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Analytical Calculations of the Quantum Tsallis Thermodynamic Variables

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Abstract: In this paper, we provide an account of analytical results related to the Tsallis thermodynamics that have been the subject matter of a lot of studies in the field of high-energy collisions. After reviewing the results for the classical case in the massless limit and for arbitrarily massive classical particles, we compute the quantum thermodynamic variables. For the first time, the analytical formula for the pressure of a Tsallis-like gas of massive bosons has been obtained. The study serves both as a brief review of the knowledge gathered in this area, and as original research that forwards the existing scholarship. The results of the present paper will be important in a plethora of studies in the field of high-energy collisions including the propagation of non-linear waves generated by the traversal of high-energy particles inside the quark-gluon plasma medium showing the features of non-extensivity.

Keywords: Tsallis statistics; Tsallis thermodynamics; thermodynamic variables; integral representation

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1. Introduction

Power-law distributions have been routinely used to describe particle yields in high-energy collision physics. It has been observed that the pions, kaons, protons (and other hadrons) originated in these collision events follow a power-law distribution in the transverse momentum (p_T) space. The power-law formula, utilized by experiments such as STAR [1], PHENIX [2], ALICE [3] and CMS [4], use the following form of a power-law transverse momentum distribution,

$$\frac{d^2N}{dp_\mathrm{T}dy} = p_\mathrm{T}\frac{dN}{dy}\frac{(n-1)(n-2)}{nC(nC+m_0(n-2))}\left(1+\frac{m_\mathrm{T}-m_0}{nC}\right)^{-n}, \tag{1}$$

which has some correspondence with the form of the Tsallis transverse momentum distribution, proposed by Cleymans and Worku in 2012 [5,6],

$$\frac{d^2N}{dp_\mathrm{T}dy} = \frac{gV}{(2\pi)^2}p_\mathrm{T}m_\mathrm{T}\cosh y\left(1+(q-1)\frac{m_\mathrm{T}\cosh y-\mu}{T}\right)^{-\frac{q}{q-1}}. \tag{2}$$

In Equations (1) and (2), n, C, m_0, V (volume), q (Tsallis parameter), T (Tsallis temperature) and μ (chemical potential) are fit parameters, g is degeneracy, $m_\mathrm{T} = \sqrt{p_\mathrm{T}^2+m^2}$ is the transverse mass of a particle with the mass m and y is rapidity. These distributions can be associated with the Tsallis statistical mechanics, developed by C. Tsallis in 1988 [7], a statistics that has long been used to tackle a medium with fluctuation, long-range correlation [8–11], small system size [12] and fractal structure [13]. It has been shown that the Tsallis-like distribution, proposed in Refs. [5,6], obeys thermodynamic relations. Later on, from the definition of the Tsallis entropy, the distribution (2) has been shown to be the zeroth-order approximation of the exact Tsallis-like transverse momentum distribution in the Tsallis-2 prescription [14] that was shown to be rather useful for the Large Hadron

Collider (LHC) phenomenology [15–19]. The same distribution can also be obtained from the q-dual statistics, proposed in Ref. [20]. In the papers that spearheaded the study of Tsallis thermodynamics, there were comparisons between the Tsallis-like classical and quantum distributions and their Boltzmann–Gibbs (BG) counterparts (see, e.g., [5]) which is achieved once the q-parameter approaches unity.

A reader may also have felt an implicit necessity to be able to compare the thermodynamic variables in the two formulations—Tsallis and BG. One might always take a numerical approach to provide an answer, as the analytical formulae of the Tsallis thermodynamic variables were not widely available, as opposed to their Boltzmann–Gibbs counterparts, which are expressible in terms of the modified Bessel functions. However, this does not mean that there were no attempts to find analytical results. Lavagno already in 2002 provided these expressions in terms of the q-modified Bessel functions of the second kind [21]. However, the properties of this group of q-modified special functions is not widely available and known to physicists. Hence, it was necessary to explore this question further. Thermodynamic variables are important as their relationships form the equation of state that is an important input to study, for example, the evolution of the quark-gluon plasma (QGP) medium [22–28], propagation of non-linear waves in the QGP using the hydrodynamic equation [29–35].

There was a renewed interest in this attempt during Professor Jean Cleymans' visit to India in 2015. This attempt was based on the observation that the Tsallis-like distributions can bewritten using the Taylor's series expansion in the increasing order of $(q-1)^n$, $n \in \mathbb{Z}^{\geq}$ ($\mathbb{Z}^{\geq}$ represents the set of non-negative integers). In a joint paper [36], the Tsallis thermodynamic variables for an ideal gas of massive particles were explored. However, eventually it was realized that the results were restricted by the fact that the Tsallis distribution was truncated at $\mathcal{O}(q-1)^2$. Such an early truncation led to restrictions in the phase space dictated by the ratios involving q, T and the single-particle energy, E_p; see also [10]. Therefore, the question of finding an unapproximated analytical expression of the Tsallis thermodynamic variables was still open. In the meantime, one of the authors (T.B.) joined the group of Jean Cleymans in Cape Town, and some progress ensued. Cleymans proved that the calculations for the massless case can be performed analytically, and that was a breakthrough. This was the inspiration behind another paper in collaboration with him [37] that elaborated a method to analytically calculate the Tsallis thermodynamic variables for the massive particles without an approximation (like the Taylor's series expansion, or considering the massless case) using the Mellin–Barnes contour integral representation of the Tsallis distribution. It was found that the interesting features (like poles) that were missed in the Taylor approximated calculations are intact in the massless as well as the massive case. In this paper, we extend the existing knowledge to the quantum domain, and propose a method to calculate analytical formulae of the quantum Tsallis thermodynamic variables. In the present study, isotropic momentum distributions are considered. So, the results can also be useful for the quark-gluon plasma medium formed in the early universe [38] or other branches of physics that use isotropic distributions [39–43]. For a more realistic scenario to treat high-energy collision physics, anisotropy may be considered but we reserve that for future studies.

Being involved in such a journey with Jean Cleymans as a friend, philosopher and guide is truly rewarding. The present paper serves as our tribute to the memory and inspiring scientific curiosity of Jean Cleymans.

2. Review: Tsallis Thermodynamics: $m = 0$, $\mu = 0$

The Tsallis thermodynamic variables can be written in terms of the Tsallis single-particle distribution. The single-particle distributions can be obtained following three different averaging schemes [44], which we name Tsallis-1, 2 and 3. These schemes differ in the definition of the mean values (e.g., the mean energy), utilized for the constrained maximization of the Tsallis entropy. In the first scheme, the mean is defined as $\langle O \rangle = \sum_i p_i O_i$, in the second scheme $\langle O \rangle = \sum_i p_i^q O_i$ and in the third scheme

$\langle O \rangle = \sum_i p_i^q O_i / \sum p_i^q$, where $\{p_i\}$ are the probabilities of micro-states. It is worthwhile to mention that following the previous studies [5,6], we follow the second averaging scheme. The present paper focuses entirely on the analytical method to calculate the Tsallis thermodynamic variables, and based on this prescription, it is relatively straightforward to extend the calculations for other forms of the single-particle distributions. With this understanding, we consider the following isotropic quantum single-particle distributions (positive sign for fermions (f), negative sign for bosons (b)):

$$n_{\mathrm{b/f}} = \frac{1}{\left(1 + (q-1)\frac{E_p - \mu}{T}\right)^{\frac{q}{q-1}} \pm 1}, \tag{3}$$

where $E_p = \sqrt{p^2 + m^2}$ is the single-particle energy of a particle with the mass, m, and the three-momentum, $\boldsymbol{p}$ (with magnitude p). This distribution is not entirely phenomenological because it can be obtained (after certain approximations) from the constrained maximization of the Tsallis entropy, as shown in [45]. The distribution (3) is similar to (but not exactly the same as) the one, proposed in [46,47]. For the sake of completeness, we also quote the popular classical (Maxwell–Boltzmann, MB) Tsallis-like single-particle distribution that gives rise to Equation (2):

$$n_{\mathrm{MB}} = \left(1 + (q-1)\frac{E_p - \mu}{T}\right)^{-\frac{q}{q-1}}. \tag{4}$$

With the help of the single-particle distributions (n_{s}; s = b, f, MB), the thermodynamic variables such as the pressure, P, the mean energy, E, and the mean number of particles, N, can be expressed as follows:

$$P = g\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{p^2}{3\,E_p}\,n_{\mathrm{s}};\quad E = gV\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,E_p\,n_{\mathrm{s}};\quad N = gV\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,n_{\mathrm{s}}. \tag{5}$$

2.1. Classical Case

In this Section we tabulate the results for the classical (Tsallis Maxwell–Boltzmann) case in the massless approximation. The pressure, P, the energy density, $\epsilon = E/V$, and the number density, $\rho = N/V$, with $\mu = 0$ are given by [37]

$$P = \frac{gT^4}{6\pi^2}\frac{1}{(2-q)(3/2-q)(4/3-q)} \tag{6}$$

$$\epsilon = \frac{gT^4}{2\pi^2}\frac{1}{(2-q)(3/2-q)(4/3-q)} = 3P \tag{7}$$

$$\rho = \frac{gT^3}{2\pi^2}\frac{1}{(2-q)(3/2-q)}. \tag{8}$$

Interestingly, from the above expressions (e.g., for P), the first pole of q appears at $q = 4/3$, which is close to $1.\bar{3}$ [48]. This puts an upper-bound on the q value that is a parameter to be determined from the experimental data. Experimental observations indeed show that q values do obey this upper-bound which is imposed because of the finite values of the thermodynamic variables. However, other considerations may further shrink the range [49]. It is also noteworthy that the upper-bound of q (denoted as $q_{\max}^{(D)}$), obtained from the thermodynamic considerations, changes with the dimension of the system as $q_{\max}^{(D)} < 1 + 1/(D-1)$. Hence, for $D = 4$, the value of $q_{\max}^{(D)}$ is $4/3$.

2.2. Quantum Case

Tsallis quantum thermodynamic variables in the massless limit are given by the following closed analytic formulae; see Ref. [34] for deatils.

2.2.1. Bosons

$$P_{\rm b} = \frac{gT^4}{6\pi^2(q-1)^3 q}\left[3\psi^{(0)}\left(\frac{3}{q}-2\right)+\psi^{(0)}\left(\frac{1}{q}\right)-3\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{4}{q}-3\right)\right], \tag{9}$$

$$\epsilon_{\rm b} = \frac{gT^4}{2\pi^2(q-1)^3 q}\left[3\psi^{(0)}\left(\frac{3}{q}-2\right)+\psi^{(0)}\left(\frac{1}{q}\right)-3\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{4}{q}-3\right)\right], \tag{10}$$

$$\rho_{\rm b} = \frac{gT^3}{2\pi^2(q-1)^2 q}\left[2\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{3}{q}-2\right)-\psi^{(0)}\left(\frac{1}{q}\right)\right]. \tag{11}$$

2.2.2. Fermions

$$P_{\rm f} = \frac{gT^4}{6\pi^2(q-1)^3 q}\Bigg[3\Phi\left(-1,1,\frac{2}{q}-1\right)-3\Phi\left(-1,1,\frac{3}{q}-2\right)+\Phi\left(-1,1,\frac{4}{q}-3\right) - \Phi\left(-1,1,\frac{1}{q}\right)\Bigg], \tag{12}$$

$$\epsilon_{\rm f} = \frac{gT^4}{2\pi^2(q-1)^3 q}\Bigg[3\Phi\left(-1,1,\frac{2}{q}-1\right)-3\Phi\left(-1,1,\frac{3}{q}-2\right)+\Phi\left(-1,1,\frac{4}{q}-3\right) - \Phi\left(-1,1,\frac{1}{q}\right)\Bigg], \tag{13}$$

$$\rho_{\rm f} = \frac{gT^3}{2\pi^2(q-1)^2 q}\left[-2\Phi\left(-1,1,\frac{2}{q}-1\right)+\Phi\left(-1,1,\frac{3}{q}-2\right)+\Phi\left(-1,1,\frac{1}{q}\right)\right]. \tag{14}$$

Here $\psi^{(0)}(z)$ is the digamma function, and $\Phi(a,b,z)$ is Lerch's transcendent [50], both having the poles at $z = 0$. One observes that, similar to the classical case, the first pole in the thermodynamic variables (for example, pressure) appears at $q = 4/3$. Hence, the upper-bound $q < 4/3$ is still relevant. Before moving to the next Section, let us comment that the above results can be extended to treat very light particles, as discussed in Ref. [34]. It is possible to obtain the $\mathcal{O}(m^2T^2)$ correction to the above approximated results, which may also work for the light quarks such as up and down.

3. Review and New Results: Tsallis Thermodynamics: $m \neq 0, \mu = 0$

In this Secton, we quote the closed analytical formula of the pressure in a gas of massive classical and quantum particles without utilizing any approximation. The classical case has already been considered earlier [37]. However, we are not aware of any other results for the quantum case with arbitrarily massive particles (albeit results are available for slightly massive particles). In this Secton, we only quote the obtained results for classical and quantum (boson) particles. Detailed mathematical steps for obtaining quantum results are described in Section 4.

3.1. Classical Case (Review)

In this Section, we tabulate analytical results of classical Tsallis thermodynamics for arbitrarily massive particles. The calculation involves an integral representation of a power-law function that appears in the Tsallis statistics. Due to the nature of the integrals involved, the convergence conditions lead to two separate formulae for the thermodynamic variables for two regions $q > 1 + T/m$, which we call the "upper region" and $q \leq 1 + T/m$, which we call the "lower region". The origin of these regions are explained in Section 4, where the quantum case is considered.

3.1.1. Upper Region: $q > 1 + T/m$

The analytical expression valid for the upper region is:

$$P_{\rm U} = \frac{g\,m^4}{16\pi^{\frac{3}{2}}}\left(\frac{T}{(q-1)\,m}\right)^{\frac{q}{q-1}}\left[\frac{\Gamma\left(\frac{4-3q}{2(q-1)}\right)}{\Gamma\left(\frac{2q-1}{2(q-1)}\right)}\,{}_2F_1\left(\frac{q}{2(q-1)},\frac{4-3q}{2(q-1)},\frac{1}{2};\frac{T^2}{(q-1)^2m^2}\right)\right.$$
$$\left.-\frac{2T}{(q-1)m}\times\frac{\Gamma\left(\frac{3-2q}{2(q-1)}\right)}{\Gamma\left(\frac{q}{2(q-1)}\right)}\,{}_2F_1\left(\frac{2q-1}{2(q-1)},\frac{3-2q}{2(q-1)},\frac{3}{2};\frac{T^2}{(q-1)^2m^2}\right)\right], \tag{15}$$

where $\Gamma(z)$ is the Gamma function, and ${}_2F_1(a,b,c;z)$ is the hypergeometric function [50].

3.1.2. Lower Region: $q \leq 1 + T/m$

The analytical expression valid for the lower region is:

$$P_{\rm L} = \frac{g\,m^4}{2^{\frac{q}{q-1}}\pi^{\frac{3}{2}}}\left[\frac{(q-1)^2\,(3-q)\,\Gamma\left(\frac{1}{q-1}\right)}{(4-3q)\,(3-2q)\,(2-q)\Gamma\left(\frac{1+q}{2(q-1)}\right)}\right]$$
$$\times\,{}_2F_1\left(\frac{2q-1}{2(q-1)},\frac{q}{2(q-1)},\frac{3-q}{2(q-1)},1-\frac{(q-1)^2\,m^2}{T^2}\right). \tag{16}$$

It is worth noticing that both the expressions in general require $q < 4/3$ for the consistency of the framework apart from the (upper or lower) limits, based on the convergence criterion.

3.2. Quantum Case: Bosons (New Results)

In this Section, we tabulate the newly found analytical results of the Tsallis thermodynamics for arbitrarily massive bosons. In comparison with the classical case, there is an extra step in the quantum calculations that entails expressing the quantum single-particle distributions as a superposition of an infinite number of classical distributions. However, all the other procedures are the same as those in the classical case. In the quantum case also, two analytical formulae for the upper and the lower regions are obtained.

3.2.1. Upper Region: $q > 1 + T/m$

$$P_{\rm U,b} = \sum_{s=1}^{s_0}\frac{gm^4}{32\pi^2\Gamma\left(\frac{qs}{q-1}\right)}\left(\frac{2T}{m(q-1)}\right)^{\frac{qs}{q-1}}\left[\Gamma\left(\frac{qs}{2(q-1)}\right)\Gamma\left(\frac{qs}{2(q-1)}-2\right)\right.$$
$$\times\,{}_2F_1\left(\frac{qs}{2(q-1)},\frac{qs}{2(q-1)}-2;\frac{1}{2};\frac{T^2}{m^2(q-1)^2}\right)-\frac{2T}{m(q-1)}\Gamma\left(\frac{qs-3q+3}{2(q-1)}\right)$$
$$\left.\times\,\Gamma\left(\frac{qs+q-1}{2(q-1)}\right){}_2F_1\left(\frac{qs-3q+3}{2(q-1)},\frac{qs+q-1}{2(q-1)};\frac{3}{2};\frac{T^2}{m^2(q-1)^2}\right)\right]. \tag{17}$$

3.2.2. Lower Region: $q \leq 1 + T/m$

$$P_{\rm L,b} = \frac{gT^4}{16(q-1)^4}\sum_{s=1}^{s_0}{}_2\tilde{F}_1\left(\frac{qs}{2(q-1)}-2,\frac{qs}{2(q-1)}-\frac{3}{2};\frac{qs}{q-1}-\frac{3}{2};1-\frac{m^2(q-1)^2}{T^2}\right)$$
$$\times\sec\left(\frac{\pi qs}{q-1}\right)\left[\frac{\Gamma\left(\frac{qs-4q+4}{2(q-1)}\right)}{\Gamma\left(\frac{q(s-5)+5}{2(1-q)}\right)\Gamma\left(\frac{qs}{2(1-q)}+\frac{1}{2}\right)\Gamma\left(\frac{qs}{2(q-1)}+\frac{1}{2}\right)}\right.$$
$$\left.-\frac{\Gamma\left(\frac{qs}{2(q-1)}-\frac{3}{2}\right)}{\Gamma\left(\frac{qs}{2(q-1)}\right)\Gamma\left(\frac{qs}{2(1-q)}+1\right)\Gamma\left(\frac{qs}{2(1-q)}+3\right)}\right]. \tag{18}$$

Here, the regularized hypergeometric function ${}_2\tilde{F}_1(a,b;c;z) = {}_2F_1(a,b;c;z)/\Gamma(c)$. Ideally, s_0 is a very large number, depending on the desired agreement between the numerical and analytical results. Equations (17) and (18) (also repeated in Equations (30) and (31) below) are the main results of the paper.

4. Methodology: The Pressure of a Gas of Bosons Following the Tsallis Distribution

From Equation (5), the pressure for the bosons is:

$$P_{\mathrm{b}} = g\int\frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\,\frac{p^2}{3\,E_p}\,n_{\mathrm{b}}, \tag{19}$$

where n_{b} is the Tsallis Bose–Einstein single-particle distribution given by Equation (3). The spherical symmetry of the integrand implies that:

$$P_{\mathrm{b}} = \frac{g}{6\pi^2}\int_0^\infty \frac{p^4}{\sqrt{m^2+p^2}}\,\frac{1}{\left[1+(q-1)\frac{\sqrt{m^2+p^2}}{T}\right]^{\frac{q}{q-1}}-1}\,dp, \tag{20}$$

where $\mu = 0$ is set. Now, we describe the steps to obtain Equations (17) and (18).

4.1. Rescaling the Integration Variable

To simplify the calculations, the $k = p/m$ is defined, so that the pressure reads:

$$P_{\mathrm{b}} = \frac{gm^4}{6\pi^2}\int_0^\infty \frac{k^4}{\sqrt{1+k^2}}\,\frac{1}{\left[1+\frac{m(q-1)}{T}\sqrt{1+k^2}\right]^{\frac{q}{q-1}}-1}\,dk. \tag{21}$$

4.2. Infinite Summation

Now, one observes that, similar to the Boltzmann–Gibbs case, the Tsallis quantum distributions can be expressed in terms of an infinite summation of the Tsallis MB distributions:

$$\frac{1}{\left[1+\frac{m(q-1)}{T}\sqrt{1+k^2}\right]^{\frac{q}{q-1}}\pm 1} = \sum_{s=1}^{\infty}(-1)^{a(s+1)}\left(1+\frac{m(q-1)}{T}\sqrt{1+k^2}\right)^{-\frac{qs}{q-1}}, \tag{22}$$

where $a = 0$ ($a = 1$) yields the bosonic (fermionic) distribution. This step allows us to write down the Tsallis pressure in a bosonic gas in a form similar to its classical counterpart, except for a summation sign in front and a power index s in the denominator. Hence, one obtains:

$$P_{\mathrm{b}} = \frac{gm^4}{6\pi^2}\sum_{s=1}^{\infty}\int_0^\infty \frac{k^4}{\sqrt{1+k^2}}\,\frac{1}{\left(1+\frac{m(q-1)}{T}\sqrt{1+k^2}\right)^{\frac{qs}{q-1}}}\,dk. \tag{23}$$

4.3. Contour Integral Representation

Next, we use the Mellin–Barnes contour representation [51–53] of the power-law function appearing in the integrand in Equation (23), and follow the procedure, described in Ref. [37]. A power-law function can be written as a Mellin–Barnes contour integration:

$$\frac{1}{(X+Y)^\lambda} = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\Gamma(-z)\Gamma(z+\lambda)}{\Gamma(\lambda)}\,\frac{Y^z}{X^{\lambda+z}}\,dz, \tag{24}$$

where $\mathrm{Re}(\lambda) > 0$ and $\mathrm{Re}(\epsilon) \in (-\mathrm{Re}(\lambda), 0)$, which is the case here since $\lambda = qs/(q-1) > 0 \Leftrightarrow q, s \geq 1$. Moreover, one can take $X = m(q-1)\sqrt{1+k^2}/T$ and $Y = 1$, or other combinations, all of which yield the following expression:

$$P_{\mathrm{b}} = \frac{gm^4}{12\pi^3 i}\sum_{s=1}^{\infty}\left(\frac{T}{m(q-1)}\right)^{\frac{qs}{q-1}}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\Gamma(-z)\,\Gamma\left(z+\frac{qs}{q-1}\right)}{\Gamma\left(\frac{qs}{q-1}\right)}\left(\frac{T}{m(q-1)}\right)^{z} dz \\ \times\int_0^{\infty} k^4(1+k^2)^{-\frac{z}{2}-\frac{1}{2}-\frac{qs}{2(q-1)}}\,dk. \tag{25}$$

After performing the k-integration, the pressure reads:

$$P_{\mathrm{b}} = \frac{gm^4}{32\pi^{\frac{5}{2}} i}\sum_{s=1}^{\infty}\left(\frac{T}{m(q-1)}\right)^{\frac{qs}{q-1}}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{\Gamma(-z)\Gamma\left(z+\frac{qs}{q-1}\right)\Gamma\left(\frac{qs}{2(q-1)}+\frac{z}{2}-2\right)}{\Gamma\left(\frac{qs}{q-1}\right)\Gamma\left(\frac{qs}{2(q-1)}+\frac{z}{2}+\frac{1}{2}\right)} \\ \times\left(\frac{T}{m(q-1)}\right)^{z} dz. \tag{26}$$

The convergence of the scaled momentum integration requires $\mathrm{Re}(z) \geq 0$.

4.4. Wrapping Contour Clockwise: $q > 1 + T/m$

To identify the poles, in order to obtain the residues of the integrand, the transformation $z \to 2z$ is made along with the use of the Legendre's duplication formula [54] and Cauchy's residue formula [55], so that the pressure now reads:

$$\begin{aligned} P_{\mathrm{b}} =& \frac{gm^4}{64\pi^{\frac{7}{2}} i}\sum_{s=1}^{\infty}\frac{2^{\frac{qs}{q-1}}}{\Gamma\left(\frac{qs}{q-1}\right)}\left(\frac{T}{m(q-1)}\right)^{\frac{qs}{q-1}}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\Gamma(-z)\Gamma\left(-z+\frac{1}{2}\right) \\ &\times\Gamma\left(z+\frac{qs}{2(q-1)}\right)\Gamma\left(\frac{qs}{2(q-1)}+z-2\right)\left(\frac{T}{m(q-1)}\right)^{2z} dz \\ =&(-2\pi i)\times\frac{gm^4}{64\pi^{\frac{7}{2}} i}\sum_{s=1}^{\infty}\frac{2^{\frac{qs}{q-1}}}{\Gamma\left(\frac{qs}{q-1}\right)}\left(\frac{T}{m(q-1)}\right)^{\frac{qs}{q-1}} \\ &\times\sum_{\ell=0}^{\infty}\left\{\mathrm{Res}^{(1)}[f(z), z=\ell] + \mathrm{Res}^{(2)}\left[f(z), z=\ell+\frac{1}{2}\right]\right\}, \end{aligned} \tag{27}$$

where $f(z)$ is defined as:

$$f(z) \equiv \Gamma(-z)\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{qs}{2(q-1)}+z\right)\Gamma\left(\frac{qs}{2(q-1)}+z-2\right)\left(\frac{T}{m(q-1)}\right)^{2z},$$

and the contour is wrapped clockwise so that residues receive a contribution only from the poles of $\Gamma(-z)$ at the positive integers ($\mathrm{Res}^{(1)}$) including zero, and the poles of $\Gamma(-z+1/2)$

at the positive half-integers ($\mathrm{Res}^{(2)}$). This clockwise wrapping of contour imposes the convergence condition $q > 1 + T/m$ when $z \to \infty$. $\mathrm{Res}^{(1)}$ and $\mathrm{Res}^{(2)}$ are defined as follows:

$$\begin{aligned}
\mathrm{Res}^{(1)} =& \mathrm{Res}\left[f(z), \{z = \ell \ni \ell \in \mathbb{Z}^{\geq}\}\right] \\
=& \frac{(-1)^{\ell+1}\left(\frac{T}{m(q-1)}\right)^{2\ell}}{\ell!}\Gamma\left(\frac{1}{2} - \ell\right)\Gamma\left(\ell + \frac{qs}{2(q-1)} - 2\right) \\
& \times \Gamma\left(\ell + \frac{qs}{2(q-1)}\right) \qquad (28) \\
\mathrm{Res}^{(2)} =& \mathrm{Res}\left[f(z), \{z = \ell + \frac{1}{2} \ni \ell \in \mathbb{Z}^{\geq}\}\right] \\
=& \frac{(-1)^{\ell+1}\left(\frac{T}{m(q-1)}\right)^{2\ell+1}}{\ell!}\Gamma\left(-\ell - \frac{1}{2}\right)\Gamma\left(\ell + \frac{qs}{2(q-1)} - \frac{3}{2}\right) \\
& \times \Gamma\left(\ell + \frac{qs}{2(q-1)} + \frac{1}{2}\right). \qquad (29)
\end{aligned}$$

In substituting Equations (28) and (29) into Equation (27), the infinite summation over ℓ can be expressed in terms of the hypergeometric function ${}_2F_1$ [50], and the pressure in the region $q > 1 + T/m$, given by Equation (17), reads:

$$\begin{aligned}
P_{\mathrm{U,b}} =& \sum_{s=1}^{s_0} \frac{g m^4}{32\pi^2 \Gamma\left(\frac{qs}{q-1}\right)}\left(\frac{2T}{m(q-1)}\right)^{\frac{qs}{q-1}}\left[\Gamma\left(\frac{qs}{2(q-1)}\right)\Gamma\left(\frac{qs}{2(q-1)} - 2\right)\right. \\
& \times {}_2F_1\left(\frac{qs}{2(q-1)}, \frac{qs}{2(q-1)} - 2; \frac{1}{2}; \frac{T^2}{m^2(q-1)^2}\right) - \frac{2T}{m(q-1)}\Gamma\left(\frac{qs-3q+3}{2(q-1)}\right) \\
& \left. \times \Gamma\left(\frac{qs+q-1}{2(q-1)}\right) {}_2F_1\left(\frac{qs-3q+3}{2(q-1)}, \frac{qs+q-1}{2(q-1)}; \frac{3}{2}; \frac{T^2}{m^2(q-1)^2}\right)\right], \qquad (30)
\end{aligned}$$

when the infinite summation is truncated at $s = s_0$.

4.5. Analytic Continuation: $q \leq 1 + T/m$

Instead of keeping the dimension of the momentum space arbitrary and analytically continuing the integrand prior to wrapping (since it does not lead to a closed form), the result, obtained in Equation (30) using Ref. [50], is analytically continued, and one obtains the following result in the complementary (lower) region:

$$\begin{aligned}
P_{\mathrm{L,b}} =& \frac{g T^4}{16(q-1)^4}\sum_{s=1}^{s_0} {}_2\tilde{F}_1\left(\frac{qs}{2(q-1)} - 2, \frac{qs}{2(q-1)} - \frac{3}{2}; \frac{qs}{q-1} - \frac{3}{2}; 1 - \frac{m^2(q-1)^2}{T^2}\right) \\
& \times \sec\left(\frac{\pi qs}{q-1}\right)\left[\frac{\Gamma\left(\frac{qs-4q+4}{2(q-1)}\right)}{\Gamma\left(\frac{q(s-5)+5}{2(1-q)}\right)\Gamma\left(\frac{qs}{2(1-q)} + \frac{1}{2}\right)\Gamma\left(\frac{qs}{2(q-1)} + \frac{1}{2}\right)}\right. \\
& \left. - \frac{\Gamma\left(\frac{qs}{2(q-1)} - \frac{3}{2}\right)}{\Gamma\left(\frac{qs}{2(q-1)}\right)\Gamma\left(\frac{qs}{2(1-q)} + 1\right)\Gamma\left(\frac{qs}{2(1-q)} + 3\right)}\right], \qquad (31)
\end{aligned}$$

where the regularized hypergeometric function, ${}_2\tilde{F}_1(a, b; c; z)$, is defined as for Equation (18).

5. Numerical Results and Discussion

Here, some comments about the comparison of numerical results with the results, obtained from the analytical formulae, are in order. Let us check the massless limit first. We notice that the final result works rather well for the case of massless particles ($m = 0$), as it should. To check this we substitute numerical values in Equation (18) and compare the result with the numerical value of the integral in Equation (20). We take, for example, $q = 1.2$, $g = 1$, $T = 0.08$ GeV, $m = 10^{-7}$ GeV, such that $q = 1.2 \ll 1 + T/m$. This condition implies that Equation (31) was used. For $s_0 = 20$, both the numerical and the analytical results (obtained also from Equation (9)) agree up to eleven significant digits and the value of pressure is 2.20098×10^{-5} GeV^4.

Next, let us consider light particles like the positively-charged pions (of the mass of 0.140 GeV), produced in proton-proton (p-p) collisions at the LHC. We observe that for $q = 1.154$, $g = 1$ and $T = 0.0682$ GeV (values taken from [5]), the value of s_0 significantly differs from the massless case when we consider the pions. For the pions ($q = 1.154 < 1 + T/m = 1.487$), a similar agreement between the analytical and numerical results can be reached for $s_0 = 5$ and the pressure turns out to be 5.4318×10^{-6} GeV^4.

We also consider more massive particles like the protons (of the mass of 0.938 GeV), produced in p-p collisions at the LHC. For $q = 1.107$, $g = 2$ and $T = 0.073$ GeV (the values are taken from [5]), a similar agreement between the numerical and analytical results, both of which are 3.5597×10^{-7} GeV^4 , can be obtained including just two terms, i.e., $s_0 = 2$. It is noteworthy that the protons are fermions, and the corresponding formula for pressure can be obtained by multiplying $(-1)^{s+1}$ with each term of, for example, Equation (30), as $q = 1.107 > 1 + T/m \approx 1.078$. In these examples, the heavier the particle, the faster the infinite summation convergence. This trend is repeated when we change only mass, while keeping q and T values unaltered.

For a gas of positively charged pions produced at the RHIC (Relativistic Heavy Ion Collider) (center-of-mass energy, $\sqrt{s_{\mathrm{NN}}} = 200$ GeV, Au-Au collisions, $q = 1.090$ and $T = 0.117$ GeV [56]), the pressure is found to be 3.0089×10^{-5} GeV^4. Considering all the charged particles produced at the LHC ($\sqrt{s_{\mathrm{NN}}} = 2760$ GeV Pb-Pb collisions, $q = 1.135$ and $T = 0.096$ GeV [19]), the pressure is found to be 6.5733×10^{-5} GeV^4

We conclude from the comparison of numerical and analytical results that the latter works considerably well. We hope that the main results, reported in this paper, will sufficiently reduce the overall computation time. We have checked that for some of the above examples, computation time is almost ten times reduced when the analytical formulae are used.

6. Summary, Conclusions and Outlook

In summary, we presented a brief review of the studies, related to the Tsallis thermodynamics, that may be important in the further studies of the quark-gluon plasma and many other systems that display fluctuation and long-range correlation. We also presented a detailed description of how to extend those existing findings to the quantum domain (Equations (30) and (31)). We used the contour integral representation of the power-law function and followed the ritual proposed in [37], after expressing the quantum distributions in terms of an infinite summation of classical MB distributions. We elaborated the analytical computation of the pressure of a bosonic gas following the Tsallis statistics, and the final result can be expressed as a summation that appears from the superposition of classical distributions. However, we noticed that in the examples discussed, only a finite number of terms are needed, and the number of required terms for convergence decreases with mass (when q, and T, are kept unaltered). The integral representation, also known as the Mellin–Barnes representation, has extensively been used in the studies involving loop calculations in quantum field theory [53]. Hence, in a way, this is one of the examples where techniques established in one field of research benefit another. Although not mentioned in the paper, extension to the fermionic case is straightforward. The only difference in summation comes owing to a factor $(-1)^{s+1}$ appearing with each term. In this paper, we provided the results only of the pressure of a Tsallis-like bosonic gas. Other thermodynamic variables of such a system can be calculated by appropriately differentiating pressure. Extension to the $\mu \neq 0$ case can be performed with a proper identification of the variables

X and Y in Equation (24). Moreover, in this case, the convergence condition for clockwise wrapping is modified [37].

There may be many different applications of the present study but we would like to mention a particular field that has caught some recent interest. Of late, there have been studies [34,35] reporting the propagation of non-linear waves in the quark-gluon plasma fluid (both ideal and viscous) in which constituents follow the Tsallis-like distributions. In studies [34,35], a Tsallis-like MIT bag equation of state, considering massless (or very light) particles, was used. It will be interesting to modify the equation of state incorporating the present findings. It will also be interesting to extend the study for hadronic gases. It has been shown [14] that the exact Tsallis single-particle distribution is expressed in terms of a series summation, and the distributions, used in Equation (3) are only the approximations. For low-energy collisions (e.g., in the future experiments at the NICA (Nuclotron-based Ion Collider fAcility) and FAIR (Facility for Antiproton and Ion Research) facilities, terms beyond the one, used in the present paper, may be important. It will be worthwhile to investigate how those additional terms would affect the present results, and hence, the studies utilizing them.

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