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Article

New Inequalities Using Multiple Erdélyi–Kober Fractional Integral Operators

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Abstract: The role of fractional integral inequalities is vital in fractional calculus to develop new models and techniques in the most trending sciences. Taking motivation from this fact, we use multiple Erdélyi–Kober (M-E-K) fractional integral operators to establish Minkowski fractional inequalities. Several other new and novel fractional integral inequalities are also established. Compared to the existing results, these fractional integral inequalities are more general and substantial enough to create new and novel results. M-E-K fractional integral operators have been previously applied for other purposes but have never been applied to the subject of this paper. These operators generalize a popular class of fractional integrals; therefore, this approach will open an avenue for new research. The smart properties of these operators urge us to investigate more results using them.

Keywords: fractional integral inequalities; positive functions; Fox-*H*-function; mathematical operators

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1. Introduction and Motivation

Integral inequalities are a crucial tool in basic mathematical analysis. Various names of fundamental inequalities can be found in the literature, for example, Cauchy–Schwarz, Hölder, and Minkowski inequalities. Moreover, in the past few years, fractional integral inequalities have emerged as one of the most practical and extensive instruments for the advancement of numerous topics in both pure and applied mathematics [1]. Therefore, several scholars have presented a variety of generalized inequalities involving fractional integral operators [2–11]. In addition to its generalizations and extensions in one or more variables, fractional calculus covers a variety of important problems involving unique mathematical physics functions and offers various potentially helpful methods for solving differential and integral equations [12]. Several extensions of the Riemann–Liouville (R-L) and Erdélyi–Kober (E-K) operators have been studied in the literature. These extensions include the Bessel function J_μ, as well as the *H*- and *G*-functions in the integrand [13]. The work of Srivastava is mentionable for developing a rigorous and more general theory of the operators of fractional calculus. For example, many generic families of operators of fractional integration, including Fox's H-function and its extensions in two or more variables, are discussed in [14] (see also [15]). A distinct class of fractional calculus operators and their uses concerning higher transcendental functions have been examined in [16]. Furthermore, numerous variations in both parameters and arguments for the fractional calculus operators, together with associated special functions and integral transformations, are offered in [17]. Among these, we apply the multiple Erdélyi–Kober fractional operators [12–14] in this research to explore the new fractional inequalities that generalize the earlier works cited in [11]. To the best of our knowledge, these multiple operators have never been used for this purpose. Therefore, before continuing with further discussion about our new results, we first present the basic preliminaries and required facts in the subsequent section.

The plan of this paper is as follows: After presenting the necessary definitions and required facts in Section 2, we proceed to prove the reverse Minkowski inequalities using multiple Erdélyi–Kober fractional operators in Section 3. A novel and new class of inequalities using the multiple Erdélyi–Kober fractional integral operator is presented in Section 4. Finally, Section 5 contains a summary of the results.

2. Multiple Erdélyi–Kober Fractional Integral Operators

This section is complimentary to this research as it contains all necessary preliminaries, concepts, and definitions about M-E-K integral operators. The Fox-H-function is defined by (see [12–14]).

$$H^{l,m}_{p,q}(\omega) = H^{l,m}_{p,q}\left[\omega\middle|\begin{matrix}(a_i, A_i)\\(b_j, B_j)\end{matrix}\right] = \frac{1}{2\pi\iota}\int_L \frac{\prod_{j=1}^{l}\Gamma(b_j, +B_j s)\prod_{i=1}^{m}\Gamma(1-a_i-A_i s)}{\prod_{j=l+1}^{p}\Gamma(1-b_j-B_j s)\prod_{i=m+1}^{p}\Gamma(a_i+A_i s)}\omega^{-s}ds \quad (1)$$
$$(i = 1,\dots,p; j = 1,\dots,q; 0 \le m \le p; 1 \le l \le q; A_i > 0; B_j > 0; a_i \in C; b_j \in C).$$

The poles of the gamma function in the numerator of the above equation are split by making use of an appropriate contour L. Furthermore, the H-function reduces to the Meijer G-function [12] by considering the value of all $A_i = 1 = B_j$. The Meijer G-function is further related to many other special functions like Fox–Wright, hypergeometric, and Mittag Leffler functions, which makes the operators defined in the following definition very significant.

Definition 1. *Multiple Erdélyi–Kober fractional integral operators, $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}$ are defined as*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} f(x) = x^{-1}\int_0^x H^{m,0}_{m,m}\left[\frac{\sigma}{x}\middle|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m\end{matrix}\right] f(x\sigma)d\sigma. \quad (2)$$

In the above equation, the various parameters used, i.e., ($\delta_i \ge 0$), provide the order of integration, while ($\gamma_i \ge 0$) are multi-weight, whereas ($\beta_i \ge 0$) are also some additional multi-parameters. We can also note that $\sum_{k=1}^{m}\delta_k > 0$ and for all $\delta_i = 0$, we obtain the identity operator, i.e., $I^{(\gamma_k),(0,\dots,0)}_{(\beta_k),m} f(x) := f(x)$, from the above equation.

Under such circumstances, the generalized fractional integrals can be broken down into commutative products of conventional operators (namely, operators). Consequently, the generalized fractional calculus with a well-developed thorough theory and numerous established applications combines the capabilities of the special functions with the widespread use of conventional fractional calculus [12,13]. Similar to this, a specific number of compositions of R-L and E-K operators are also taken into consideration [18,19]. $H^{m,0}_{m,m}$ (or $H^{m,m}_{n+n,m+n}$ if compositions of left as well as right operators are taken) serve as the kernels for these compositions. In [12], several subjects including classes of differential and integral equations, geometric function theory, special functions, integral transformations, and operational calculus have all been addressed using the proposed theory.

In [20], Dimovski presented the spaces $C_\mu([0,\infty])$ over real variable $x > 0$ of good functions. This is where we work in this study.

Definition 2. *For $x > 0, \mu \in R$ and $f \in C_\mu$, we have $\widetilde{f}(x) = x^p f(x), p > \mu$, where $\widetilde{f} \in C[0,\infty]$ is continuous. Similar mapping holds for $f \in C^{(n)}_\mu, n \in N$ with $\widetilde{f} \in C^n([0,\infty))$.*

More properties of such spaces can be seen in [12]. Similarly, the elements of Lebesgue integrable spaces ($L^p_\mu(0,\infty), 1 \leq p < \infty$) satisfying $f_{\mu,p} = \int_0^\infty x^{\mu-1}|f(x)|^p dx]^{1/p} < \infty$ also preserve the power function. Furthermore, our results depend on the following assumptions:

$$\begin{gathered} \delta_k \geq 0; k = 1, \ldots, m; \\ \beta_k(\gamma_k + 1) > -\mu, (f \in C_\mu([0,\infty)); \\ \beta_k(\gamma_k + 1) > \mu/\rho; (f \in L_{\mu,\rho}([0,\infty)). \end{gathered} \tag{3}$$

Remark 1. *It is evident that the kernel in the above definition of the multiple E-K fractional integral remains positive [21–23].*

Lemma 1. *For $z^\mu = f \in C_\mu$, we have the following useful result:*

$$\begin{gathered} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left\{x^{\mu-1}\right\} = \prod_{i=1}^m \frac{\Gamma\left(\gamma_i + 1 + \frac{\mu-1}{\beta_i}\right)}{\Gamma\left(\gamma_i + \delta_i + 1 + \frac{\mu-1}{\beta_i}\right)} x^{\mu-1}, \\ (\delta_k \geq 0, \mu - 1 > -\beta(\gamma_k + 1), k = 1, \ldots, m). \end{gathered} \tag{4}$$

If we consider $\mu = 1$ in Lemma 1, then we obtain the following relation:

$$\begin{gathered} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\{1\} = \prod_{i=1}^m \frac{\Gamma(\gamma_i + 1)}{\Gamma(\gamma_i + \delta_i + 1)}, \\ \delta_k \geq 0, k = 1, \ldots, m, \end{gathered} \tag{5}$$

and $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} : C_\mu \mapsto C_\mu$.

Furthermore, the class of weighted analytic functions is also preserved using these operators [20]. These operators are bilinear, commutative, invertible, and satisfy a semigroup property [20]. It is also proved that under the assumptions (3), they act as bounded linear operators over L^p_μ. For $f \in C^{(N)}_\mu$ under the assumptions in (3), M-E-K satisfy the initial conditions given by

$$\begin{gathered} \left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\right\}^{(j)}(0) = c_j f^{(j)}(0); c_j = \prod_{i=1}^m \frac{\Gamma(\gamma_i + 1 + p/\beta_i)}{\Gamma(\gamma_i + \delta_i + 1 + j/\beta_i)}, \\ \delta_k \geq 0, j > -\beta(\gamma_k + 1), k = 1, \ldots, m. \end{gathered} \tag{6}$$

The Caputo and Riemann–Liouville versions of the M-E-K fractional operators are also investigated in [20].

Another interesting feature of these operators is how they relate to a wide range of widely used fractional integrals, as listed below [12,13], when all $\beta_k' s$ are equal, i.e., $\beta_k = \beta$.

1. The Meijer G-function of the form can be used to describe the multiple E-K fractional integrals more simply (see [12], Chapter 1).
2. $m = 3$ and $\beta = 1; 1 = \alpha_1 = \alpha_2 = \alpha_3$, and M-E-K reduce to the Marichev–Saigo–Maeda (M-S-M) fractional operator [24,25].
3. $m = 2$ and $\beta = 1; \alpha_1 = \alpha_2 = \alpha > 0; \sigma = t/x; \sigma = x/t)$, it reduces to the Saigo fractional operator [26,27].
4. For $m = 1 = \beta$, it reduces to the E-K fractional operator.
5. $m = 1$ and $(\beta = 1; \alpha = 1; t/x = \sigma; x/t = \sigma)$, it reduces to the (R-L) fractional operator [12,13].
6. $m \geq 0; (\alpha_1 = \alpha_2 = \alpha_3 = \alpha)$, then we obtain the Bessel and hyper-Bessel operators [12,13].

Furthermore, these operators also contain other well-known fractional integrals, such as Weyl, and the Hadamard and Katugampola can also be obtained (see [28] and references therein).

Section 2 specifies that the conditions on the parameters will be considered typical unless specifically stated otherwise during this investigation.

3. Reverse Minkowski Inequalities Using Multiple Erdélyi–Kober Fractional Operator

The M-E-K fractional operators are used in this section to state and prove reverse Minkowski integral inequalities, and the following theorem is our main result about the reverse Minkowski fractional integral inequalities.

Theorem 1. *For $m > 1$, let $\delta_k \geq 0, \beta_k > 0$ and $\gamma_k > -1 - \frac{\mu}{\beta_k}$. Furthermore, for $x > 0$, consider two positive functions Ξ and Θ on the interval $[0, \infty)$ satisfying $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Xi^\lambda(x) < \infty$, $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Theta^\lambda(x) < \infty$. Then, for $0 < m \leq \frac{\Xi(t)}{\Theta(t)} \leq M$, $t \in [0, x]; \lambda \geq 1$, we prove the subsequent inclusion*

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Xi^\lambda(x)\right)^{\frac{1}{\lambda}} + \left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Theta^\lambda(x)\right)^{\frac{1}{\lambda}} \leq \frac{1+M(m+2)}{(m+1)(M+1)} \left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} (\Xi + \Theta)^\lambda(x)\right)^{\frac{1}{\lambda}}. \tag{7}$$

Proof. For $t \in [0, x]; x > 0$, and $\frac{\Xi(t)}{\Theta(t)} < M$, we obtain the following inequality:

$$(M+1)^\lambda \Xi^\lambda(t) \leq M^\lambda (\Xi + \Theta)^\lambda(t). \tag{8}$$

Next, for $x > 0$ we take the following expression:

$$\mathfrak{F}(x, t) = \frac{1}{x} H^{m,0}_{m,m} \left[\frac{t}{x} \middle| \begin{array}{c} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{array} \right]. \tag{9}$$

Then, by multiplying it on both sides of (8) because of Remark 1 and integrating the resultant inequality for $t \in [0, x]; x > 0$, we obtain

$$\begin{aligned} &\frac{(M+1)^\lambda}{x} \int_0^x H^{m,0}_{m,m} \left[\frac{t}{x} \middle| \begin{array}{c} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{array} \right] \Xi^\lambda(t) dt \\ \leq & M^\lambda \int_0^x H^{m,0}_{m,m} \left[\frac{t}{x} \middle| \begin{array}{c} \left(\gamma_k + \delta_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k + 1 - \frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \end{array} \right] (\Xi + \Theta)^\lambda(t) dt. \end{aligned}$$

This implies that

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Xi^\lambda(x) \leq \frac{M^\lambda}{(M+1)^\lambda} I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} (\Xi + \Theta)^\lambda(x).$$

and

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} \Xi^\lambda(x)\right)^{\frac{1}{\lambda}} \leq \frac{M}{(M+1)} \left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m} (\Xi + \Theta)^\lambda(x)\right)^{\frac{1}{\lambda}}. \tag{10}$$

Next, by making use of the inequality $m\Theta(t) \leq \Xi(t)$, one can obtain

$$\left(1 + \frac{1}{m}\right) \Theta(t) \leq \frac{1}{m} (\Xi(t) + \Theta(t)),$$

leading to the following:

$$\left(1+\frac{1}{m}\right)^{\lambda}\Theta^{\lambda}(t) \leq \left(\frac{1}{m}\right)^{\lambda}(\Xi(t)+\Theta(t))^{\lambda}. \tag{11}$$

Now, if we multiply (9) with the above expression (11) and integrate the resultant inequality for $t \in [0,x]; x > 0$, we obtain

$$\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{1}{\lambda}} \leq \frac{1}{(m+1)}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}(\Xi+\Theta)^{\lambda}(x)\right)^{\frac{1}{\lambda}}. \tag{12}$$

We can compute the required result as stated in Theorem 1 with the addition of inequality (10) and inequality (12). □

Theorem 2. *For $m > 1$, let $\delta_k \geq 0, \beta_k > 0$ and $\gamma_k > -1-\frac{\mu}{\beta_k}$. Furthermore, for $x > 0$, consider two positive functions Ξ and Θ on the interval $[0,\infty)$ satisfying, $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x) < \infty$, $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x) < \infty$. Then, for $0 < m \leq \frac{\Xi(t)}{\Theta(t)} \leq M$, $t \in [0,x]; \lambda \geq 1$, we prove the subsequent inclusion*

$$\begin{aligned}&\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{2}{\lambda}}+\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{2}{\lambda}}\\ &\geq\left(\frac{(M+1)(m+1)}{M}-2\right)\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{1}{\lambda}}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{1}{\lambda}}.\end{aligned} \tag{13}$$

Proof. We can achieve this in two steps. Firstly, we will multiply the inequalities (10) and (12). This will lead to the following:

$$\left(\frac{(M+1)(m+1)}{M}\right)\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{1}{\lambda}}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{1}{\lambda}} \leq \left[\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}(\Xi(x)+\Theta(x))^{\lambda}\right)^{\frac{1}{\lambda}}\right]^{2}. \tag{14}$$

In the second step, we achieve the subsequent result by making use of the well-known Minkowski inequality on the right-hand side (RHS) of (14):

$$\begin{aligned}&\left[\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}(\Xi(x)+\Theta(x))^{\lambda}\right)^{\frac{1}{\lambda}}\right]^{2} \leq \left[\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{1}{\lambda}}+\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{1}{\lambda}}\right]^{2}\\ &\leq\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{2}{\lambda}}+\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{2}{\lambda}}+2\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\right)^{\frac{1}{\lambda}}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta^{\lambda}(x)\right)^{\frac{1}{\lambda}}.\end{aligned} \tag{15}$$

Hence, the required result, i.e., (13), of the stated theorem follows from inequalities (14) and (15). □

4. New Inequalities Using Multiple Erdélyi–Kober Fractional Integral Operator

This section contains the proof of the novel inequalities involving the M-E-K operators.

Theorem 3. *For $\frac{1}{r}+\frac{1}{s}=1; r,s > 1$, consider Ξ,Θ to be a pair of positive functions on the interval $[0,\infty)$ such that $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)] < \infty$, $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Theta(x)] < \infty$. If $0 < m \leq \frac{\Xi(t)}{\Theta(t)} \leq M < \infty$, then we can obtain*

$$\begin{aligned}&\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi(x)\right)^{\frac{1}{r}}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta(x)\right)^{\frac{1}{s}} \leq \left(\frac{M}{m}\right)^{\frac{1}{rs}}\left(I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)]^{\frac{1}{r}}[\Theta(x)]^{\frac{1}{s}}\right)\\ &\left(t \in [0,x], x > 0, m > 1, \delta_k \geq 0, \beta_k > 0, \gamma_k > -1-\frac{\mu}{\beta_k}\right).\end{aligned} \tag{16}$$

Proof. For $t \in [0,x], x > 0$ and $\frac{\Xi(t)}{\Theta(t)} \leq M < \infty$, we obtain the following:

$$[\Theta(t)]^{\frac{1}{s}} \geq M^{\frac{-1}{s}} [\Xi(t)]^{\frac{1}{s}}. \tag{17}$$

It will lead to the following:

$$\begin{aligned}[\Xi(t)]^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{s}} &\geq M^{\frac{-1}{s}} [\Xi(t)]^{\frac{1}{r}} [\Xi(t)]^{\frac{1}{s}} \\ &\geq M^{\frac{-1}{s}} [\Xi(t)]^{\frac{1}{r}+\frac{1}{s}} \\ &\geq M^{\frac{-1}{s}} [\Xi(t)].\end{aligned} \tag{18}$$

Next, we multiply (9) and (18) then integrate the result for $t \in [0,x], x > 0$

$$\begin{aligned}&\frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle| \begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1 \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right][\Xi(t)]^{\frac{1}{r}}[\Theta(t)]^{\frac{1}{s}}dt \\ \geq &\frac{M^{\frac{-1}{s}}}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle| \begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1 \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right]\Xi(t)dt.\end{aligned} \tag{19}$$

This implies that

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[[\Xi(t)]^{\frac{1}{r}}[\Theta(t)]^{\frac{1}{s}}\right] \geq M^{\frac{-1}{r}}\left[I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi(t)\right]. \tag{20}$$

Finally, we compute

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[[\Xi(t)]^{\frac{1}{r}}[\Theta(t)]^{\frac{1}{s}}\right]\right)^{\frac{1}{r}} \geq M^{\frac{-1}{rs}}\left[I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi(t)\right]^{\frac{1}{r}}. \tag{21}$$

Furthermore, by considering $m\Theta(t) \leq \Xi(t)$, we obtain

$$[\Xi(t)]^{\frac{1}{r}} \geq m^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{r}}. \tag{22}$$

This implies that

$$\begin{aligned}[\Xi(t)]^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{s}} &\geq m^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{s}} \\ &\geq m^{\frac{1}{r}} [\Theta(t)]^{\frac{1}{r}+\frac{1}{s}} \\ &\geq m^{\frac{1}{r}} [\Theta(t)].\end{aligned} \tag{23}$$

Next, we multiply (23) and (9) and then integrate the result for $t \in [0,x], x > 0$ to obtain

$$\begin{aligned}&\frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle| \begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1 \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right][\Xi(t)]^{\frac{1}{r}}[\Theta(t)]^{\frac{1}{s}}dt \\ \geq &\frac{m^{\frac{1}{r}}}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle| \begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1 \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right]\Theta(t)dt.\end{aligned} \tag{24}$$

Therefore, we have

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[[\Xi(t)]^{\frac{1}{r}}[\Theta(t)]^{\frac{1}{s}}\right]\right)^{\frac{1}{r}} \geq m^{\frac{1}{rs}}\left[I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi(t)\right]^{\frac{1}{s}}. \tag{25}$$

The required result is obtained when we multiply Equations (21) and (25). □

Theorem 4. *Consider $\frac{1}{r}+\frac{1}{s}=1; r,s>1$ and a pair of positive functions Ξ,Θ on the interval $[0,\infty)$ satisfying $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi^r(x)]<\infty$, $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta^s(x)]<\infty$. Furthermore, if $0<m\le\frac{\Xi(t)^r}{\Theta(t)^s}\le M<\infty$, then the following inequality can be obtained:*

$$\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi^r(x)\right)^{\frac{1}{r}}\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Theta^s(x)\right)^{\frac{1}{s}}\le\left(\frac{M}{m}\right)^{\frac{1}{rs}}\left(I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi(x)]^{\frac{1}{r}}[\Theta(x)]^{\frac{1}{s}}\right)$$
$$\left(t\in[0,x], x>0, m>1, \delta_k\ge 0, \beta_k>0, \gamma_k>-1-\frac{\mu}{\beta_k}\right). \tag{26}$$

Proof. For $x>0; t\in[0,x]$, by making a replacement of $\Xi(t)$ and $\Theta(t)$ with $\Xi(t)^r$ and $\Theta(t)^s$, in Theorem 3, we can compute the required result (26). □

Theorem 5. *Consider a pair of positive functions Ξ and Θ on $[0,\infty)$, where Ξ is non-decreasing and Θ is non-increasing. Therefore, we can obtain*

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\Theta^\nu(x)\right]\le\frac{1}{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1]}I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\right]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta^\nu(x)], x>0, \tag{27}$$

where $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1]$ is defined by (5) and $\lambda,\nu>0$.

Proof. Suppose $\lambda>0$, $\nu>0$, and $t,u\in[0,x]$, $x>0$; then, we obtain

$$\left(\Xi^\lambda(t)-\Xi^\lambda(u)\right)(\Theta^\nu(t)-\Theta^\nu(u))\ge 0. \tag{28}$$

This implies that

$$\Xi^\lambda(t)\Theta^\nu(t)+\Xi^\lambda(u)\Theta^\nu(u)\le\Xi^\lambda(u)\Theta^\nu(t)+\Xi^\lambda(t)\Theta^\nu(u). \tag{29}$$

We can integrate the product of (29) and (9) for the variable t on the interval $[0,x]$ to obtain the following

$$\begin{aligned}&\frac{1}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]\Xi^\lambda(t)\Theta^\nu(t)dt\\ &+\frac{\Xi^\lambda(u)\Theta^\nu(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right][1]dt\\ &\le\frac{\Theta^\nu(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]\Xi^\lambda(t)dt\\ &+\frac{\Xi^\lambda(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\middle|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]dt.\end{aligned} \tag{30}$$

This implies that

$$\begin{aligned}&I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\Theta^\nu(x)\right]+\Xi^\lambda(u)\Theta^\nu(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1]\\ \le&\Xi^\lambda(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta^\nu(x)]+\Theta^\nu(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\right].\end{aligned} \tag{31}$$

A product of (9) and (31), after integration over the variable $u\in[0,x]$, yields

$$\begin{aligned}&I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\Theta^\nu(x)\right]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\Xi^\lambda(x)\Theta^\nu(x)\right]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}1]\\ \le&I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi^\lambda(x)]\,I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta^\nu(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta^\nu(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\Xi^\lambda(x)\Big],\end{aligned} \tag{32}$$

which leads to the required result. □

Theorem 6. *Consider a pair of positive functions Ξ and Θ on the interval* $[0,\infty)$ *so that Ξ is non-decreasing and Θ is non-increasing. These assumptions will lead to the following statement:*

$$\begin{aligned} & I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}\left[\Xi^{\lambda}(x)\Theta^{\nu}(x)\right]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[1]+I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\Theta^{\nu}(x)\right]I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[1] \\ \le & I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\right]I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[\Theta^{\nu}(x)]+I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[\Theta^{\nu}(x)]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\right], \end{aligned} \tag{33}$$

where $x>0;\lambda;\nu>0$ *and* $I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi(x)\ge I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[1]$ *is defined by (5).*

Proof. Consider the product of

$$\mathfrak{F}(x,u)=\frac{1}{x}H_{m,m}^{m,0}\left[\frac{u}{x}\,\middle|\,\begin{matrix}\left(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right] \tag{34}$$

with (31) and then integrate the result for the variable $u\in(0,x)$ to obtain the following:

$$\begin{aligned} & \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\Theta^{\nu}(x)\right]}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right][1]du \\ + & \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[1]}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]\left[\Xi^{\lambda}(u)\Theta^{\nu}(u)\right]du \\ \le & \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Theta^{\nu}(x)]}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]\Xi^{\lambda}(u)du \\ + & \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\right]}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\,\middle|\,\begin{matrix}\left(\varepsilon_k+\eta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\varepsilon_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right]\Theta^{\nu}(u)du. \end{aligned} \tag{35}$$

This provides the required result (33). □

Remark 2. *The stated inequalities (27) and (33) can be reversed by making use of the functions*

$$\left(\Xi^{\lambda}(t)-\Xi^{\lambda}(u)\right)(\Theta^{\nu}(t)-\Theta^{\nu}(u))\ge 0.$$

Remark 3. *Considering Theorem 6 using* $\varepsilon_k=\gamma_k$ *and* $\eta_k=\delta_k$*, we obtain the statement of Theorem 5.*

Theorem 7. *Consider a pair of functions such that* $\Xi\ge 0$ *and* $\Theta\ge 0$ *on the interval* $[0,\infty)$ *so that Θ is non-decreasing. Then, the assumption*

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi(x)\ge I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Theta(x),x>0 \tag{36}$$

leads to the following result:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda-\nu}(x)\le I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\Xi^{\lambda}(x)\Theta^{-\nu}(x);x>0,\lambda>0,\nu>0,\lambda-\nu>0. \tag{37}$$

Proof. Since, $\lambda>0$, $\nu>0$; therefore, by making use of an arithmetic–geometric inequality, we obtain

$$\tfrac{\lambda}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(t) - \tfrac{\nu}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Theta^{\lambda-\nu}(t) \leq I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda}(t)\Theta^{-\nu}(t), t \in (0,x), x > 0. \tag{38}$$

Next, by considering the product of (9) and (38) and integrating it for the variable $t \in (0,x)$, we obtain

$$\begin{aligned}
&\frac{\lambda}{x(\lambda-\nu)} \int_0^x H_{m,m}^{m,0}\left[\frac{t}{x} \left| \begin{array}{c} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{array}\right.\right] \Xi^{\lambda-\nu}(t)dt \\
&-\frac{\nu}{x(\lambda-\nu)} \int_0^x H_{m,m}^{m,0}\left[\frac{t}{x} \left| \begin{array}{c} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{array}\right.\right] \Theta^{\lambda-\nu}(t)dt \\
&\leq \frac{1}{x} \int_0^x H_{m,m}^{m,0}\left[\frac{t}{x} \left| \begin{array}{c} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m \end{array}\right.\right] \Xi^{\lambda}(t)\Theta^{-\nu}(t)dt.
\end{aligned}$$

This will lead to the following:

$$\tfrac{\lambda}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(x) - \tfrac{\nu}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Theta^{\lambda-\nu}(x) \leq I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda}(x)\Theta^{-\nu}(x),$$

which can be expressed as

$$\tfrac{\lambda}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(x) \leq I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda}(x)\Theta^{-\nu}(x) + \tfrac{\nu}{\lambda-\nu} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Theta^{\lambda-\nu}(x).$$

This implies that

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(x) \leq \tfrac{\lambda-\nu}{\lambda} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda}(x)\Theta^{-\nu}(x) + \tfrac{\nu}{\lambda} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Theta^{\lambda-\nu}(x). \tag{39}$$

Next, by making use of (36), we obtain

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(x) \leq \tfrac{\lambda-\nu}{\lambda} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda}(x)\Theta^{-\nu}(x) + \tfrac{\nu}{\lambda} I_{(\beta_k),m}^{(\gamma_k),(\delta_k)} \Xi^{\lambda-\nu}(x), \tag{40}$$

which leads to the desired result. □

Theorem 8. *Consider three positive functions, namely, Ξ, Θ, and h, to also be continuous on the interval $[0,\infty)$ satisfying*

$$(\Theta(t)-\Theta(u))\left(\tfrac{\Xi(u)}{h(u)} - \tfrac{\Xi(t)}{h(t)}\right) \geq 0, t,u \in [0,x), x > 0. \tag{41}$$

Then, for every $x > 0$, we state that

$$\frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[h(x)]} \geq \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[(\Theta\Xi)(x)]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[(\Theta h)(x)]}. \tag{42}$$

Proof. Given that Ξ, Θ, and h are positive as well as continuous functions on the interval $[0,\infty)$, by making use of (41), we obtain

$$\Theta(t)\tfrac{\Xi(u)}{h(u)} + \Theta(u)\tfrac{\Xi(t)}{h(t)} - \Theta(u)\tfrac{\Xi(u)}{h(u)} - \Theta(t)\tfrac{\Xi(t)}{h(t)} \geq 0, t,u \in [0,x), x > 0. \tag{43}$$

Consider the product of (43) with $h(t)h(u)$; then, we have

$$\Theta(t)\Xi(u)h(t) + \Theta(u)\Xi(t)h(u) - \Theta(u)\Xi(u)h(t) - \Theta(t)\Xi(t)h(u) \geq 0. \tag{44}$$

Next, an integration of the product of (9) and (44) for the variable $t \in (0,x)$ leads to the following:

$$\frac{\Xi(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right.\right]\Theta(t)h(t)dt$$
$$+\frac{\Theta(u)h(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right.\right]\Xi(t)dt$$
$$-\frac{\Xi(u)\Theta(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right.\right]h(t)dt$$
$$-\frac{h(u)}{x}\int_0^x H^{m,0}_{m,m}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)^m_1\end{matrix}\right.\right]\Xi(t)\Theta(t)dt.$$

This implies that

$$\begin{aligned}&\Xi(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]+\Theta(u)h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]\\&-\Theta(u)\Xi(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]-h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]\geq 0.\end{aligned}\tag{45}$$

A product of (45) with $\mathfrak{F}(x,u)$ leads to the subsequent expression after integration over u

$$\begin{aligned}&I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]\\&-I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]-I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]\geq 0.\end{aligned}$$

Therefore, we obtain

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]\leq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)].\tag{46}$$

This leads to the required result. □

Theorem 9. *Consider three positive functions Ξ, Θ, and h to be continuous over $[0,\infty)$ so that*

$$(\Theta(t)-\Theta(u))\left(\frac{\Xi(u)}{h(u)}-\frac{\Xi(t)}{h(t)}\right)\geq 0, t,u\in[0,x), x>0.\tag{47}$$

Then, we can state that

$$\frac{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(\Theta h)(x)]+I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]}{I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[h(x)]}\geq 1.\tag{48}$$

Proof. Consider the product of (31) with (34) and obtain the subsequent result after integrating this product over $u\in(0,x)$:

$$\begin{aligned}&I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]+I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]\\&-I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]-I^{(\varepsilon_k),(\eta_k)}_{(\beta_k),m}[h(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta\Xi)(x)]\geq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[\Xi(x)]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[(\Theta h)(x)]+I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[(\Theta h)(x)]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)]\\ \geq &I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[(\Theta\Xi)(x)]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[h(x)]+I_{(\beta_k),m}^{(\varepsilon_k),(\eta_k)}[h(x)]I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[(\Theta\Xi)(x)],\end{aligned}$$

which completes the steps of the required proof. □

Remark 4. *An application of the Theorem 9 by replacing ε_k with γ_k and η_k with δ_k leads to the statement of Theorem 8.*

Theorem 10. *Consider a pair of positive continuous functions, namely Ξ and h, satisfying $\Xi \leq h$ over the interval $[0,\infty)$. Suppose $\frac{\Xi}{h}$ is decreasing while Ξ is increasing over the interval $[0,\infty)$. Then, for all $x > 0$ as well as for any $\lambda \geq 1$, we state that*

$$\frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[h(x)]} \geq \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi^{\lambda}(x)]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[h^{\lambda}(x)]}. \tag{49}$$

Proof. A substitution of $\Theta = \Xi^{\lambda-1}$ in the statement of Theorem 8 leads to the following:

$$\frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[\Xi(x)]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}[h(x)]} \geq \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(\Xi\Xi^{\lambda-1}\right)(x)\right]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(h\Xi^{\lambda-1}\right)(x)\right]}. \tag{50}$$

The inequality $\Xi \leq h$ leads to the following:

$$h\Xi^{\lambda-1}(x) \leq h^{\lambda}(x). \tag{51}$$

Integrating the product of (9) and (51) over the interval $t \in (0,x)$, we obtain

$$\begin{aligned}&\frac{1}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]\Xi^{\lambda-1}h(t)dt\\ \leq&\frac{1}{x}\int_0^x H_{m,m}^{m,0}\left[\frac{t}{x}\left|\begin{matrix}\left(\gamma_k+\delta_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\\ \left(\gamma_k+1-\frac{1}{\beta_k},\frac{1}{\beta_k}\right)_1^m\end{matrix}\right.\right]h^{\lambda}(t)dt,\end{aligned}$$

which leads to the following:

$$I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] \leq I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(h^{\lambda}(x)\right]\right.. \tag{52}$$

Making use of (52), we state that

$$\frac{1}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(h\Xi^{\lambda-1}\right)(x)\right]} \geq \frac{1}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[h^{\lambda}(x)\right]},$$

and therefore we obtain

$$\frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(\Xi\Xi^{\lambda-1}\right)(x)\right]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\left(h\Xi^{\lambda-1}\right)(x)\right]} \geq \frac{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[\Xi^{\lambda}(x)\right]}{I_{(\beta_k),m}^{(\gamma_k),(\delta_k)}\left[h^{\lambda}(x)\right]}. \tag{53}$$

Hence, a combination of (50) and (53) leads to the required result. □

Theorem 11. *Consider a pair of positive continuous functions, namely Ξ and h, satisfying $\Xi \leq h$ over the interval $[0,\infty)$. Suppose $\frac{\Xi}{h}$ is decreasing and Ξ is increasing on $[0,\infty)$, then for all $x > 0$ as well as for any $\lambda \geq 1$, we obtain*

$$\frac{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h^{\lambda}(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h^{\lambda}(x)]}{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi^{\lambda}(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi^{\lambda}(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]} \geq 1. \tag{54}$$

Proof. Substituting $\Theta = \Xi^{\lambda-1}$ in the statement of Theorem (9), we obtain

$$\frac{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(h\Xi^{\lambda-1})(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(h\Xi^{\lambda-1})(x)]}{I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi^{\lambda}(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi^{\lambda}(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]} \geq 1. \tag{55}$$

According to our assumption $\Xi \leq h$, we obtain the following:

$$h\Xi^{\lambda-1}(x) \leq h^{\lambda}(x). \tag{56}$$

Next, considering the product of $F(u,t)$ as defined in (9) with (56) and then integrating it over $u \in (0,x)$ will lead to the following result:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] \leq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\lambda}(x)\right]. \tag{57}$$

Next, considering the product of (57) with $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]$ leads to the following:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] \leq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\lambda}(x)\right]. \tag{58}$$

Following the same procedure, we obtain the subsequent inequality:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] \leq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\lambda}(x)\right]. \tag{59}$$

Therefore a combined effect of (58) and (59) leads to the following:

$$\begin{aligned} &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]\mathfrak{I}^{\alpha,\beta,\zeta,\zeta',\lambda}_{0,x}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[\left(h\Xi^{\lambda-1}\right)(x)\right] \\ \leq &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\lambda}(x)\right] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}\left[h^{\lambda}(x)\right]. \end{aligned} \tag{60}$$

Finally, (55) and (60) lead to the required result. □

Theorem 12. *For all $t, u \in (0,x)$, $x > 0$, we consider three positive functions Ξ, Θ, and h to also be continuous over the interval $[0,\infty)$ satisfying*

$$(\Xi(t) - \Xi(u))(\Theta(t) - \Theta(u))(h(u) + h(t)). \tag{61}$$

In this way, we obtain the following:

$$\begin{aligned} &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)] \\ \geq &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi h)(x)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)], x > 0, \end{aligned} \tag{62}$$

where $I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1]$ is defined as (5).

Proof. Following the conditions of Theorem 12, we have

$$\begin{aligned} &\Xi(t)\Theta(t)h(t) + \Xi(t)\Theta(u)h(u) - \Xi(t)\Theta(u)h(t) - \Xi(t)\Theta(u)h(u) - \Xi(u)\Theta(t)h(t) \\ &-\Xi(u)\Theta(t)h(u) + \Xi(u)\Theta(u)h(t) + \Xi(u)\Theta(u)h(u) \geq 0. \end{aligned} \tag{63}$$

Integrating the product of (9) and (63) over the interval $t \in (0, x)$, we obtain

$$\begin{aligned} &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta h)(t)] + \Theta(u)h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(t)] - \Theta(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi h)(t)] \\ &-\Theta(u)h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(t)] - \Xi(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(t)] - \Xi(u)h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta(t)] \\ &+\Xi(u)\Theta(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(t)] + \Xi(u)\Theta(u)h(u)I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1] \geq 0. \end{aligned} \tag{64}$$

Consider the product of (9) with (62) and then integrate the result over $u \in (0, x)$ to obtain the following:

$$\begin{aligned} &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta h)(t)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(t)] \\ &+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(t)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[1] \\ &\geq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(t)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi h)(t)] \\ &+I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(t)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi h(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta(t)]. \end{aligned} \tag{65}$$

Hence, the required proof of the theorem is completed. □

Theorem 13. *Consider three positive functions* Ξ, Θ, *and* h *to also be continuous over* $[0, \infty)$ *and satisfy*

$$(\Xi(t) - \Xi(u))(\Theta(t) + \Theta(u))(h(t) + h(u)), t, u \in (0, x), x > 0. \tag{66}$$

Then, we obtain the followingforx > 0*:*

$$\begin{aligned} &I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Theta(x)] \\ &\geq I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Theta h)(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[\Xi(x)] + I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[h(x)]I^{(\gamma_k),(\delta_k)}_{(\beta_k),m}[(\Xi\Theta)(x)]. \end{aligned} \tag{67}$$

Proof. For any t and u, we have the following relation by making use of the assumption stated in Theorem 13:

$$\begin{aligned} &\Xi(t)\Theta(t)h(t) + \Xi(t)\Theta(u)h(u) + \Xi(t)\Theta(u)h(t) + \Xi(t)\Theta(u)h(u) \\ &\geq \Xi(u)\Theta(t)h(t) + \Xi(u)\Theta(t)h(u) + \Xi(u)\Theta(u)h(t) + \Xi(u)\Theta(u)h(u) \geq 0. \end{aligned} \tag{68}$$

Similarly by making use of the Theorem 12, we can prove the Theorem 13. □

5. Conclusions

Using the M-E-K fractional integral operators, the authors of this paper present a novel class of inequalities. The existing classical inequalities cited herein can be obtained as special cases of the inequalities produced in this paper. As a result, as was discussed in Section 2 of this article, these inequalities can be reduced in terms of the other non-trivial integral inequalities involving Saigo, M-S-M, R-L [5,10], and so forth. M-E-K fractional integral operators have been effectively used by authors to investigate a novel special function representation [29,30]. These operators' smart characteristics compel us to look at more outcomes for them including the classes of differential and integral equations, geometric function theory, special functions, integral transformations, and operational calculus.

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Abbreviations

The following abbreviations are used in this manuscript:

M-E-K	Multiple Erdélyi–Kober
M-S-M	Marichev–Saigo–Maeda
E-K	Erdélyi–Kober
R-L	Riemann–Liouville

References

1. Tassaddiq, A. General escape criteria for the generation of fractals in extended Jungck–Noor orbit. *Math. Comput. Simul.* **2022**, *196*, 1–14. [CrossRef]
2. Dahmani, Z.; Tabharit, L. On weighted Gruss type inequalities via fractional integration. *J. Adv. Res. Pure Math.* **2010**, *2*, 31–38. [CrossRef]
3. Dahmani, Z. New inequalities in fractional integrals. *Int. J. Nonlinear Sci.* **2010**, *9*, 493–497.
4. Dahmani, Z. On Minkowski and Hermite–Hadamard integral inequalities via fractional integral. *Ann. Funct. Anal.* **2010**, *1*, 51–58. [CrossRef]
5. Chinchane, V.L. New approach to Minkowski fractional inequalities using generalized K-fractional integral operator. *J. Indian Math. Soc.* **2018**, *85*, 32–41. [CrossRef]
6. Chinchane, V.L.; Pachpatte, D.B. New fractional inequalities involving Saigo fractional integral operator. *Math. Sci. Lett.* **2014**, *3*, 133–139. [CrossRef]
7. Houas, M. Some integral inequalities involving Saigo fractional integral operators. *J. Interdiscip. Math.* **2018**, *21*, 681–694. [CrossRef]
8. Purohit, S.D.; Raina, R.K. Chebyshev type inequalities for the Saigo fractional integral and their q-analogues. *J. Math. Inequal.* **2013**, *7*, 239–249. [CrossRef]
9. Yang, H.; Qaisar, S.; Munir, A.; Naeem, M. New inequalities via Caputo-Fabrizio integral operator with application. *Aims Math.* **2023**, *8*, 19391–19412. [CrossRef]
10. Singhal, M.; Mitta, E. On new fractional integral inequalities using Marichev–Saigo–Maeda operator. *Math. Methods Appl. Sci.* **2023**, *46*, 2055–2071. [CrossRef]
11. Tassaddiq, A.; Khan, A.; Rahman, G.; Nisar, K.S.; Abouzaid, M.S.; Khan, I. Fractional integral inequalities involving Marichev–Saigo–Maeda fractional integral operator. *J. Inequal. Appl.* **2020**, *2020*, 185. [CrossRef]
12. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Wiley: Harlow, UK; New York, NY, USA, 1994.
13. Kiryakova, V. Unified Approach to Fractional Calculus Images of Special Functions—A Survey. *Mathematics* **2020**, *8*, 2260. [CrossRef]
14. Srivastava, H.M.; Saxena, R.K. Operators of fractional integration and their applications. *Appl. Math. Comput.* **2001**, *118*, 1–52. [CrossRef]
15. Srivastava, H.M. An introductory overview of fractional-calculus operators based upon the Fox-Wright and related higher transcendental functions. *J. Adv. Engrg. Comput.* **2021**, *5*, 135–166. [CrossRef]
16. Srivastava, H.M. A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics. *Symmetry* **2021**, *13*, 2294. [CrossRef]
17. Srivastava, H.M. Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations. *J. Nonlinear Convex Anal.* **2021**, *22*, 1501–1520.
18. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies*; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204.
19. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon & Breach Science Publishers: London, UK; New York, NY, USA, 1993.
20. Dimovski, I. Operational calculus for a class of differential operators. *CR Acad. Bulg. Sci.* **1966**, *19*, 1111–1114.
21. Karp, D.B.; Prilepkina, E.G. Completely Monotonic Gamma Ratio and Infinitely Divisible H-Function of Fox. *Comput. Methods Funct. Theory* **2016**, *16*, 135–153. [CrossRef]

22. Mehrez, K. New integral representations for the Fox–Wright functions and its applications. *J. Math. Anal. Appl.* **2018**, *468*, 650–673. [CrossRef]
23. Mehrez, K. Positivity of certain classes of functions related to the Fox H-functions with applications. *Anal. Math. Phys.* **2021**, *11*, 114. [CrossRef]
24. Marichev, O.I. Volterra equation of Mellin convolution type with a horn function in the kernel. *Vescì Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk* **1974**, *1*, 128–129.
25. Raina, R.K. Solution of Abel-type integral equation involving the Appell hypergeometric function. *Integral Transforms Spec. Funct.* **2010**, *21*, 515–522. [CrossRef]
26. Saigo, M. A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Kyushu Univ.* **1978**, *11*, 135–143.
27. Saigo, M.; Maeda, N. More generalization of fractional calculus. In *Transform Methods and Special Functions, Proceedings of Second International Workshop, Varna, Bulgaria, 23–30 August 1996*; Rusev, P., Dimovski, I., Kiryakova, V., Eds.; IMI-BAS: Sofia, Bulgaria, 1998; Volume 19, pp. 386–400.
28. Sousa, J.V.d.C.; Oliveira, D.S.; de Oliveira, E.C. Grüss-Type Inequalities by Means of Generalized Fractional Integrals. *Bull. Braz. Math. Soc. New Ser.* **2019**, *50*, 1029–1047. [CrossRef]
29. Tassaddiq, A.; Srivastava, R.; Kasmani, R.M.; Alharbi, R. Complex Generalized Representation of Gamma Function Leading to the Distributional Solution of a Singular Fractional Integral Equation. *Axioms* **2023**, *12*, 1046. [CrossRef]
30. Tassaddiq, A.; Cattani, C. Fractional distributional representation of gamma function and the generalized kinetic equation. *Alex. Eng. J.* **2023**, *82*, 577–586. [CrossRef]