000000 |
| 24 | 3.0000000574 | 3.0000000041 | 3.0000000000 | 3.0000000000 |
| 25 | 3.0000000278 | 3.0000000017 | 3.0000000000 | 3.0000000000 |
| 26 | 3.0000000135 | 3.0000000008 | 3.0000000000 | 3.0000000000 |
| 27 | 3.0000000065 | 3.0000000003 | 3.0000000000 | 3.0000000000 |
| 28 | 3.0000000032 | 3.0000000001 | 3.0000000000 | 3.0000000000 |
| 29 | 3.0000000015 | 3.0000000001 | 3.0000000000 | 3.0000000000 |
| 30 | 3.0000000007 | 3.0000000000 | 3.0000000000 | 3.0000000000 |

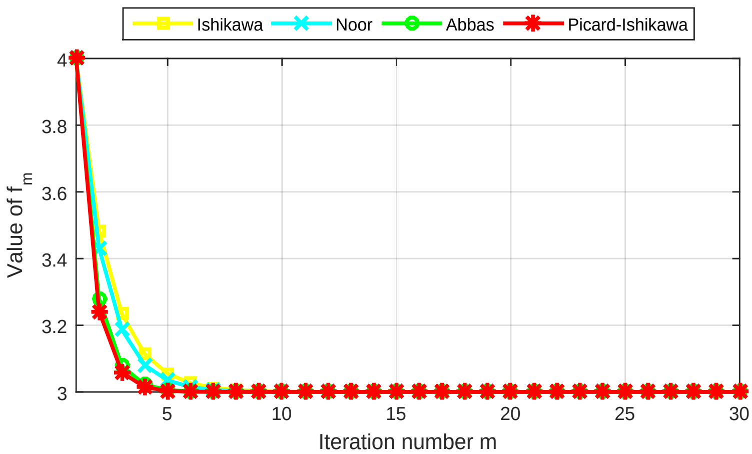


**Figure 2.** Graph corresponding to Table 2.

It is not hard to see from the above figures and tables that the Picard–Ishikawa iterative method convergences faster to 3 (the fixed point of $\mathcal{L}$) than the other compared iterative methods.

Now, we provide an example of a mapping satisfying (2) in a higher dimensional space as follows:

**Example 2.** *Let* $\mathcal{M} = \mathbb{R}^2$ *and* $\mathcal{G} = \{v = (v_1, v_2) : (v_1, v_2) \in [0,1] \times [0,1]\}$ *be a subset of* $\mathcal{M}$ *with the taxicab norm*

$$\|v\| = \|(v_1, v_2)\| = |v_1| + |v_2|.$$

*Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be defined by*

$$\mathcal{L}(v_1, v_2) = \begin{cases} (1 - v_1, 1 - v_2), & \text{if } (v_1, v_2) \in [0, \frac{1}{2}] \times [0, 1], \\ \left(\frac{1+v_1}{3}, \frac{1+v_2}{3}\right), & \text{if } (v_1, v_2) \in (\frac{1}{2}, 1] \times [0, 1]. \end{cases}$$

*If $v = (0, 0)$ and $w = \left(\frac{51}{100}, \frac{1}{4}\right)$ we obtain $\mathcal{L}v = (1, 1)$, $\mathcal{L}w = \left(\frac{151}{300}, \frac{5}{12}\right)$, $\frac{1}{2}\|w - \mathcal{L}w\| = \frac{13}{150} \leq \frac{19}{25} = \|v - w\|$. But $\|\mathcal{L}v - \mathcal{L}w\| = \frac{27}{25} > \frac{19}{25} = \|v - w\|$. Thus, $\mathcal{L}$ is not enriched with condition (C). Next, we show that $\mathcal{L}$ satisfies the inequality (10). The following cases will be considered:*

*(i)* *If $v = (v_1, v_2)$, $w = (w_1, w_2) \in [0, \frac{1}{2}] \times [0, 1]$;*

$$\begin{aligned} \|v - \mathcal{L}w\| &\leq \|v - \mathcal{L}v\| + \|\mathcal{L}v - \mathcal{L}w\| \\ &= \|v - \mathcal{L}v\| + (|v_1 - w_1| + |v_2 - w_2|) \\ &= \|v - \mathcal{L}v\| + \|v - w\|. \end{aligned}$$

*(ii)* *If $v = (v_1, v_2)$, $w = (w_1, w_2) \in (\frac{1}{2}, 1] \times [0, 1]$;*

$$\begin{aligned} \|v - \mathcal{L}w\| &\leq \|v - \mathcal{L}v\| + \|\mathcal{L}v - \mathcal{L}w\| \\ &= \|v - \mathcal{L}v\| + \frac{1}{3}(|v_1 - w_1| + |v_2 - w_2|) \\ &\leq \|v - \mathcal{L}v\| + (|v_1 - w_1| + |v_2 - w_2|) \\ &= \|v - \mathcal{L}v\| + \|v - w\|. \end{aligned}$$

*(iii)* *If $v = (v_1, v_2) \in [0, \frac{1}{2}] \times [0, 1]$ and $w = (w_1, w_2) \in (\frac{1}{2}, 1] \times [0, 1]$;*

$$\begin{aligned} \|v - \mathcal{L}w\| &= \left|\frac{3v_1 - w_1 - 1}{3}\right| + \left|\frac{3v_2 - w_2 - 1}{3}\right| \\ &= \left|\frac{v_1 - w_1 + 2v_1 - 1}{3}\right| + \left|\frac{v_2 - w_2 + 2v_2 - 1}{3}\right| \\ &\leq \left|\frac{v_1 - w_1}{3}\right| + \left|\frac{2v_1 - 1}{3}\right| + \left|\frac{v_2 - w_2}{3}\right| + \left|\frac{2v_2 - 1}{3}\right| \\ &\leq (|2v_1 - 1| + |2v_2 - 1) + (|v_1 - w_1| + |v_2 - w_2|) \\ &= \|v - \mathcal{L}v\| + \|v - w\|. \end{aligned}$$

*Thus, the mapping $\mathcal{L}$ satisfies the inequality (10) with $\left(\frac{3+\alpha}{1-\alpha}\right) \geq 1$ and has a fixed point $\left(\frac{1}{2}, \frac{1}{2}\right)$. Hence, $\mathcal{L}$ is a generalized $\alpha$-nonexpansive mapping.*

## 4. An Application to Fractional Delay Differential Equations in the Caputo Sense

Fractional derivatives and integrals have been widely studied in various fields that have been in the developing stage especially for some decades now [34]. The techniques based on fractional calculus establish models of engineering systems better than the ordinary derivatives approaches [34]. Particularly, fractional differential equations as an important research branch of fractional calculus have been extensively studied by many authors as a result of fractional calculus' numerous applications in engineering and applied sciences. Numerous schemes have been developed for numerical solutions of fractional differential equations.

In this article, we will approximate the solution of the following fractional delay differential equation in the Caputo sense via the Picard–Ishikawa iterative method (9):

$$^{c}\mathcal{D}u(z) = f(z, u(z), u(z - p)), \ z \in [d, J], \tag{25}$$

with the initial conditions

$$u(z) = \varrho(z), \ z \in [d-q, d], \tag{26}$$

where $p > 0$, $\gamma \in (0,1)$, $J > 0$, $q > 0$, the mapping $\varrho \in C([d-p, d] : \Re^m)$, $u \in \Re^m$ is continuous and the mapping $f : [d, J] \times \Re^m \times \Re^m \to \Re^m$ is also continuous. We assume that the following conditions hold:

($C_1$) The Lipschitz constant $L_f > 0$ exists with

$$\|f(z, u_1, v_1) - f(z, u_2, v_2)\| \leq L_f(\|u_1 - v_1\| + \|u_2 - v_2\|)$$

for each $z \in \Re^+$ and $u_1, v_1, u_2, v_2 \in \Re^m$;

($C_2$) A constant $\delta_L > 0$ exists with $\frac{2L}{\delta_L} < 1$.

Let $u^* \in C([d-p, J] : \Re^m) \cap C^1([d, J] : \Re^m)$ be a function satisfying (25) and (26); then, $u^*$ is a solution of (25) and (26). It is shown in [35] that the solution to the problem (25) and (26) is equivalent to the solution of the following integral equation:

$$w(z) = \varrho(d) + \frac{1}{\Gamma(\gamma)} \int_d^z (z-q)^{(\gamma-1)} f(q, u(q), u(q-p))dq, \ \forall \ z \in [d, J], \tag{27}$$

where $u(z) = \varrho(z), \forall \in [d-q, d]$. We define the norm $\| \cdot \|_{\delta_L}$ on $C([d-p, J] : \Re^k)$ by

$$\|\varrho\|_{\delta_L} = \frac{\sup \|\varrho(z)\|}{E_\gamma(\delta_L w_\gamma)} \text{ for all } \varrho \in C([d-p, d] : \Re^m), \tag{28}$$

where $E_\gamma : \Re \to \Re$ is the Mittag-Leffler function and it is defined by

$$E_\gamma(z) = \sum_{m=0}^{\infty} \frac{z_k}{\Gamma(\gamma m + 1)}, \text{ for all } z \in \Re.$$

Clearly, $C([d-p, d] : \Re^m, \| \cdot \|_{\delta_L})$ is a Banach space.

From condition ($C_1$), Wang et al. [33] showed that the solution of problem (25) and (26) exists and it is also unique. Now, we utilize the Picard–Ishikawa iterative method (9) to estimate the solution of problem (25) and (26).

The main result in this section is given as follows:

**Theorem 7.** *Let $z$ and $\varrho$ be the same functions defined above. If the conditions ($C_1$)–($C_2$) are performed, then the sequence generated by* (9) *converges to a unique solution to problem* (25) *and* (26)*, say $u^*$, in $\mathcal{G} = C([d-p, J] : \Re^m) \cap C^1([d, J] : \Re^m)$.*

**Proof.** Let $\mathcal{L} : \mathcal{G} \to \mathcal{G}$ be an operator defined by

$$\mathcal{L}u(z) = \begin{cases} \varrho(d) + \frac{1}{\Gamma(\gamma)} \int_d^w (w-q)^{(\gamma-1)} f(q, u(q), u(q-p))dq, & w \in [d, J], \\ \varrho(z), & z \in [d-q, d]. \end{cases}$$

We now show that $u_m \to u^*$ as $m \to \infty$. Suppose $z \in [d-q, d]$; then, evidently, $u_m \to u^*$ as $m \to \infty$. So, if $\in [d, J]$, by (9) and the conditions ($C_1$)–($C_2$) one obtains

$$\begin{aligned} \|w_m - u^*\| &= \|(1-\beta_m)u_m + \beta_m \mathcal{L}u_m - u^*\| \\ &\leq (1-\beta_m)\|u_m - u^*\| + \beta_m \|\mathcal{L}u_m - u^*\|. \end{aligned} \tag{29}$$

We take the supremum over $[d-p, J]$ on both sides of (29) and we obtain

$$
\begin{aligned}
\sup_{z\in[d-p,J]} \|w_m - u^*\| &\leq (1-\beta_m)\sup_{z\in[d-p,J]}\|u_m - u^*\| + \beta_m \sup_{z\in[d-p,J]}\|\mathcal{L}u_m - u^*\| \\
&\leq (1-\beta_m)\sup_{z\in[d-p,J]}\|u_m - u^*\| \\
&\quad +\beta_m \sup_{z\in[d-p,J]}\|\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)} f(q,u_m(q),u_m(q-p))dq \\
&\quad -\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)} f(q,u^*(q),u^*(q-p))dq\| \\
&\leq (1-\beta_m)\sup_{z\in[d-p,J]}\|u_m - u^*\| + \beta_m \sup_{z\in[d-p,J]}\frac{1}{\Gamma(\gamma)}\int_d^z (z-d)^{(\gamma-1)} \\
&\quad \times L_f(\|u_m(q)-u^*(q)\| + \|u_m(q-p)-u^*(q-p)\|)dq \\
&\leq (1-\beta_m)\sup_{z\in[d-p,J]}\|u_m - u^*\| + \beta_m\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \,\times \\
&\quad L_f(\sup_{z\in[d-p,J]}\|u_m(q)-u^*(q)\| \\
&\quad + \sup_{z\in[d-p,J]}\|u_m(q-p)-u^*(q-p)\|)
\end{aligned} \tag{30}
$$

If both sides of (30) are divided by $E_\gamma(\delta_L w_\gamma)$ we obtain

$$
\begin{aligned}
\frac{\sup_{z\in[d-p,J]}\|u_m - u^*\|}{E_\gamma(\delta_L z_\gamma)} &\leq \frac{(1-\beta_m)\sup_{z\in[d-p,J]}\|u_m - u^*\|}{E_\gamma(\delta_L z_\gamma)} + \beta_m\frac{L_f}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \,\times \\
&\quad \left(\frac{\sup_{z\in[d-p,J]}\|u_m(q)-u^*(q)\|}{E_\gamma(\delta_L z_\gamma)}\right. \\
&\quad \left. + \frac{\sup_{z\in[d-p,J]}\|u_m(q-p)-u^*(q-p)\|}{E_\gamma(\delta_L z_\gamma)}\right).
\end{aligned} \tag{31}
$$

By (28), we then reduce (31) to

$$
\begin{aligned}
\|w_m - u^*\|_{\delta_L} &\leq (1-\beta_m)\|u_m-u^*\|_{\delta_L} + \frac{\beta_m}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \,\times \\
&\quad L_f(\|u_m(q)-u^*(q)\|_{\delta_L} + \|u_m(q-p)-u^*(q-p)\|_{\delta_L}) \\
&= (1-\beta_m)\|u_m-u^*\|_{\delta_L} + \beta_m(2L_f)\|u_m-u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-\eta)^{(\gamma-1)}dq \\
&= (1-\beta_m)\|u_m-u^*\|_{\delta_L} \\
&\quad + \frac{\beta_m(2L_f)}{E_\gamma(\delta_L w_\gamma)}\|u_m-u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}E_\gamma(\delta_L z_\gamma)dq \\
&= (1-\beta_m)\|u_m-u^*\|_{\delta_L} + \frac{\beta_m(2L_f)}{E_\gamma(\delta_L z_\gamma)}\|u_m-u^*\|_{\delta_L}.{}^{\mathrm{c}}\mathcal{I}^\gamma\left({}^{\mathrm{c}}\mathcal{D}\left(\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\right)\right) \\
&= (1-\beta_m)\|u_m-u^*\|_{\delta_L} + \frac{\beta_m(2L_f)}{E_\gamma(\delta_L z_\gamma)}.\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\|u_m-u^*\|_{\delta_L} \\
&= (1-\beta_m)\|u_m-u^*\|_{\delta_L} + \frac{\beta_m(2L_f)}{\delta_L}\|u_m-u^*\|_{\delta_L}.
\end{aligned} \tag{32}
$$

Since $\frac{2L_f}{\delta_L} < 1$, it follows that

$$\|w_m - u^*\|_{\delta_L} \leq \|u_m - u^*\|_{\delta_L}. \tag{33}$$

Again, from (9) we obtain

$$\begin{aligned} \|v_m - u^*\| &= \|(1-\alpha_m)u_m + \alpha_m \mathcal{L}w_m - u^*\| \\ &\leq (1-\alpha_m)\|u_m - u^*\| + \alpha_m\|\mathcal{L}w_m - u^*\|. \end{aligned} \tag{34}$$

If we take the supremum over $[d-p, J]$ on both sides of (34) we obtain

$$\begin{aligned} \sup_{z\in[d-p,J]} \|v_m - u^*\| &\leq (1-\alpha_m) \sup_{z\in[d-p,J]} \|u_m - u^*\| + \alpha_m \sup_{z\in[d-p,J]} \|\mathcal{L}w_m - u^*\| \\ &\leq (1-\alpha_m) \sup_{z\in[d-p,J]} \|u_m - u^*\| \\ &\quad +\alpha_m \sup_{z\in[d-p,J]} \|\frac{1}{\Gamma(\gamma)} \int_d^z (z-q)^{(\gamma-1)} f(q, w_m(q), w_m(q-p))dq \\ &\quad -\frac{1}{\Gamma(\gamma)} \int_d^z (z-q)^{(\gamma-1)} f(q, u^*(q), u^*(q-p))dq\| \\ &\leq (1-\alpha_m) \sup_{z\in[d-p,J]} \|u_m - u^*\| + \alpha_m \sup_{z\in[d-p,J]} \frac{1}{\Gamma(\gamma)} \int_d^z (z-d)^{(\gamma-1)} \\ &\quad \times L_f(\|w_m(q) - u^*(q)\| + \|w_m(q-p) - u^*(q-p)\|)dq \\ &\leq (1-\alpha_m) \sup_{z\in[d-p,J]} \|u_m - u^*\| + \alpha_m \frac{1}{\Gamma(\gamma)} \int_d^z (z-q)^{(\gamma-1)} dq \times \\ &\quad L_f( \sup_{z\in[d-p,J]} \|w_m(q) - u^*(q)\| \\ &\quad + \sup_{z\in[d-p,J]} \|w_m(q-p) - u^*(q-p)\|) \end{aligned} \tag{35}$$

If both sides of (35) are divided by $E_\gamma(\delta_L w_\gamma)$ we have

$$\begin{aligned} \frac{\sup\limits_{z\in[d-p,J]} \|v_m - u^*\|}{E_\gamma(\delta_L z_\gamma)} &\leq \frac{(1-\alpha_m) \sup\limits_{z\in[d-p,J]} \|u_m - u^*\|}{E_\gamma(\delta_L z_\gamma)} + \alpha_m \frac{L_f}{\Gamma(\gamma)} \int_d^z (z-q)^{(\gamma-1)} dq \times \\ &\quad \left( \frac{\sup\limits_{z\in[d-p,J]} \|w_m(q) - u^*(q)\|}{E_\gamma(\delta_L z_\gamma)} \right. \\ &\quad \left. + \frac{\sup\limits_{z\in[d-p,J]} \|w_m(q-p) - u^*(q-p)\|}{E_\gamma(\delta_L z_\gamma)} \right). \end{aligned} \tag{36}$$

By (28), we then reduce (36) to

$$\begin{aligned}
\|v_m - u^*\|_{\delta_L} &\leq (1-\alpha_m)\|u_m - u^*\|_{\delta_L} + \frac{\alpha_m}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \times \\
&\quad L_f(\|u_m(q) - u^*(q)\|_{\delta_L} + \|u_m(q-p) - u^*(q-p)\|_{\delta_L}) \\
&= (1-\alpha_m)\|u_m - u^*\|_{\delta_L} + \alpha_m(2L_f)\|u_m - u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-\eta)^{(\gamma-1)}dq \\
&= (1-\alpha_m)\|u_m - u^*\|_{\delta_L} \\
&\quad + \frac{\alpha_m(2L_f)}{E_\gamma(\delta_L z_\gamma)}\|w_m - u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}E_\gamma(\delta_L z_\gamma)dq \\
&= (1-\alpha_m)\|u_m - u^*\|_{\delta_L} + \frac{\alpha_m(2L_f)}{E_\gamma(\delta_L z_\gamma)}\|w_m - u^*\|_{\delta_L}.{}^{c}\mathcal{I}^{\gamma}\left({}^{c}\mathcal{D}\left(\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\right)\right) \\
&= (1-\alpha_m)\|u_m - u^*\|_{\delta_L} + \frac{\alpha_m(2L_f)}{E_\gamma(\delta_L z_\gamma)}.\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\|u_m - u^*\|_{\delta_L} \\
&= (1-\alpha_m)\|u_m - u^*\|_{\delta_L} + \frac{\alpha_m(2L_f)}{\delta_L}\|w_m - u^*\|_{\delta_L}. \qquad (37)
\end{aligned}$$

Since $\frac{2L_f}{\delta_L} < 1$, then from (33) and (37) it follows that

$$\|v_m - u^*\|_{\delta_L} \leq \|u_m - u^*\|_{\delta_L}. \qquad (38)$$

Finally, from (9) we obtain

$$\|u_{m+1} - u^*\| = \|\mathcal{L}v_m - u^*\|.$$

If we take the supremum over $[d-p, J]$ on both sides of (39) we obtain

$$\begin{aligned}
\sup_{z\in[d-p,J]}\|u_{m+1} - u^*\| &\leq \sup_{z\in[d-p,J]}\|\mathcal{L}v_m - u^*\| \\
&\leq \sup_{z\in[d-p,J]}\|\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}f(q, v_m(q), v_m(q-p))dq \\
&\quad -\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}f(q, u^*(q), u^*(q-p))dq\| \\
&\leq \sup_{z\in[d-p,J]}\frac{1}{\Gamma(\gamma)}\int_d^z (z-d)^{(\gamma-1)} \\
&\quad \times L_f(\|v_m(q) - u^*(q)\| + \|v_m(q-p) - u^*(q-p)\|)dq \\
&\leq \frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \times L_f(\sup_{z\in[d-p,J]}\|v_m(q) - u^*(q)\| \\
&\quad + \sup_{z\in[d-p,J]}\|v_m(q-p) - u^*(q-p)\|) \qquad (39)
\end{aligned}$$

If both sides of (39) are divided by $E_\gamma(\delta_L w_\gamma)$ we obtain

$$\begin{aligned}
\frac{\sup_{z\in[d-p,J]}\|u_{m+1} - u^*\|}{E_\gamma(\delta_L z_\gamma)} &\leq \frac{L_f}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \times \\
&\left(\frac{\sup_{z\in[d-p,J]}\|v_m(q) - u^*(q)\|}{E_\gamma(\delta_L z_\gamma)} + \frac{\sup_{z\in[d-p,J]}\|v_m(q-p) - u^*(q-p)\|}{E_\gamma(\delta_L z_\gamma)}\right).
\end{aligned}$$

By (28), we then reduce (40) to

$$\begin{aligned}
\|u_{m+1}-u^*\|_{\delta_L} &\leq \frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}dq \times \\
& \quad L_f(\|u_m(q)-u^*(q)\|_{\delta_L}+\|u_m(q-p)-u^*(q-p)\|_{\delta_L}) \\
&= 2L_f\|u_m-u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-\eta)^{(\gamma-1)}dq \\
&= \frac{2L_f}{E_\gamma(\delta_L z_\gamma)}\|v_m-u^*\|_{\delta_L}\frac{1}{\Gamma(\gamma)}\int_d^z (z-q)^{(\gamma-1)}E_\gamma(\delta_L z_\gamma)dq \\
&= \frac{2L_f}{E_\gamma(\delta_L z_\gamma)}\|v_m-u^*\|_{\delta_L}.{}^{c}\mathcal{I}^\gamma\left({}^{c}\mathcal{D}^\gamma\left(\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\right)\right) \\
&= \frac{2L_f}{E_\gamma(\delta_L z_\gamma)}.\frac{E_\gamma(\delta_L z_\gamma)}{\delta_L}\|u_m-u^*\|_{\delta_L} \\
&= \frac{2L_f}{\delta_L}\|v_m-u^*\|_{\delta_L}.
\end{aligned} \tag{40}$$

Combining (38) and (40) we obtain

$$\|u_{m+1}-u^*\|_{\delta_L} \leq \|u_m-u^*\|_{\delta_L}. \tag{41}$$

If we set $\|u_{m+1}-u^*\|_{\delta_L}=\psi_m$ then we obtain

$$\psi_{m+1} \leq \psi_m, \ \forall \ m \in \mathbb{N}.$$

Thus, $\{\psi_m\}$ is a monotonically decreasing sequence of real numbers. This implies that it is a bounded sequence, so we obtain

$$\lim_{m\to\infty}\psi_m = \inf\{\psi_m\} = 0.$$

Hence,

$$\lim_{m\to\infty}\|u_m-u^*\|_{\delta_L} = 0.$$

□

## 5. Conclusions

In this work, we have studied an efficient iterative method known as the Picard–Ishikawa method, as defined in (9). We used this method to approximate the fixed points of generalized $\alpha$-nonexpansive mappings. Under standard and mild conditions imposed on the control parameters of the Picard–Ishikawa method, we proved several weak and convergence theorems of the method. We have shown numerically that the studied method has a better rate of convergence than some well-known methods for generalized $\alpha$-nonexpansive mappings. We solved a problem involving fractional delay differential equations via the Picard–Ishikawa iterative method. Since the class of mappings studied in this work is more general than those in many existing results, our findings extend, generalize and unify several existing results in the literature.

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