fractal and fractional

MDPI

Article

Exploring Impulsive and Delay Differential Systems Using Piecewise Fractional Derivatives

Hicham Saber [1], Arshad Ali [2], Khaled Aldwoah [3,*], Tariq Alraqad [1], Abdelkader Moumen [1], Amer Alsulami [4] and Nidal Eljaneid [5]

1 Department of Mathematics, College of Science, University of Ha'il, Ha'il 55473, Saudi Arabia; hi.saber@uoh.edu.sa (H.S.); t.alraqad@uoh.edu.sa (T.A.); mo.abdelkader@uoh.edu.sa (A.M.)
2 Department of Mathematics, University of Malakand, Lower Dir, Chakdara 18000, Khyber Pakhtunkhwa, Pakistan; arshad.swatpk@gmail.com
3 Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah 42351, Saudi Arabia
4 Department of Mathematics, Turabah University College, Taif University, Taif 21944, Saudi Arabia
5 Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia
* Correspondence: aldwoah@yahoo.com or aldwoah@iu.edu.sa

Abstract: This paper investigates a general class of variable-kernel discrete delay differential equations (DDDEs) with integral boundary conditions and impulsive effects, analyzed using Caputo piecewise derivatives. We establish results for the existence and uniqueness of solutions, as well as their stability. The existence of at least one solution is proven using Schaefer's fixed-point theorem, while uniqueness is established via Banach's fixed-point theorem. Stability is examined through the lens of Ulam–Hyers (U-H) stability. Finally, we illustrate the application of our theoretical findings with a numerical example.

Keywords: impulsive and integral boundary conditions; fractional piecewise derivatives; nonlinear methods; variable kernel; discrete delay differential equations; existence and stability results

check for updates

Academic Editor: Gani Stamov

Received: 26 November 2024
Revised: 27 January 2025
Accepted: 6 February 2025
Published: 10 February 2025

Citation: Saber, H.; Ali, A.; Aldwoah, K.; Alraqad, T.; Moumen, A.; Alsulami, A.; Eljaneid, N. Exploring Impulsive and Delay Differential Systems Using Piecewise Fractional Derivatives. *Fractal Fract.* **2025**, *9*, 105. https://doi.org/10.3390/fractalfract9020105

1. Introduction

Non-integer-order calculus and its applications have gained significant attention in engineering and physical sciences due to its ability to describe both global and nonlocal behaviors. This facilitates the understanding and controlling of the behavior of natural and physical phenomena (see [1,2]). For a considerable period, many scientific disciplines have incorporated the fundamentals of fractional calculus into their curricula. Aeronautics, astronautics, bioengineering, chemical engineering, mechanical engineering, and marine engineering are among the fields in which it is applied (see [3]). A significant amount of research focuses on its applications (see [4]). Additionally, the tools of fractional calculus have been employed to study various problems in nonlocal elasticity [5], mechanics [6], solid mechanics [7], and diffusion processes [8]. Given the above importance and applications, researchers have extensively studied the theoretical and numerical aspects of various fractional-order problems (see, for instance, [9,10]).

Over time, new concepts in fractional calculus have been introduced. For example, Caputo and Fabrizio proposed fractional differentiation (FD), incorporating an exponential decay kernel [11], while Atangana and Baleanu introduced definitions based on the Mittag–Leffler kernel [12]. This field has also found applications in various disciplines.

Certain physical systems rely on dynamics with memory effects, causing them to behave differently over time and even transition from one fractional order to another. As a result, the piecewise derivative was introduced [13–15]. With the piecewise derivative, short-memory principles that enhance productivity and efficiency have been examined. Piecewise fractional derivatives generalize classical fractional derivatives, allowing the order of differentiation to vary over different intervals. This provides greater flexibility in modeling complex systems with multiple scales. In summary, this type of operator plays a crucial role in equations and mathematical modeling, particularly in describing complex phenomena that exhibit non-integer, nonlocal, or anomalous behavior. Due to their significance, piecewise derivatives with non-integer orders have been widely studied in recent years (see [16,17]). Fractional DEs corresponding to impulsive situations also have significant and intriguing applications in many scientific domains. Impulsive DEs, for instance, are used to model physical phenomena that exhibit discontinuous jumps and abrupt changes (see [18,19]).

In the literature, researchers have investigated numerous mathematical models with singular, non-singular, and constant kernels. Fractional derivatives and integrals with variable kernels are particularly interesting due to their adaptability in modifying the kernel (see [20]).

Moreover, implicit DEs are essential for modeling many physical phenomena. The literature contains numerous studies on implicit DEs. Similarly, DEs subject to integral boundary conditions describe various physical processes, including nonlinear gas propagation and biological problems. The author of [21] introduced new results for a Caputo non-integer-order derivative of a function concerning another function. Researchers [22] have examined fixed-time sliding mode control for robotic manipulators using non-integer-order derivatives. More recently, the authors of [23] studied a coupled system of DEs involving a power-law kernel and piecewise order, deriving theoretical conditions for their solutions.

In general, delay DEs exhibit more complex dynamics. Time delays can cause population fluctuations and contribute to unstable equilibrium states. Numerous scientists have incorporated various types of time delays into biological models to simulate feeding cycles, resource regeneration durations, maturation periods, and reaction times. We refer to important articles [24–27] discussing biological models of general delay DEs.

On the other hand, integral boundary value problems hold significant importance. Such problems arise in electromagnetic applications, fluid mechanics, and hydrodynamical phenomena. For further applications in seismology, microscopy, radio astronomy, electron emission, X-ray radiography, atomic scattering, and radar ranging, we refer to [28].

Certainly, the dynamics of evolutionary processes are sometimes subjected to sudden changes, such as shocks, harvesting, natural disasters, and earthquakes. The concept of impulsive differential equations (DEs) plays a significant role in modeling such processes. When the derivatives involved in impulsive problems are expressed using fractional calculus concepts, the operators exhibit global behavior compared to integer-order derivatives. However, traditional integer-order and conventional fractional-order derivatives have proven insufficient in accurately capturing the multi-faceted behaviors of dynamical systems.

To achieve more realistic representations, researchers have recently employed fractional piecewise derivatives. Moreover, incorporating a variable piecewise order extends the concept of fixed piecewise fractional-order derivatives. This approach provides a more precise description of crossover behaviors in evolutionary processes. In [29], researchers explored existence results and numerical methods, along with asymptotic stability conditions, for piecewise fractional-order problems. Additionally, the authors of [30] applied fractional variable-order chaotic systems for fast image encryption, while the authors of [31] utilized the piecewise concept to study short-memory non-integer-order DEs in the design of novel memristors and neural networks.

Inspired by the cited work, in this research, we consider a general integral boundary value problem of variable-kernel discrete delay differential equations (DDDEs) subjected to impulsive effects:

$$\begin{cases} {}^{C}D_{[t]}^{\alpha(t)}\theta(t) = g_1\Big(t, \theta(t), {}^{C}D_{[t]}^{\alpha(t)}\theta(t)\Big), \quad t \in V = [0, t_1), \\ {}^{C}D_{[t]}^{\alpha(t)}\theta(t) = g_2\Big(t, \theta(t), \theta(t-\lambda), {}^{C}D_{[t]}^{\alpha(t)}\theta(t)\Big), \quad t \in [t_1, T]\backslash\{t_j\}, \\ j = 2, \ldots, m, \quad 0 < \alpha(t) \le 1, \\ \Delta\theta(t_j) = \theta\Big(t_j^+\Big) - \theta\Big(t_j^-\Big) = \mathcal{W}_j\Big(\theta\Big(t_j^-\Big)\Big), \ j = 1, 2, \ldots, m, \\ \theta(0) = \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \theta_0, \quad 0 < \delta \le 1, \end{cases} \tag{1}$$

where the functions $g_1 : V \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g_2 : V \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $h : V \to \mathbb{R}$ are given to be continuous. In the following, functions g_1 and g_2 will be denoted by g. Also, $\lambda \le t_1$ stands for the discrete delay that represents the time lapses after which the past values of θ affect the current behavior of the system. $\mathcal{W}_j : \mathbb{R} \to \mathbb{R}$, $j = 1, 2, \ldots, m$, where t_j holds inequalities $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta\theta(t_j) = \theta\Big(t_j^+\Big) - \theta\Big(t_j^-\Big) = \theta\Big(t_j^+\Big) - \theta(t_j) = \mathcal{W}_j\Big(\theta\Big(t_j^-\Big)\Big)$, $\theta\Big(t_j^+\Big) = \lim_{x\to 0^+}\theta(t_j + x)$, and $\theta\Big(t_j^-\Big) = \lim_{x\to 0^-}\theta(t_j + x)$.

It is important to note that our proposed framework and results hold for both delay and non-delay systems. Specifically, in the absence of time delay, the system remains well defined, and the derived stability conditions still apply, reducing to the special case where $\lambda = 0$. This generalization highlights the flexibility of our approach in modeling various dynamical behaviors.

Our work uses a novel approach to modeling complex systems incorporating the impulsive effects using piecewise fractional derivatives. The model can handle sudden changes and memory effects, leading to more accurate predictions. Unlike the traditional methods, our approach handles the non-integer order and nonlocal behavior of complex systems, providing a more accurate and comprehensive understanding of their dynamics. The main motivation for carrying out this research is to develop more accurate models. Because many real-world systems exhibit complex, nonlinear dynamics cannot be accurately captured with the traditional integer-order or fractional-order models.

We define

$$\alpha(t) = \begin{cases} \alpha_0, & 0 < t \le t_1, \\ \alpha_1, & t_1 < t \le t_2 \\ \vdots \\ \alpha_m, & t_m < t \le T, \end{cases} \tag{2}$$

where $\alpha_j \in (0, 1]$ is a finite sequence of real numbers, with $j = 0, 1, \ldots, m$. For non-negative increasing functions $y_0, y_1, \ldots, y_m$, we define

$${}^{C}D_{[t]}^{\alpha(t)}\theta(t) = \begin{cases} {}^{C}D_{[t]}^{\alpha_0, y_0}\theta(t), & 0 < t \le t_1, \\ {}^{C}D_{[t]}^{\alpha_1, y_1}\theta(t), & t_1 < t \le t_2 \\ \vdots \\ {}^{C}D_{[t]}^{\alpha_m, y_m}\theta(t), & t_m < t \le T, \end{cases} \tag{3}$$

where ${}^{C}D[t]^{\alpha j, yj}\theta(t)$ represents the variable CFD of order αj of $\theta(t)$ with respect to y_j. In (3), we observe that the order of the derivative changes and is defined for m subintervals; therefore, our problem can be treated as a problem of piecewise derivatives.

The remainder of this manuscript is structured as follows: In Section 2, preliminary results are presented. In Section 3, an auxiliary result providing a solution representation is given. In Section 4, the main results concerning the existence of solutions are discussed. In Section 5, stability results are derived. In Section 6, the obtained results are applied to a general problem to validate their effectiveness. Finally, in Section 7, the conclusion is presented.

2. Basic Results

In this section, some results and basic definitions are given. We define

$$\mathbb{E} = \Big\{\theta : V \to \mathbb{R} : \theta \in C(V_q, \mathbb{R}) \text{ and } \theta\big(t_q^+\big),\ \theta\big(t_q^-\big) \text{ exist so that } \Delta\theta(t_q) = \theta\big(t_q^+\big) - \theta\big(t_q^-\big), \text{ for } q = 1, 2, \ldots, m\Big\}. \tag{4}$$

As mentioned above, $[0, T] =: V$, while V_q is the set of impulsive points. The space $(\mathbb{E}, \| \theta \|_{\mathbb{E}})$ is a Banach space with the norm $\| \theta \|_{\mathbb{E}} = \max_{t \in V} |\theta(t)|$. We set $V' := V \backslash \{t_1, \ldots, t_m\}$.

Definition 1 ([1,2]). *The usual fractional-order integral of function $\theta \in C[0, T]$ in Riemann–Liouville sense is defined by*

$$I_{0^+}^{\alpha} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \theta(u) du. \tag{5}$$

Definition 2 ([21]). *The fractional-order integral of a function $\theta \in C[0, T]$ with respect to the function $y \in C[0, T]$ in Riemann–Liouville sense is defined by*

$$I_{0^+}^{\alpha, y} \theta(t) = \frac{1}{\Gamma(\alpha)} \int_0^t y'(u)(y(t) - y(u))^{\alpha-1} \theta(u) du.$$

Definition 2 can be generalized to variable order by replacing the constant order α with a function $\alpha : [0, T] \subset \mathbb{R}^+ \to (0, 1)$, [32].

Definition 3 ([1,2]). *The usual fractional-order derivative of function $\theta \in C[0, T]$ in Caputo sense is defined by*

$$^C D_{0^+}^{\alpha} \theta(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-u)^{n-\alpha-1} \theta^{(n)}(u) du,$$

where $n-1 < \alpha \le n$. Moreover, for $\alpha \in (0, 1]$, we have

$$^C D_{0^+}^{\alpha} \theta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-u)^{-\alpha} \theta'(u) du.$$

Definition 4 ([21]). *The fractional-order derivative of $\theta \in C[0, T]$ with respect to $y \in C[0, T]$ in Caputo sense is defined by*

$$^C D_{0^+}^{\alpha, y} \theta(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t y'(u)(y(t) - y(u))^{n-\alpha-1} \theta^{(n)}(u) du,$$

where $n-1 < \alpha \le n$. Further, if $\alpha \in (0, 1]$, then one has

$$^C D_{0^+}^{\alpha, y} \theta(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t y'(u)(y(t) - y(u))^{-\alpha} \theta'(u) du.$$

Definition 4 can be generalized to variable order by replacing the constant order α with a function $\alpha : [0,T] \subset \mathbb{R}^+ \to (0,1)$, [32].

Lemma 1 ([21]). *Let $\theta \in C[0,T]$. Then, for $0 < \alpha \leq 1$, one has*

$$ {}^{C}D_{0^+}^{\alpha,y}\left[I_{0^+}^{\alpha,y}\theta(t)\right] = \theta(t), $$

and

$$ I_{0^+}^{\alpha,y}\left[{}^{C}D_{0^+}^{\alpha,y}\theta(t)\right] = \theta(t) - \theta(0). $$

In addition, ${}^{C}D_{0^+}^{\alpha,y}\theta(t) = 0$ iff the function θ is constant.

Lemma 2 ([21]). *For $\alpha \in (0,1]$, the problem*

$$ \begin{cases} {}^{C}D_{0^+}^{\alpha,y}\theta(t) = \Phi(t), \\ \theta(0) = \theta_0, \end{cases} $$

has the following solution:

$$ \theta(t) = \theta_0 + \frac{1}{\Gamma(\alpha)}\int_0^t y'(u)(y(t)-y(u))^{\alpha-1}\phi(u)du, $$

where $\Phi(t) \in C[0,T]$.

Theorem 1 ([33]). *Let $\mathcal{M}$ be a closed and non-empty subset of a Banach space, say X. If $\mathcal{N} : \mathcal{M} \longrightarrow \mathcal{M}$ is a contraction, then there exists a unique fixed point of $\mathcal{N}$.*

Theorem 2 ([34]). *Let S be a norm linear space and $\mathcal{W}$ be its convex subset with $0 \in \mathcal{W}$. Assume that $\mathcal{N} : \mathcal{W} \to \mathcal{W}$ is a completely continuous operator. Then, either the set $\mathcal{X} = \{\theta \in \mathcal{W} : \theta = \xi\mathcal{N}\theta;\ 0 < \xi < 1\}$ is unbounded or $\mathcal{N}$ has a fixed point in $\mathcal{W}$.*

3. Solution Representation of Problem (1)

Lemma 3. *Let $\alpha \in (0,1]$ and let $\Phi : V \to \mathbb{R}$ be continuous. A function $\theta \in \mathbb{E}$ is the solution of the fractional integral equation*

$$ \theta(t) = \begin{cases} \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1} \\ \quad \Phi(u)du, \quad if\ t \in [0,t_1], \\ \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1} \\ \quad \Phi(u)du \ + \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1} \\ \quad \Phi(u)du + \mathcal{W}_1\theta(t_1^-), \quad if\ t \in (t_1,t_2], \\ \quad \vdots \\ \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \sum_{j=1}^q \mathcal{W}_j\theta\left(t_j^-\right) + \sum_{j=1}^q \frac{1}{\Gamma(\alpha_{j-1})} \\ \quad \int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j) - y_{j-1}(u)\right)^{\alpha_{j-1}-1}\Phi(u)du \\ \quad + \frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\Phi(u)du, \\ \quad if\ \ t \in \left(t_q, t_{q+1}\right],\ q = 1,2,\ldots,m, \end{cases} \tag{6} $$

if and only if it is a solution of (7):

$$\begin{cases} {}^{C}D_{[t]}^{\alpha(t)}\theta(t) = \Phi(t), \quad t \in [0,T]\backslash\{t_q\}, \quad q = 1,2,\ldots,m, \\ \Delta\theta(t_q) = \theta\left(t_q^+\right) - \theta\left(t_q^-\right) = \mathcal{W}_q\theta\left(t_q^-\right), \quad q = 1,2,\ldots,m, \\ \theta(0) = \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \theta_0, \end{cases} \tag{7}$$

where $[t] = t_q$ *if* $t \in (t_q, t_{q+1}]$, $q = 0,1,\ldots,$ *and* $t_0 = 0$.

Proof. Assume that θ satisfies (7). If $t \in [0, t_1]$, then

$${}^{C}D_{[t]}^{\alpha_0,y_0}\theta(t) = \Phi(t), [t] = 0,$$

Using Lemma 2, we obtain

$$\theta(t) = \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)} \int_0^t y_0'(u)(y_0(t) - y_0(u))^{\alpha_0-1}\Phi(u)du.$$

This gives

$$\theta(t_1^-) = \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} y_0'(u)(y_0(t_1) - y_0(u))^{\alpha_0-1}\Phi(u)du.$$

Applying the impulse $\theta(t_1^-) = \theta(t_1^+) - \mathcal{W}_1\theta(t_1^-)$, we obtain

$$\theta(t_1^+) = \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)} \int_0^{t_1} y_0'(u)(y_0(t_1) - y_0(u))^{\alpha_0-1}\Phi(u)du + \mathcal{W}_1\theta(t_1^-).$$

If $t \in (t_1, t_2]$, then

$${}^{C}D_{[t]}^{\alpha_1,y_1}\theta(t) = \Phi(t), \ [t] = t_1.$$

Using Lemma 2, we obtain

$$\begin{aligned} \theta(t) &= \theta(t_1^+) + \tfrac{1}{\Gamma(\alpha_1)} \int_{t_1}^t y_1'(u)(y_1(t) - y_1(u))^{\alpha_1-1}\Phi(u)du \\ &= \theta(t_1^-) + \mathcal{W}_1\theta(t_1^-) + \tfrac{1}{\Gamma(\alpha_1)} \int_{t_1}^t y_1'(u)(y_1(t) - y_1(u))^{\alpha_1-1}\Phi(u)du \\ &= \theta_0 + \int_0^T \tfrac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \tfrac{1}{\Gamma(\alpha_0)} \int_0^{t_1} y_0'(u)(y_0(t_1) - y_0(u))^{\alpha_0-1}\Phi(u)du \\ &\quad + \tfrac{1}{\Gamma(\alpha_1)} \int_{t_1}^t y_1'(u)(y_1(t) - y_1(u))^{\alpha_1-1}\Phi(u)du + \mathcal{W}_1\theta(t_1^-). \end{aligned}$$

This gives

$$\begin{aligned} \theta(t_2^-) &= \theta_0 + \int_0^T \tfrac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \tfrac{1}{\Gamma(\alpha_0)} \int_0^{t_1} y_0'(u)(y_0(t_1) - y_0(u))^{\alpha_0-1}\Phi(u)du \\ &\quad + \tfrac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t_2} y_1'(u)(y_1(t_2) - y_1(u))^{\alpha_1-1}\Phi(u)du + \mathcal{W}_1\theta(t_1^-). \end{aligned}$$

Applying the impulse $\theta(t_2^-) = \theta(t_2^+) - \mathcal{W}_2\theta(t_2^-)$, we obtain

$$\begin{aligned} \theta(t_2^+) &= \theta_0 + \int_0^T \tfrac{(T-u)^{\delta-1}}{\Gamma(\delta)} h(\theta(u))du + \tfrac{1}{\Gamma(\alpha_0)} \int_0^{t_1} y_0'(u)(y_0(t_1) - y_0(u))^{\alpha_0-1}\Phi(u)du \\ &\quad + \tfrac{1}{\Gamma(\alpha_1)} \int_{t_1}^{t_2} y_1'(u)(y_1(t_2) - y_1(u))^{\alpha_1-1}\Phi(u)du + \mathcal{W}_1\theta(t_1^-) + \mathcal{W}_2\theta(t_2^-). \end{aligned}$$

If $t \in (t_2, t_3]$, then

$${}^{C}D_{[t]}^{\alpha_2,y_2}\theta(t) = \Phi(t), \ [t] = t_2.$$

Using Lemma 2, we obtain

$$\begin{aligned}
\theta(t) &= \theta\left(t_2^+\right) + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t y_2'(u)(y_2(t)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&= \theta\left(t_2^-\right) + \mathcal{W}_2\theta\left(t_2^-\right) + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t y_2'(u)(y_2(t)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&= \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\Phi(u)du \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2} y_1'(u)(y_1(t_2)-y_1(u))^{\alpha_1-1}\Phi(u)du + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^t y_2'(u)(y_2(t)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&+ \mathcal{W}_1\theta\left(t_1^-\right) + \mathcal{W}_2\theta\left(t_2^-\right),
\end{aligned}$$

which implies that

$$\begin{aligned}
\theta\left(t_3^-\right) &= \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\Phi(u)du \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2} y_1'(u)(y_1(t_2)-y_1(u))^{\alpha_1-1}\Phi(u)du + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^{t_3} y_2'(u)(y_2(t_3)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&+ \mathcal{W}_1\theta\left(t_1^-\right) + \mathcal{W}_2\theta\left(t_2^-\right).
\end{aligned}$$

Applying the impulse $\theta\left(t_3^-\right) = \theta\left(t_3^+\right) - \mathcal{W}_3\theta\left(t_3^-\right)$, we obtain

$$\begin{aligned}
\theta\left(t_3^+\right) &= \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\Phi(u)du \\
&+\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2} y_1'(u)(y_1(t_2)-y_1(u))^{\alpha_1-1}\Phi(u)du + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^{t_3} y_2'(u)(y_2(t_3)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&+\mathcal{W}_1\theta\left(t_1^-\right) + \mathcal{W}_2\theta\left(t_2^-\right) + \mathcal{W}_3\theta\left(t_3^-\right).
\end{aligned}$$

Let

$$\begin{aligned}
\theta\left(t_q^+\right) &= \theta_0 + \mathcal{W}_1\theta\left(t_1^-\right) + \mathcal{W}_2\theta\left(t_2^-\right) + \mathcal{W}_3\theta\left(t_3^-\right) + \cdots + \mathcal{W}_q\theta\left(t_q^-\right) \\
&+ \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\Phi(u)du \\
&+ \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^{t_2} y_1'(u)(y_1(t_2)-y_1(u))^{\alpha_1-1}\Phi(u)du + \frac{1}{\Gamma(\alpha_2)}\int_{t_2}^{t_3} y_2'(u)(y_2(t_3)-y_2(u))^{\alpha_2-1}\Phi(u)du \\
&+ \cdots + \frac{1}{\Gamma(\alpha_{q-1})}\int_{t_{q-1}}^{t_q} y_{q-1}'(u)\left(y_{q-1}(t_q)-y_{q-1}(u)\right)^{\alpha_{q-1}-1}\Phi(u)du.
\end{aligned}$$

Then, inductively, for $t \in \left(t_q, t_{q+1}\right]$, we have

$$^{C}D_{[t]}^{\alpha_q, y_q}\theta(t) = \Phi(t), \ [t] = t_q.$$

By Lemma 2, the solution becomes

$$\begin{aligned}
\theta(t) &= \theta\left(t_q^+\right) + \frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\Phi(u)du \\
&= \theta_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du + \sum_{j=1}^{q}\mathcal{W}_j\theta\left(t_j^-\right) \\
&+ \sum_{j=1}^{q}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}\Phi(u)du \\
&+ \frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\Phi(u)du.
\end{aligned}$$

Hence, (6) holds.

Conversely, if θ satisfies (6) with $t \in [0, t_1]$, then $\theta(0) = \theta_0$ since $^{C}D_{[t]}^{\alpha(t)}$ is the left inverse of $\mathbb{I}_{[t]}^{\alpha(t)}$. Therefore, Lemma 1 yields

$$^{C}D_0^{\alpha_0, y_0}\theta(t) = \Phi(t), \ t \in [0, t_1].$$

If $t \in \left[t_q, t_{q+1}\right)$, $q = 1, \ldots, m$, then

$$^{C}D_{[t]}^{\alpha_q, y_q}\theta(t) = \Phi(t).$$

We can infer that

$$\theta\left(t_q^+\right)-\theta\left(t_q^-\right)=\mathcal{W}_q\theta\left(t_q^-\right),\ q=1,\ldots,m.$$

□

Corollary 1. *In light of Lemma 3, we give the solution for problem (1) as*

$$\theta(t)=\begin{cases}\theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}\\ \quad\times g\left(u,\theta(u),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du,\quad if\ t\in[0,t_1],\\ \\ \theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\\ \times g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du\ +\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}\\ \times g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du+\mathcal{W}_1\theta(t_1^-),\quad t\in(t_1,t_2],\\ \vdots\\ \theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\sum_{j=1}^q\mathcal{W}_j\theta\left(t_j^-\right)+\sum_{j=1}^q\frac{1}{\Gamma(\alpha_{j-1})}\\ \times\int_{t_{j-1}}^{t_j}y_{j-1}'(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du\\ +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du,\\ \qquad t\in\left(t_q,t_{q+1}\right],\ q=1,2,\ldots,m.\end{cases}\tag{8}$$

4. Existence of Solutions

In this section, we investigate two main existence results, which are at least one solution and one unique solution of the proposed problem. We proceed with defining an operator $\mathcal{N}$ as

$$(\mathcal{N}\theta)(t)=\begin{cases}\theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}\\ \quad\times g\left(u,\theta(u),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du,\quad if\ t\in[0,t_1],\\ \\ \theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\\ \times g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du\ +\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}\\ \times g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du+\mathcal{W}_1\theta(t_1^-),\quad t\in(t_1,t_2],\\ \vdots\\ \theta_0+\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}h(\theta(u))du+\sum_{j=1}^q\mathcal{W}_j\theta\left(t_j^-\right)+\sum_{j=1}^q\frac{1}{\Gamma(\alpha_{j-1})}\\ \times\int_{t_{j-1}}^{t_j}y_{j-1}'(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du\\ +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}g\left(u,\theta(u),\theta(u-\lambda),{}^C D_{[u]}^{\alpha(u)}\theta(u)\right)du,\\ \qquad t\in\left(t_q,t_{q+1}\right],\ q=1,2,\ldots,m.\end{cases}\tag{9}$$

For simplicity, we use the following notation:

$$Y_\theta(t)=g(t,\theta(t),{}^C D_{[t]}^{\alpha(t)}\theta(t)=g(t,\theta(t),v_\theta(t)),$$

$$v_\theta(t)=g(t,\theta(t),\theta(t-\lambda),{}^C D_{[t]}^{\alpha(t)}\theta(t)=g(t,\theta(t),\theta(t-\lambda),v_\theta(t)).$$

Next, we assume the following:

(H$_1$) For function g, the constants $\ell_1, \ell_2, k_1, k_2 > 0$ satisfy the Lipschitz condition

$$|g(t,\theta(t),v_\theta(t)) - g(t,\theta^*(t),v_{\theta^*}(t))| \leq \ell_1|\theta(t) - \theta^*(t)| + \ell_2|v_\theta(t) - v_{\theta^*}(t)|,$$
$$|g(t,\theta(t),\theta(t-\lambda),v_\theta(t)) - g(t,\theta^*(t),\theta^*(t-\lambda),v_{\theta^*}(t))|$$
$$\leq k_1(|\theta(t) - \theta^*(t)| + |\theta(t-\lambda) - \theta^*(t-\lambda)|) + k_2|v_\theta(t) - v_{\theta^*}(t)|, \text{for each } t \in V,\ \theta, \theta^* \in \mathbb{R}.$$

For the term $|\theta(t-\lambda) - \theta^*(t-\lambda)|$, we use the following relations:

$$|\theta(t-\lambda) - \theta^*(t-\lambda)| \quad \leq \quad \sup\{|\theta(z) - \theta^*(z)| : t-\lambda \leq z \leq t\}. \tag{10}$$

(H$_2$) Let $\mathcal{W}_q : \mathbb{R} \to \mathbb{R}$ be continuous such that

$$\left|\mathcal{W}_q(\theta) - \mathcal{W}_q(\theta^*)\right| \leq k_{\mathcal{W}}|\theta - \theta^*|; \quad k_{\mathcal{W}} > 0,\ q = 1, \ldots, m,\ \text{for } \theta, \theta^* \in \mathbb{R}.$$

(H$_3$) There exist bounded functions $\gamma_1, \gamma_2, \gamma_3, \mu_1, \mu_2, \mu_3 \in C(V, \mathbb{R})$ such that

$$|g(t,\theta(t),v_\theta(t))| \leq \gamma_1(t) + \gamma_2(t)|\theta(t)| + \gamma_3(t)|v_\theta(t)|, \text{for each } t \in V \textit{ and } \theta, v_\theta \in \mathbb{R},$$
$$|g(t,\theta(t),\theta(t-\lambda),v_\theta(t))| \leq \mu_1(t) + \mu_2(t)(|\theta(t)| + |\theta(t-\lambda)|) + \mu_3(t)|v_\theta(t)|, \text{for each } t \in V \textit{ and } \theta, v_\theta \in \mathbb{R},$$

where $\gamma_1^* = \sup_{t\in V}\gamma_1(t)$, $\gamma_2^* = \sup_{t\in V}\gamma_2(t)$, and $\gamma_3^* = \sup_{t\in V}\gamma_3(t) < 1$. And similarly, $\mu_1^* = \sup_{t\in V}\mu_1(t)$, $\mu_2^* = \sup_{t\in V}\mu_2(t)$, and $\mu_3^* = \sup_{t\in V}\mu_3(t) < 1$.

(H$_4$) There exist $\eta_1, \eta_2 > 0$ such that

$$\left|\mathcal{W}_q(\theta)\right| \leq \eta_1 + \eta_2|\theta|,\ q = 1, 2, \ldots, m,\ \theta \in \mathbb{R}$$

(H$_5$) There exists constant $k_h > 0$ such that

$$|h(\theta(t))| \leq k_h;$$

(H$_6$) We assume that

$$|h(\theta(t)) - h(\theta^*(t))| \leq k_h^*|\theta - \theta^*|; \quad k_h^* > 0.$$

Theorem 3. *Let $g : V \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and assume that $(H_3) - (H_4)$ hold. If*

$$\max\left(\frac{\gamma_2^*(y_0(T) - y_0(0))^{\alpha_0}}{(1-\gamma_3^*)\Gamma(\alpha_0+1)}, m\eta_2 + \frac{2\mu_2^*}{(1-\mu_3^*)}\sum_{j=0}^{q}\frac{(y_j(T) - y_j(t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)}\right) < 1. \tag{11}$$

then problem (1) has at least one confirmed solution in $\mathbb{E}$.

Proof. The proof of this result is based on Schaefer's fixed-point theorem. For

$$\zeta \geq \max\left(\frac{|\theta_0| + \frac{k_h T^\delta}{\Gamma(\delta+1)} + \frac{\gamma_1^*(y_0(T)-y_0(0))^{\alpha_0}}{(1-\gamma_3^*)\Gamma(\alpha_0+1)}}{1 - \frac{\gamma_2^*(y_0(T)-y_0(0))^{\alpha_0}}{(1-\gamma_3^*)\Gamma(\alpha_0+1)}}, \frac{|\theta_0| + m\eta_1 + \frac{k_h T^\delta}{\Gamma(\delta+1)} + \frac{\mu_1^*}{(1-\mu_3^*)}\sum_{j=0}^{q}\frac{(y_j(T)-y_j(t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)}}{1 - \left(m\eta_2 + \frac{2\mu_2^*}{(1-\mu_3^*)}\sum_{j=0}^{q}\frac{(y_j(T)-y_j(t_j))^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)}\right),$$

we set $\mathcal{B}_\zeta = \{\theta \in \mathbb{E} : \|\theta\|_{\mathbb{E}} \leq \zeta\}$.

Step 1:

By (9), for $\theta \in \mathcal{B}_\zeta$, we have

$$|\mathcal{N}\theta(t)| \quad \leq \quad |\theta_0| + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\theta(u))|du + \frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t) - y_0(u))^{\alpha_0-1}|\Upsilon_\theta(u)|du, \quad t \in [0, t_1].$$

By assumption (H_3), we have

$$\begin{aligned} |Y_\theta(t)| = |g(t,\theta(t),v_\theta(t))| &\leq \gamma_1(t) + \gamma_2(t)|\theta(t)| + \gamma_3(t)|Y_\theta(t)| \\ |Y_\theta(t)| &\leq \frac{\left(\sup_{t\in V}\gamma_1(t)+\sup_{t\in V}\gamma_2(t)|\theta(t)|\right)}{1-\sup_{t\in V}\gamma_3(t)} \\ &\leq \frac{\left(\gamma_1^*+\zeta\gamma_2^*\right)}{\left(1-\gamma_3^*\right)}. \end{aligned} \tag{12}$$

Also, using assumption (H_5), we have

$$|h(\theta(t))| \leq k_h.$$

Thus, we write

$$\begin{aligned} |\mathcal{N}\theta(t)| &\leq |\theta_0| + k_h \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}du + \frac{\left(\gamma_1^*+\zeta\gamma_2^*\right)}{\left(1-\gamma_3^*\right)} \int_0^t \frac{y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}}{\Gamma(\alpha_0)}du \\ &\leq |\theta_0| + \frac{k_h T^\delta}{\Gamma(\delta+1)} + \frac{\left(\gamma_1^*+\zeta\gamma_2^*\right)}{\left(1-\gamma_3^*\right)\Gamma(\alpha_0+1)}(y_0(t_1)-y_0(0))^{\alpha_0} \\ &\leq |\theta_0| + \frac{k_h T^\delta}{\Gamma(\delta+1)} + \frac{\gamma_1^*}{\left(1-\gamma_3^*\right)\Gamma(\alpha_0+1)}(y_0(T)-y_0(0))^{\alpha_0} \\ &\quad +\zeta\frac{\gamma_2^*}{\left(1-\gamma_3^*\right)\Gamma(\alpha_0+1)}(y_0(T)-y_0(0))^{\alpha_0} \\ &\leq \zeta. \end{aligned}$$

Also, for interval $\left(t_q, t_{q+1}\right],\ q = 1,2,\ldots,m$, we have

$$\begin{aligned} &|\mathcal{N}\theta(t)| \leq |\theta_0| + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\theta(u))|du + \sum_{0<t_q<t}\left|\mathcal{W}_q\theta\left(t_q^-\right)\right| \\ &+\sum_{j=1}^{q}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}\left(t_j\right)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|v_\theta(u)|du \\ &+\frac{1}{\Gamma\left(\alpha_q\right)}\int_{t_q}^{t} y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|v_\theta(u)|du \end{aligned}$$

Using assumptions (H_3) and (H_5) and result (12), we have

$$\begin{aligned} &|\mathcal{N}\theta(t)| \leq |\theta_0| + k_h\int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}du + \sum_{0<t_q<t}\left(\eta_1+\eta_2\left|\theta\left(t_q^-\right)\right|\right) \\ &+\sum_{j=1}^{q}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}\left(t_j\right)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}\frac{(\mu_1(u)+2\mu_2(u)|\theta(u)|)}{1-\mu_3(u)}du \\ &+\frac{1}{\Gamma\left(\alpha_q\right)}\int_{t_q}^{t} y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\frac{(\mu_1(u)+2\mu_2(u)|\theta(u)|)}{1-\mu_3(u)}du \\ &\leq |\theta_0| + \frac{k_h T^\delta}{\Gamma(\delta+1)} + m(\eta_1+\eta_2\|\ \theta\ \|) + \left(\frac{\left(\mu_1^*+2\mu_2^*\|\theta\|\right)}{\left(1-\mu_3^*\right)}\right) \\ &\times\left(\sum_{j=1}^{q}\frac{\left(y_{j-1}\left(t_j\right)-y_{j-1}\left(t_{j-1}\right)\right)^{\alpha_{j-1}}}{\Gamma\left(\alpha_{j-1}+1\right)} + \frac{\left(y_q(t)-y_q\left(t_q\right)\right)^{\alpha_q}}{\Gamma\left(\alpha_q+1\right)}\right) \\ &\leq |\theta_0| + \frac{k_h T^\delta}{\Gamma(\delta+1)} + m(\eta_1+\eta_2\zeta) + \left(\frac{\left(\mu_1^*+2\mu_2^*\zeta\right)}{\left(1-\mu_3^*\right)}\right) \\ &\times\left(\sum_{j=1}^{q}\frac{\left(y_{j-1}\left(t_j\right)-y_{j-1}\left(t_{j-1}\right)\right)^{\alpha_{j-1}}}{\Gamma\left(\alpha_{j-1}+1\right)} + \frac{\left(y_q(t)-y_q\left(t_q\right)\right)^{\alpha_q}}{\Gamma\left(\alpha_q+1\right)}\right) \\ &\leq |\theta_0| + m\eta_1 + \frac{k_h T^\delta}{\Gamma(\delta+1)} + \frac{\mu_1^*}{\left(1-\mu_3^*\right)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j\left(t_j\right)\right)^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)} \\ &+\left(m\eta_2 + \frac{2\mu_2^*}{\left(1-\mu_3^*\right)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j\left(t_j\right)\right)^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right)\zeta \\ &\leq \zeta. \end{aligned}$$

Thus, $\|\ \mathcal{N}\theta\ \|_{\mathbb{E}} \leq \zeta$. This shows that $\mathcal{N}$ maps $\mathcal{B}_\zeta$ into itself.

Step 2:

Consider a sequence $\{\theta_s\}_{s\in\mathbb{N}}$ such that $\theta_s \to \theta$ on $\mathcal{B}_\zeta$. The continuity of g, $\mathcal{W}_q(\theta)$, and $h(\theta)$ implies that $g(\cdot,\theta_s(t),Y_{\theta,s}(t)) \to g(\cdot,\theta(t),Y_\theta(t))$, $g(\cdot,\theta_s(t),\theta_s(t-\lambda),v_{\theta,s}(t)) \to g(\cdot,\theta(t),\theta(t-\lambda),v_\theta(t))$, and $\mathcal{W}_q(\theta_s) \to \mathcal{W}_q(\theta)$ $h(\theta_s) \to h(\theta)$ as $s \to \infty$ and $q = 1,...,m$. Moreover, for each $t \in [0,t_1]$,

$$\begin{aligned}
& |\mathcal{N}\theta_s(t) - \mathcal{N}\theta(t)| \\
\leq\ & \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} |h(\theta_s(u)) - h(\theta(u))| du + \frac{1}{\Gamma(\alpha_0)} \int_0^t y_0'(u)(y_0(t) - y_0(u))^{\alpha_0-1} \\
\times\ & |Y_{\theta,s}(u) - Y_\theta(u)| du \\
\leq\ & k_h^* \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} \| \theta_s - \theta \|_{\mathbb{E}} du + \frac{\ell_1}{(1-\ell_2)\Gamma(\alpha_0)} \int_0^t y_0'(u)(y_0(t) - y_0(u))^{\alpha_0-1} \\
\times\ & \| \theta_s - \theta \|_{\mathbb{E}} du \\
\leq\ & \frac{k_h^* T^\delta}{\Gamma(\delta+1)} \| \theta_s - \theta \|_{\mathbb{E}} + \frac{\ell_1 (y_0(t_1) - y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)} \| \theta_s - \theta \|_{\mathbb{E}}.
\end{aligned} \tag{13}$$

Also, for $t \in \left(t_q, t_{q+1}\right],\ q = 1,2,\ldots,m$,

$$\begin{aligned}
& |\mathcal{N}\theta_s(t) - \mathcal{N}\theta(t)| \\
\leq\ & \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} |h(\theta_s(u)) - h(\theta(u))| du + \sum_{0<t_q<t} \left| \mathcal{W}_q\theta_s\left(t_q^-\right) - \mathcal{W}_q\theta\left(t_q^-\right) \right| \\
& + \sum_{j=1}^{q} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j) - y_{j-1}(u)\right)^{\alpha_{j-1}-1} |v_{\theta,s}(u) - v_\theta(u)| du \\
& + \frac{1}{\Gamma(\alpha_q)} \int_{t_q}^t y_q'(u)\left(y_q(t) - y_q(u)\right)^{\alpha_q-1} |v_{\theta,s}(u) - v_\theta(u)| du \\
& \leq k_h^* \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} \| \theta_s - \theta \|_{\mathbb{E}} du + \sum_{j=1}^{q} \| \mathcal{W}_j\theta_s(\cdot) - \mathcal{W}_j\theta(\cdot) \|_{\mathcal{PC}} \\
& + \frac{2k_1}{(1-k_2)} \sum_{j=1}^{q} \frac{1}{\Gamma(\alpha_{j-1})} \int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j) - y_{j-1}(u)\right)^{\alpha_{j-1}-1} \| \theta_s - \theta \|_{\mathbb{E}} du \\
& + \frac{2k_1}{(1-k_2)\Gamma(\alpha_q)} \int_{t_q}^t y_q'(u)\left(y_q(t) - y_q(u)\right)^{\alpha_q-1} \| \theta_s - \theta \|_{\mathbb{E}} du \\
\leq\ & \frac{k_h^* T^\delta}{\Gamma(\delta+1)} \| \theta_s - \theta \|_{\mathbb{E}} + \sum_{j=1}^{q} \| \mathcal{W}_j\theta_s(\cdot) - \mathcal{W}_j\theta(\cdot) \|_{\mathbb{E}} \\
& + \frac{2k_1}{(1-k_2)} \sum_{j=0}^{q} \frac{\left(y_j(T) - y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)} \| \theta_s - \theta \|_{\mathbb{E}}.
\end{aligned} \tag{14}$$

As $s \to \infty$, θ_s and $\mathcal{W}_q(\theta_s)$ are convergent to θ and $\mathcal{W}_q(\theta)$, $q = 1,\ldots,m$, respectively. In combining (13) and (14), it follows that $\| \mathcal{N}\theta_s - \mathcal{N}\theta \|_{\mathbb{E}} \to 0,$ as $s \to \infty$. Thus, $\mathcal{N}$ is continuous.

Step 3: For arbitrary $\tau_1, \tau_2 \in [0,t_1],\ \tau_1 < \tau_2$, we have

$$\begin{aligned}
& |\mathcal{N}\theta(\tau_2) - \mathcal{N}\theta(\tau_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha_0)} \int_0^{\tau_1} y_0'(u) \left[(y_0(\tau_2) - y_0(u))^{\alpha_0-1} - (y_0(\tau_1) - y_0(u))^{\alpha_0-1} \right] |Y_\theta(u)| du \\
& + \frac{1}{\Gamma(\alpha_0)} \int_{\tau_1}^{\tau_2} y_0'(u)(y_0(\tau_2) - y_0(u))^{\alpha_0-1} |Y_\theta(u)| du \\
\leq\ & \frac{(\gamma_1^* + \gamma_2^* \|\theta\|)}{(1-\gamma_3^*)\Gamma(\alpha_0)} \int_0^{\tau_1} y_0'(u) \left[(y_0(\tau_1) - y_0(u))^{\alpha_0-1} - (y_0(\tau_2) - y_0(u))^{\alpha_0-1} \right] du \\
& + \frac{(\gamma_1^* + \gamma_2^* \|\theta\|)}{(1-\gamma_3^*)\Gamma(\alpha_0)} \int_{\tau_1}^{\tau_2} y_0'(u)(y_0(\tau_2) - y_0(u))^{\alpha_0-1} du \\
\leq\ & \frac{(\gamma_1^* + \gamma_2^* \zeta)}{(1-\gamma_3^*)\Gamma(\alpha_0+1)} \left[(y_0(\tau_2) - y_0(\tau_1))^{\alpha_0} + (y_0(\tau_1) - y_0(0))^{\alpha_0} - (y_0(\tau_2) - y_0(0))^{\alpha_0} \right] \\
+\ & \frac{(\gamma_1^* + \gamma_2^* \zeta)}{(1-\gamma_3^*)\Gamma(\alpha_0+1)} (y_0(\tau_2) - y_0(\tau_1))^{\alpha_0}
\end{aligned} \tag{15}$$

$$\leq \frac{2\left(\gamma_1^* + \gamma_2^* \zeta\right)}{(1-\gamma_3^*)\Gamma(\alpha_0+1)} (y_0(\tau_2) - y_0(\tau_1))^{\alpha_0}. \tag{16}$$

Since y_0 is continuous, $|\mathcal{N}\theta(\tau_2) - \mathcal{N}\theta(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$. Similarly, for a large interval $\left(t_q, t_{q+1}\right],\ q = 1,2,\ldots,m$, we obtain the accompanying inequality

$$
\begin{aligned}
& |\mathcal{N}\theta(\tau_2)-\mathcal{N}\theta(\tau_1)| \\
\leq\ & \frac{1}{\Gamma(\alpha_q)}\int_{t_q}^{\tau_1} y_q'(u)\Big[\big(y_q(\tau_1)-y_q(u)\big)^{\alpha_q-1}-\big(y_q(\tau_2)-y_q(u)\big)^{\alpha_q-1}\Big] \\
& \times|v_\theta(u)|du+\frac{1}{\Gamma(\alpha_q)}\int_{\tau_1}^{\tau_2} y_q'(u)\big(y_q(\tau_2)-y_q(u)\big)^{\alpha_q-1}|v_\theta(u)|du \\
\leq\ & \frac{(\mu_1^*+2\mu_2^*\zeta)}{(1-\mu_3^*)\Gamma(\alpha_q+1)}\Big[\big(y_q(\tau_2)-y_q(\tau_1)\big)^{\alpha_q} \\
& +\big(y_q(\tau_1)-y_q(t_q)\big)^{\alpha_q}-\big(y_q(\tau_2)-y_q(t_q)\big)^{\alpha_q}\Big]+\frac{(\mu_1^*+2\mu_2^*\zeta)}{(1-\mu_3^*)\Gamma(\alpha_q+1)}\Big[\big(y_q(\tau_2)-y_q(\tau_1)\big)^{\alpha_q}\Big] \\
\leq\ & \frac{2(\mu_1^*+2\mu_2^*\zeta)}{(1-\mu_3^*)\Gamma(\alpha_q+1)}\big(y_q(\tau_2)-y_q(\tau_1)\big)^{\alpha_q}.
\end{aligned} \tag{17}
$$

Since y_q ($q=1,2,\ldots,m$) is continuous, $|\mathcal{N}\theta(\tau_2)-\mathcal{N}\theta(\tau_1)|\to 0$ as $\tau_2\to\tau_1$.

Hence, $\mathcal{N}\theta$ is equi-continuous on V.

On the other hand, according to Step 1, $\mathcal{A}\mathcal{B}_\zeta\subset\mathcal{B}_\zeta$ is uniformly bounded. Thus, $\mathcal{N}$ is completely continuous, and hence, problem (1) has a solution. □

Theorem 4. *Assume $g: V\times\mathbb{R}\times\mathbb{R}$ is continuous. If (H_1), (H_2), and (H_6) hold with*

$$
Y:=\max\left(\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+\ell_1\frac{(y_0(t_1)-y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)},\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+mk_{\mathcal{W}}+\frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\big(y_j(T)-y_j(t_j)\big)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)<1. \tag{18}
$$

then problem (1) has a unique solution in $\mathbb{E}$.

Proof. The operator $\mathcal{N}:\mathbb{E}\to\mathbb{E}$ is well defined by Theorem 3.

For arbitrary $\theta,\theta^*\in\mathbb{E}$ and $t\in[0,t_1]$. Then, we obtain

$$
\begin{aligned}
|\mathcal{N}\theta(t)-\mathcal{N}\theta^*(t)| \leq\ & \int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\theta(u))-h(\theta^*(u))|du+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1} \\
& \times|Y_\theta(u)-Y_{\theta^*}(u)|du \\
\leq\ & k_h^*\int_0^T\frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|\theta(u)-\theta^*(u)|du+\frac{\ell_1}{(1-\ell_2)\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1} \\
& \times|\theta(u)-\theta^*(u)|du \\
\leq\ & \frac{k_h^*T^\delta}{\Gamma(\delta+1)}\|\theta-\theta^*\|+\ell_1\frac{(y_0(t_1)-y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)}\|\theta-\theta^*\|.
\end{aligned}
$$

Thus,

$$
\|\mathcal{N}\theta-\mathcal{N}\theta^*\|_{\mathbb{E}}\leq Y\|\theta-\theta^*\|_{\mathbb{E}} \text{ on } [0,t_1]. \tag{19}
$$

Similarly, for $t\in\big(t_q,t_{q+1}\big]$, $q=1,2,\ldots,m$, we have

$$
\begin{aligned}
&|\mathcal{N}\theta(t)-\mathcal{N}\theta^*(t)| \\
&\leq \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\theta(u))-h(\theta^*(u))|du+\sum_{0<t_q<t}\left|\mathcal{W}_q\theta\left(t_q^-\right)-\mathcal{W}_q\theta^*\left(t_q^-\right)\right| \\
&+\sum_{j=1}^{q}\frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|v_\theta(u)-v_{\theta^*}(u))|du \\
&+\frac{1}{\Gamma\left(\alpha_q\right)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|v_\theta(u)-v_{\theta^*}(u)|du \\
\leq\ & k_h^*\int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|\theta(u)-\theta^*(u)|du+\sum_{0<t_q<t}k_{\mathcal{W}}\left|\theta\left(t_q^-\right)-\theta^*\left(t_q^-\right)\right| \\
&+\sum_{j=1}^{q}\frac{2k_1}{(1-k_2)\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|\theta(u)-\theta^*(u)|du \\
&+\frac{2k_1}{(1-k_2)\Gamma\left(\alpha_q\right)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|\theta(u)-\theta^*(u)|du \\
\leq\ & \frac{k_h^*T^\delta}{\Gamma(\delta+1)}\|\,\theta-\theta^*\,\|+mk_{\mathcal{W}}\|\,\theta-\theta^*\,\|+\frac{2k_1}{(1-k_2)}\|\,\theta-\theta^*\,\| \\
&\times\left(\sum_{j=1}^{q}\frac{\left(y_{j-1}(t_j)-y_{j-1}(t_{j-1})\right)^{\alpha_{j-1}}}{\Gamma\left(\alpha_{j-1}+1\right)}+\frac{\left(y_q(t)-y_q(t_q)\right)^{\alpha_q}}{\Gamma\left(\alpha_q+1\right)}\right) \\
\leq\ & \left(\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+mk_{\mathcal{W}}+\frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right)\|\,\theta-\theta^*\,\|.
\end{aligned}
$$

Thus,

$$\|\,\mathcal{N}\theta-\mathcal{N}\theta^*\,\|_{\mathbb{E}}\leq Y\|\,\theta-\theta^*\,\|_{\mathbb{E}} \text{ on } \left(t_q,t_{q+1}\right],\ q=1,2,\ldots,m. \tag{20}$$

Inequality (24) with (19) and (20) shows that $\mathcal{N}$ is a strict contraction on $\mathbb{E}$. By applying Theorem 1, we obtain the result. □

5. Ulam–Hyers (U-H) Stability

In the current section, we analyze the U-H stability of the proposed problem (1). We adopted the following U-H stability definitions from [35].

Definition 5. *The solution θ of problem (1) is U-H stable if we find a constant $N_g>0$ such that for any $\epsilon>0$ and any solution $\overline{\theta}\in\mathbb{E}$ of the inequality*

$$
\begin{cases}
|{}^CD_{[t]}^{\alpha(t)}\overline{\theta}(t)-g\left(t,\overline{\theta}(t),{}^CD_{[t]}^{\alpha(t)}\overline{\theta}(t)\right)|\leq\epsilon, & t\in[0,t_1) \\
|{}^CD_{[t]}^{\alpha(t)}\overline{\theta}(t)-g\left(t,\overline{\theta}(t),\overline{\theta}(t-\lambda),{}^CD_{[t]}^{\alpha(t)}\overline{\theta}(t)\right)|\leq\epsilon, & t\in[t_1,T]\backslash\{t_j\} \\
|\Delta\overline{\theta}(t_j)-\mathcal{W}_j\left(\overline{\theta}\left(t_j^-\right)\right)|\leq\epsilon, & j=1,2,\cdots,m,
\end{cases} \tag{21}
$$

their exists a unique solution θ of problem (1) in $\mathbb{E}$ such that the following relation satisfies

$$\|\,\overline{\theta}-\theta\,\|\leq N_g\epsilon.$$

Definition 6. *The solution of problem (1) is G-U-H stable if we find*

$$\phi:(0,\infty)\to(0,\infty),\ \phi(0)=0,$$

so that for any solution of inequality (21), the following relation satisfies

$$\|\,\overline{\theta}-\theta\,\|\leq N_g\phi(\epsilon).$$

Remark 1. *$\overline{\theta}$ is the solution in $\mathbb{E}$ for inequality (21) iff there exists a function $\varkappa \in \mathbb{E}$, which depends on $\overline{\theta}$, so that for any t, the following hold:*

(i) $|\vartheta(t)| \leq \epsilon,\ |\varkappa(t)| \leq \epsilon,\ |\varkappa_j| \leq \epsilon;$

(ii) ${}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t) = g\left(t,\overline{\theta}(t),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right) + \vartheta(t),$
${}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t) = g\left(t,\overline{\theta}(t),\overline{\theta}(t-\lambda),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right) + \varkappa(t);$

(iii) $\Delta\overline{\theta}(t_j) = \mathcal{W}_j\left(\overline{\theta}\left(t_j^-\right)\right) + \varkappa_j,\ j = 1,2,\ldots,m.$

By Remark 1, we have

$$\begin{cases} {}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t) = g\left(t,\overline{\theta}(t),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right) + \vartheta(t), \quad t \in [0,t_1), \\ {}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t) = g\left(t,\overline{\theta}(t),\overline{\theta}(t-\lambda),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right) + \varkappa(t), \quad t \in [t_1,T]\backslash\{t_j\} \\ j = 1,2,\cdots,m, \quad 0 < \alpha(t) \leq 1, \\ \Delta\overline{\theta}(t_j) = \mathcal{W}_j\left(\overline{\theta}\left(t_j^-\right)\right) + \varkappa_j,\ j = 1,2,\cdots,m, \\ \overline{\theta}(0) = \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h\left(\overline{\theta}(u)\right)du + \overline{\theta}_0, \end{cases} \tag{22}$$

Lemma 4. *The solution of perturbed problem (22) is given by*

$$\overline{\theta}(t) = \begin{cases} \overline{\theta}_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h\left(\overline{\theta}(u)\right)du + \frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}Y_{\overline{\theta}}(u)du \\ \quad + \frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}\vartheta(u)du, \quad if\ t \in [0,t_1], \\ \overline{\theta}_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h\left(\overline{\theta}(u)\right)du + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}v_{\overline{\theta}}(u)du \\ \quad + \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}v_{\overline{\theta}}(u)du + \mathcal{W}_1\overline{\theta}(t_1^-) + \varkappa_1 \\ \quad + \frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}\varkappa(u)du \\ \quad + \frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}\varkappa(u)du \quad if\ t \in (t_1,t_2], \\ \vdots \\ \overline{\theta}_0 + \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)} h\left(\overline{\theta}(u)\right)du + \sum_{j=1}^q \mathcal{W}_j\overline{\theta}\left(t_j^-\right) + \sum_{j=1}^q \varkappa_j + \sum_{j=1}^q \frac{1}{\Gamma\left(\alpha_{j-1}\right)} \\ \quad \times \int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}\left(t_j\right)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}v_{\overline{\theta}}(u)du \\ \quad + \frac{1}{\Gamma\left(\alpha_q\right)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}v_{\overline{\theta}}(u)du \\ \quad + \sum_{j=1}^q \frac{1}{\Gamma\left(\alpha_{j-1}\right)}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}\left(t_j\right)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}\varkappa(u)du \\ + \frac{1}{\Gamma\left(\alpha_q\right)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\varkappa(u)du, \quad if \quad t \in \left(t_q,t_{q+1}\right],\ q = 1,2,\ldots,m, \end{cases} \tag{23}$$

where $Y_{\overline{\theta}}(t) = g\left(t,\overline{\theta}(t),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right)$ and $v_{\overline{\theta}}(t) = g\left(t,\overline{\theta}(t),\overline{\theta}(t-\lambda),{}^{C}D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right)$.

Proof. We exclude the proof as it is easy. □

Theorem 5. *Assume $g : V \times \mathbb{R} \times \mathbb{R}$ is continuous. If (H_1), (H_2), and (H_6) hold with*

$$Y := \max\left(\frac{k_h^* T^\delta}{\Gamma(\delta+1)} + \ell_1\frac{(y_0(t_1)-y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)}, \frac{k_h^* T^\delta}{\Gamma(\delta+1)} + mk_{\mathcal{W}} + \frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right) < 1. \tag{24}$$

then problem (1) is U-H stable.

Proof. The Banach fixed-point theorem applies the Ulam–Hyers stability property due to the retraction–displacement condition (see [36]).

Let $\overline{\theta}$ be any solution of the set of inequalities (21) and θ be the unique solution of problem (1). Then, from integral Equations (8) and (23), we have

$$|\overline{\theta}(t)-\theta(t)| \leq \begin{cases} \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\overline{\theta}(u))-h(\theta(u))|du+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}|\Upsilon_{\overline{\theta}}(u)-\Upsilon_\theta(u)|du \\ \quad +\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}|\vartheta(u)|du, \quad if \ t\in[0,t_1], \\ \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\overline{\theta}(u))-h(\theta(u))|du+\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du \\ \quad +\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du+\mathcal{W}_1|\overline{\theta}(t_1^-)-\theta(t_1^-)|+|\varkappa_1| \\ \quad +\frac{1}{\Gamma(\alpha_0)}\int_0^{t_1} y_0'(u)(y_0(t_1)-y_0(u))^{\alpha_0-1}|\varkappa(u)|du \\ \quad +\frac{1}{\Gamma(\alpha_1)}\int_{t_1}^t y_1'(u)(y_1(t)-y_1(u))^{\alpha_1-1}|\varkappa(u)|du \quad if \ t\in(t_1,t_2], \\ \vdots \\ \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\overline{\theta}(u))-h(\theta(u))|du+\sum_{j=1}^q \mathcal{W}_j|\overline{\theta}\left(t_j^-\right)-\theta\left(t_j^-\right)|+\sum_{j=1}^q|\varkappa_j| \\ \quad +\sum_{j=1}^q \frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du \\ \quad +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du \\ \quad +\sum_{j=1}^q \frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y_{j-1}'(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|\varkappa(u)|du \\ \quad +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^t y_q'(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|\varkappa(u)|du, \qquad if \quad t\in\left(t_q,t_{q+1}\right], \ q=1,2,\ldots,m, \end{cases} \tag{25}$$

Hence, for $t \in [0, t_1]$, we have

$$\begin{aligned} &|\overline{\theta}(t)-\theta(t)| \\ &\leq \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\overline{\theta}(u))-h(\theta(u))|du+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}|\Upsilon_{\overline{\theta}}(u)-\Upsilon_\theta(u)|du \\ &+\frac{1}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}|\vartheta(u)|du \\ &\leq k_h^*\int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}\left|\overline{\theta}(u)-\theta(u)\right|du+\frac{k_1}{(1-k_2)\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}\left|\overline{\theta}(u)-\theta(u)\right|du \\ &+\frac{\epsilon}{\Gamma(\alpha_0)}\int_0^t y_0'(u)(y_0(t)-y_0(u))^{\alpha_0-1}du. \end{aligned} \tag{26}$$

This implies

$$\begin{aligned} &\parallel \overline{\theta}-\theta \parallel_{\mathbb{E}} \\ &\leq \frac{k_h^* T^\delta}{\Gamma(\delta+1)}\parallel \overline{\theta}-\theta \parallel_{\mathbb{E}}+\frac{\ell_1(y_0(t_1)-y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)}\parallel \overline{\theta}-\theta \parallel_{\mathbb{E}}+\frac{\epsilon(y_0(t_1)-y_0(0))^{\alpha_0}}{\Gamma(\alpha_0+1)} \\ &= \left[\frac{\frac{\left(y_0\left(t_1\right)-y_0(0)\right)^{\alpha_0}}{\Gamma\left(\alpha_0+1\right)}}{1-\left(\frac{k_h^* T^\delta}{\Gamma(\delta+1)}+\frac{\ell_1\left(y_0\left(t_1\right)-y_0(0)\right)^{\alpha_0}}{(1-\ell_2)\Gamma\left(\alpha_0+1\right)}\right)}\right]\epsilon. \end{aligned} \tag{27}$$

For $t \in \left(t_q, t_{q+1}\right], q = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
|\overline{\theta}(t)- & \theta(t)| \\
& \leq \int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}|h(\overline{\theta}(u))-h(\theta(u))|du+\sum_{j=1}^{q}\mathcal{W}_j|\overline{\theta}\left(t_j^-\right)-\theta\left(t_j^-\right)|+\sum_{j=1}^{q}|\varkappa_j| \\
& +\sum_{j=1}^{q}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du \\
& +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|v_{\overline{\theta}}(u)-v_\theta(u)|du \\
& +\sum_{j=1}^{q}\frac{1}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}|\varkappa(u)|du \\
& +\frac{1}{\Gamma(\alpha_q)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}|\varkappa(u)|du \\
& \leq k_h^*\int_0^T \frac{(T-u)^{\delta-1}}{\Gamma(\delta)}\left|\overline{\theta}(u)-\theta^*(u)\right|du+\sum_{0<t_q<t}k_{\mathcal{W}}\left|\overline{\theta}\left(t_q^-\right)-\theta\left(t_q^-\right)\right| \\
& +\sum_{j=1}^{q}\frac{2k_1}{(1-k_2)\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}\left|\overline{\theta}(u)-\theta(u)\right|du \\
& +\frac{2k_1}{(1-k_2)\Gamma(\alpha_q)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}\left|\overline{\theta}(u)-\theta(u)\right|du \\
& +q\epsilon+\sum_{j=1}^{q}\frac{\epsilon}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}du \\
& +\frac{\epsilon}{\Gamma(\alpha_q)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}du \\
& \leq \left(\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+mk_{\mathcal{W}}+\frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\parallel\overline{\theta}-\theta\parallel \\
& +q\epsilon+\sum_{j=1}^{q}\frac{\epsilon}{\Gamma(\alpha_{j-1})}\int_{t_{j-1}}^{t_j} y'_{j-1}(u)\left(y_{j-1}(t_j)-y_{j-1}(u)\right)^{\alpha_{j-1}-1}du+\frac{\epsilon}{\Gamma(\alpha_q)}\int_{t_q}^{t} y'_q(u)\left(y_q(t)-y_q(u)\right)^{\alpha_q-1}du \\
& \leq \left(\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+mk_{\mathcal{W}}+\frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\parallel\overline{\theta}-\theta\parallel \\
& +\left(q+\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\epsilon \\
& =\frac{\left(q+\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)\epsilon}{1-\left(\frac{k_h^*T^\delta}{\Gamma(\delta+1)}+mk_{\mathcal{W}}+\frac{2k_1}{(1-k_2)}\sum_{j=0}^{q}\frac{\left(y_j(T)-y_j(t_j)\right)^{\alpha_j}}{\Gamma(\alpha_j+1)}\right)}:=N_g\epsilon.
\end{aligned}
\tag{28}
$$

This confirms that problem (1) is U-H stable. □

Corollary 2. *Problem (1) is G-U-H stable. In this case, we set* $\phi(\epsilon)=N_g(\epsilon);\ \phi(0)=0.$

6. Application

In this section, we apply the main results to the following general problem to illustrate their applicability.

Example

$$
\begin{cases}
{}^{C}D_{[t]}^{\alpha(t)}\theta(t)=\frac{e^{-\pi t}}{15}+\frac{\left(t-\frac{1}{3}\right)}{20}\left(\sin(|\theta(t)|)+\sin(|{}^{C}D_{[t]}^{\alpha(t)}\theta(t)\,|)\right), \quad t\in[0,t_1), \quad t_1\geq 0.30 \\
{}^{C}D_{[t]}^{\alpha(t)}\theta(t)=\frac{e^{-\pi t}}{15}+\frac{\left(t-\frac{1}{3}\right)}{20}\left(\sin(|\theta(t)|)+\frac{\theta(t-0.30)}{15+|\theta(t-0.30)|}+\sin(|{}^{C}D_{[t]}^{\alpha(t)}\theta(t)\,|)\right) \\
t\in[t_1,1]\backslash\{t_2\}, \\
\Delta\theta(t_j)=\frac{1}{50}\theta\left(t_j^-\right), \\
\theta(0)=\frac{1}{\Gamma(0.5)}\int_0^1(1-s)^{0.7}\frac{\theta(s)}{18+|\theta(s)|}ds+0.022,
\end{cases}
\tag{29}
$$

where

$$g_1(t,\theta(t),v_\theta(t)) = \frac{e^{-\pi t}}{15} + \frac{\left(t-\frac{1}{3}\right)}{20}\left(\sin(|\theta(t)|) + \sin\left({}^C D_{[t]}^{\alpha(t)}\theta(t)|\right)\right), \quad t \in [0,t_1], \quad t_1 \geq 0.30$$

$$g_2(t,\theta(t),\theta(t-\lambda),v_\theta(t)) = \frac{e^{-\pi t}}{15} + \frac{\left(t-\frac{1}{3}\right)}{20}\left(\sin(|\theta(t)|) + \frac{|\theta(t-0.30)|}{15+|\theta(t-0.30)|} + \sin\left({}^C D_{[t]}^{\alpha(t)}\theta(t)|\right)\right), \quad t \in [t_1,1],$$

$$\mathcal{W}_j(\theta) = \frac{1}{50}\theta; \theta \in \mathbb{R}^+, j = 1,2,$$

and

$$h(\theta) = \frac{\theta}{18+|\theta|}.$$

Assuming $m = 2$ ($q = 1,2$), we have

$${}^C D_{[t]}^{\alpha(t)}\theta(t) = \begin{cases} {}^C D_{[t]}^{\alpha_0,y_0}\theta(t), & 0 < t \leq t_1, \\ {}^C D_{[t]}^{\alpha_1,y_1}\theta(t), & t_1 < t \leq t_2 \\ {}^C D_{[t]}^{\alpha_2,y_2}\theta(t), & t_2 < t \leq 1 \end{cases}$$

$\alpha(t)$ is the variable piecewise order defined as

$$\alpha(t) = \begin{cases} \alpha_0 = \frac{1}{7}, & 0 < t \leq \frac{1}{3}, \\ \alpha_1 = \frac{1}{4}, & \frac{1}{3} < t \leq \frac{1}{2}, \\ \alpha_2 = \frac{1}{2}, & \frac{1}{2} < t \leq 1. \end{cases}$$

$$y(t) = \begin{cases} y_0(t) = \frac{\sin(5t)}{10}, & 0 < t \leq \frac{1}{3}, \\ y_1(t) = \left(\frac{1}{3}\right)^{-\exp(4t^2)}, & \frac{1}{3} < t \leq \frac{1}{2}, \\ y_2(t) = \frac{t^3}{2}, & \frac{1}{2} < t \leq 1. \end{cases}$$

Let $t \in [t_1,1]$. Then,

$$\begin{aligned} &\left|g_2(t,\theta(t),\theta(t-\lambda),v_\theta(t)) - g_2\left(t,\overline{\theta}(t),\overline{\theta}(t-\lambda),v_{\overline{\theta}}(t)\right)\right| \\ &\quad \leq \frac{\left(t-\frac{1}{3}\right)}{20}\left(\left|\sin(|\theta(t)|) - \sin\left(|\overline{\theta}(t)|\right)\right| + \left|\frac{|\theta(t-0.30)|}{15+|\theta(t-0.30)|} - \frac{|\overline{\theta}\,(t-0.30)|}{15+|\overline{\theta}\,(t-0.30)|}\right|\right. \\ &\quad \left. +\left|\sin\left(|{}^C D_{[t]}^{\alpha(t)}\theta(t)|\right) - \sin\left(|{}^C D_{[t]}^{\alpha(t)}\overline{\theta}(t)|\right)\right|\right) \\ &\quad \leq \frac{1}{15}\left(\left|\theta(t) - \overline{\theta}(t)\right| + \left|\theta(t-0.30) - \overline{\theta}(t-0.30)\right| + \left|{}^C D_{[t]}^{\alpha(t)}\theta(t) - {}^C D_{[t]}^{\alpha(t)}\overline{\theta}(t)\right|\right). \end{aligned} \tag{30}$$

By (H_1), we obtain $k_1 = k_2 = \frac{1}{30}$, as well as by (H_2). Similarly, for $t \in [0,t_1)$, we obtain $l_1 = l_2 = \frac{1}{30}$.

$$\left|\mathcal{W}_j(\theta) - \mathcal{W}_j(\overline{\theta})\right| \quad \leq \quad \frac{1}{50}\left|\theta - \overline{\theta}\right|.$$

implies that $k_{\mathcal{W}} = \frac{1}{50}$.

By (H_6), we have

$$\begin{aligned} \left|h(\theta) - h(\overline{\theta})\right| &= \left|\frac{\theta}{18+|\theta|} - \frac{\overline{\theta}}{18+|\overline{\theta}|}\right| \\ &\leq \frac{18\left|\theta-\overline{\theta}\right|}{(18+|\theta|)(18+|\,\overline{\theta}\,|)} \leq \frac{1}{18}\left|\theta - \overline{\theta}\right|. \end{aligned}$$

which implies $k_h^* = \frac{1}{18}$. For the above derived values, one may examine that

$$\Upsilon \;:=\; \max\left(\frac{k_h^* T^\delta}{\Gamma(\delta+1)} + \ell_1 \frac{(y_0(t_1)-y_0(0))^{\alpha_0}}{(1-\ell_2)\Gamma(\alpha_0+1)}, \frac{k_h^* T^\delta}{\Gamma(\delta+1)} + mk_{\mathcal{W}} + \frac{2k_1}{(1-k_2)}\sum_{j=0}^{2}\frac{\left(y_j(T)-y_j\left(t_j\right)\right)^{\alpha_j}}{\Gamma\left(\alpha_j+1\right)}\right) < 1.$$

Recall and apply Theorems 4 and 5, respectively. Then, problem (29) has a unique solution and is U-H stable.

7. Conclusions

In the present paper, we investigated a general problem of variable-kernel discrete delay differential equations (DDDEs) subjected to integral boundary conditions and impulsive effects. This topic is particularly intriguing due to the flexibility provided by the variable kernel.

In our main findings, we presented the solution representation of the proposed problem and derived fixed-point criteria for the existence and uniqueness of solutions using Schaefer's fixed-point theorem and the Banach contraction principle, respectively. Similarly, using U-H stability analysis, we established conditions ensuring the system's stability.

To validate and demonstrate the applicability of our results, we applied a numerical example. Our findings suggest that employing piecewise fractional derivatives enables the modeling of complex systems across multiple scales, making the results more general and widely applicable. Through the incorporation of impulsive effects, the model effectively captures sudden changes and memory effects, leading to more accurate predictions.

Thus, investigating the existence, uniqueness, and stability of the studied systems using the proposed theorems provides a more comprehensive, accurate, and general framework for analyzing complex systems. Furthermore, the proposed framework and results remain valid for both delay and non-delay systems. When the delay parameter is set to zero, our derived conditions reduce to the classical case of piecewise fractional differential systems without time delay.

In the future, our proposed system of differential equations can be extended to more complex fractional differential systems with multiple delays. Additionally, future research may explore numerical schemes for solving delay fractional systems in higher dimensions.

Author Contributions: Conceptualization, K.A.; formal analysis, A.A. (Amer Alsulami) and N.E.; funding acquisition, H.S., T.A., and A.M.; investigation, A.A. (Amer Alsulami) and N.E.; writing—original draft, A.A. (Arshad Ali); writing—review and editing, H.S., K.A., T.A., and A.M.; project administration, K.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research has been funded by Scientific Research Deanship at University of Ha'il, Saudi Arabia through project number RG-23-032.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon and Breach: New York, NY, USA, 1993.
2. Podlubny, L. Fractional Differential Equations. In *An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999; Volume 198.
3. Lazopoulos, K. Non-local continuum mechanics and fractional calculus. *Mech. Res. Commun.* **2006**, *33*, 753–757. [CrossRef]
4. Cottone, G.; Paola, M.D.; Zingales, M. Fractional mechanical model for the dynamics of non-local continuum. In *Advances in Numerical Methods*; Springer: Berlin/Heidelberg, Germany, 2009; pp. 389–423.
5. Carpinteri, A.; Cornetti, P.; Sapora, A. A fractional calculus approach to nonlocal elasticity. *Eur. Phys. J. Spec. Top.* **2011**, *193*, 193–204. [CrossRef]
6. Riewe, F. Mechanics with fractional derivatives. *Phys. Rev. E* **1997**, *55*, 3581. [CrossRef]
7. Rossikhin, Y.A.; Shitikova, M.V. Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. *Appl. Mech. Rev.* **2010**, *63*, 010801. [CrossRef]
8. Yang, X.J.; Machado, J.T. A new fractional operator of variable order: Application in the description of anomalous diffusion. *Phys. A Stat. Mech. Its Appl.* **2017**, *481*, 276–283. [CrossRef]

9. Shah, K.; Naz, H.; Sarwar, M.; Abdeljawad, T. On spectral numerical method for variable-order partial differential equations. *AIMS Math.* **2022**, *7*, 10422–10438. [CrossRef]
10. Hamza, A.; Osman, O.; Ali, A.; Alsulami, A.; Aldwoah, K.; Mustafa, A.; Saber, H. Fractal-Fractional-Order Modeling of Liver Fibrosis Disease and Its Mathematical Results with Subinterval Transitions. *Fractal Fract.* **2024**, *8*, 638. [CrossRef]
11. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
12. Khan, H.; Li, Y.; Khan, A.; Khan, A. Existence of solution for a fractional-order Lotka-Volterra reaction-diffusion model with Mittag-Leffler kernel. *Math. Methods Appl. Sci.* **2019**, *42*, 3377–3387. [CrossRef]
13. Alraqad, T.; Almalahi, M.A.; Mohammed, N.; Alahmade, A.; Aldwoah, K.A.; Saber, H. Modeling Ebola Dynamics with a Piecewise Hybrid Fractional Derivative Approach. *Fractal Fract.* **2024**, *8*, 596. [CrossRef]
14. Aldwoah, K.A.; Almalahi, M.A.; Shah, K. Theoretical and numerical simulations on the hepatitis B virus model through a piecewise fractional order. *Fractal Fract.* **2023**, *7*, 844. [CrossRef]
15. Aldwoah, K.A.; Almalahi, M.A.; Abdulwasaa, M.A.; Shah, K.; Kawale, S.V.; Awadalla, M.; Alahmadi, J. Mathematical analysis and numerical simulations of the piecewise dynamics model of Malaria transmission: A case study in Yemen. *AIMS Math.* **2024**, *9*, 4376–4408. [CrossRef]
16. Khan, H.; Ahmed, S.; Alzabut, J.; Azar, A.T. A generalized coupled system of fractional differential equations with application to finite time sliding mode control for Leukemia therapy. *Chaos Solitons Fractals* **2023**, *174*, 113901. [CrossRef]
17. Ansari, K.J.; Asma; Ilyas, F.; Shah, K.; Khan, A.; Abdeljawad, T. On new updated concept for delay differential equations with piecewise Caputo fractional-order derivative. In *Waves in Random and Complex Media*; Taylor & Francis: Abingdon, UK, 2023; pp. 1–20.
18. Feckan, M.; Wang, J.R.; Zhou, Y. On the new concept of solutions and existence results for impulsive fractional evolution equations. *Dynam. Part. Differ. Eq.* **2011**, *8*, 345–361. [CrossRef]
19. Tian, Y.; Bai, Z. Existence results for the three-point impulsive boundary value problem involving fractional differential equations. *Comput. Math. Appl.* **2010**, *59*, 2601–2609. [CrossRef]
20. Abdeljawad, T.; Mlaiki, N.; Abdo, M.S. Caputo-type fractional systems with variable order depending on the impulses and changing the kernel. *Fractals* **2022**, *30*, 2240219. [CrossRef]
21. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
22. Ahmed, S.; Azar, A.T.; Tounsi, M. Design of adaptive fractional-order fixed-time sliding mode control for robotic manipulators. *Entropy* **2022**, *24*, 1838. [CrossRef] [PubMed]
23. Ali, A.; Ansari, K.J.; Alrabaiah, H.; Aloqaily, A.; Mlaiki, N. Coupled system of fractional impulsive problem involving power-law kernel with piecewise order. *Fractal Fract.* **2023**, *7*, 436. [CrossRef]
24. MacDonald, N. *Time Lags in Biological Models*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2013; Volume 27.
25. Gopalsamy, K. *Stability and Oscillations in Delay Differential Equations of Population Dynamics*; Mathematics and its Applications; Kluwer Academic Pub.: Dordrecht, The Netherlands, 1992; Volume 74.
26. Almalahi, M.A.; Aldwoah, K.A.; Shah, K.; Abdeljawad, T. Stability and Numerical Analysis of a Coupled System of Piecewise Atangana-Baleanu Fractional Differential Equations with Delays. *Qual. Theory Dyn. Syst.* **2024**, *23*, 105. [CrossRef]
27. Kuang, Y. *Delay Differential Equations with Applications in Population Dynamics*; Academic Press: New York, NY, USA, 1993.
28. Struble, R.A. *Nonlinear Differential Equations*; McGraw-Hill: New York, NY, USA, 1962.
29. Wu, G.C.; Gu, C.Y.; Huang, L.L.; Baleanu, D. Fractional differential equations of variable order: Existence results, numerical method and asymptotic stability conditions. *Miskolc Math Notes* **2022**, *23*, 485–493. [CrossRef]
30. Wu, G.C.; Deng, Z.G.; Baleanu, D.; Zeng, D.Q. New variable-order fractional chaotic systems for fast image encryption. *Chaos Interdiscip. J. Nonlinear Sci.* **2019**, *29*. [CrossRef]
31. Wu, G.C.; Luo, M.; Huang, L.L.; Banerjee, S. Short memory fractional differential equations for new memristor and neural network design. *Nonlinear Dyn.* **2020**, *100*, 3611–3623. [CrossRef]
32. Samko, S.G.; Ross, B. Integration and differentiation to a variable fractional order. *Integral Transf. Spec. Funct.* **1993**, *1*, 277–300. [CrossRef]
33. Granas, A.; Dugundji, J. *Fixed Point Theory*; Springer: New York, NY, USA, 2003.
34. Schaefer, H. Über die methode der a priori-schranken. *Math. Ann.* **1955**, *129*, 415–416. [CrossRef]
35. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpath. J. Math.* **2010**, *26*, 103–107.
36. Rus, I.A. Some variants of contraction principle, generalizations, and applications. *Stud. Univ. Babes-Bolyai Math* **2016**, *61*, 343–358.