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Jean Cleymans, Stringy Thermal Model, Tsallis Quantum Statistics

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Abstract: My memories on Jean Cleymans and a brief advocation of the stringy thermal model, describing massless constituents with the energy-per-particle and temperature relation, $E/N = 6T = 1$ GeV, are presented. Another topic, the Kubo–Martin–Schwinger (KMS) relation applied to the Tsallis distribution in quantum statistics is also sketched, which was triggered by our discussions with Jean.

Keywords: string; quark-gluon plasma; thermal model; KMS relation; Tsallis distribution

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1. My Meetings with Jean

I briefly recapitulate my meetings with Jean Cleymans; the knowledge of his unexpected recent death is still shocking. He was a real demiurge, an active spirit in organizing a number of initiatives for the better sake of the high energy physics community. Probably the most known is his advocation of establishing the cooperation of the South African Republic with CERN (European Organization for Nuclear Research, Geneva, Switzerland) and JINR (Joint Institute for Nuclear Research, Dubna, Russia).

I met Jean in Cape Town, in 2004, when he was a main organizer of the *Strangeness in Quark Matter* meeting; and then later, at the meeting held in Stellenbosch, a nearby small town famous for wine making. Several visits had been organized for the sequel, during which I met some of his younger collaborators too. Most memorable to me were Azwinndini Muronga, and in later years Andre Peshier, whom I already knew from Giessen, Germany.

Jean also reciprocated quite a few visits in Budapest. In 2007 he took part in the *Zimányi'75* memorial workshop: he chaired a session with Greco, Hamar, Petreczky and Mócsy. He delivered a talk, entitled *Transverse energy and charged hadron production from GSI to RHIC*. A few years later, in 2011, he talked again at the *Zimányi School*, in December, on *The thermal model at the LHC*. This indicates that one of his favorite topics must have been the "thermal model" of hadrons—a minimalistic theory applied to a great number of experimental results ever since.

I remember Jean walking in the winter fare in midtown Budapest, watching the typical European activity in the pre-Christmas time. He looked over the heads in the crowd.

We have discussed with Jean several physics questions during those years. In this paper, I pick up two of the topics because they are characteristic to Jean's interests as a physicist and because these we discussed a lot and I had the feeling to have succeeded to convince him on the actuality and perspective of these. One topic was to include the presence of strings, connecting massless particles in a first-principles thermodynamical treatment, as the quark-gluon plasma (QGP) counterpart of the hadronic thermal model [1]. Another point was to lure Jean into the non-extensive thermodynamics perspective, with all its complications when using cut power-law type energy distributions instead of exponentials and treating their thermodynamical consequences [2,3]. Here, a particular question—the relation between fermion and anti-fermion quantum statistics when based on the q-generalization of the exponential function—had grown from our discussions. Jean had chosen the "cut-and-paste" solution [4], we with Gergely Barnaföldi and Keming Shen in Budapest adventured around to try "deeper" approaches, paying tribute to the particle–hole symmetry, in particular to the Kubo–Martin–Schwinger (KMS) relation [5].

In the present paper, the fundamentals of these two topics are presented. Even if this is not a first appearance of these topics, perhaps, the thoughts we shared with Jean Cleymans are worth revisiting.

2. Stringy Thermal Models

One of the most intriguing features taught to us by the thermal model of hadronization [6,7] in heavy-ion collisions is the scaling of experimental points on a single curve in the temperature vs. baryochemical potential, T–μ, plane. Among a few interpretation possibilities, the most popular was that this curve represents a constant energy per particle, $E/N \approx 1$ GeV [3,8,9].

Concentrating on the low baryon number region, typical for RHIC (Relativistic Heavy Ion Collider) and LHC (Large Hadron Collider) experiments, the E/N ratio is surpirsingly high for a thermal ensemble of massless quarks and gluon partons. On the hadron side, there is no such problem [10].

For a massive, non-relativistic ideal gas of monoatomic constituents, one expects

$$\frac{E}{N} = m + \frac{3}{2}T, \tag{1}$$

using units, where the Boltzmann constant and the speed of light values are set to unity; as well the Planck's constant is set to unity in what follows; here, m is the particle mass. In this case, the measured energy-per-particle value and the conjectured temperature, also fitted to parts of transverse momentum spectra, i.e., $T \approx 167$ MeV, lead to a conclusion of $m \approx 750$ MeV. Indeed, this average hadron mass, close to the ρ-meson mass, is not unreasonable.

On the other hand, for extreme relativistic pointlike particle plasmas without interaction in the Boltzmann limit, acute at high temperatures, one obtains:

$$\frac{E}{N} = 3T, \tag{2}$$

which is modified by only ten percent when assuming Bose–Einstein distribution. Meantime, one can see that this relation does not satisfy the experimentally fitted values, cited above. Conclusively, the QGP side at around the color deconfinement, known from lattice QCD (quantum chromodynamics) simulations, cannot consist of an ideal, non-interacting plasma of massless partons.

Motivated by this, an interacting model of massless QGP particles and the consequent thermodynamics were considered [1,11]. The interaction energy was modeled as strings with individual contributions of $\varepsilon_{\text{int}} = \sigma\langle\ell\rangle$. The average length, $\langle\ell\rangle$, of such strings is a function of density, n, for straight strings, which are optimal: $\langle\ell\rangle \sim n^{-1/3}$. This gives rise to a free energy density as follows:

$$f(n,T) = f_{\text{id}}(n,T) + An^{2/3}, \tag{3}$$

where the subscript "id" denotes the ideal gas formula and A is proportional to the string tension.

The thermodynamical consequences of such a term are multiple. Due to the homogeneity assumption, one has the known relation, connecting the energy density, ε, the pressure, p, the entropy density, s and the chemical potential, μ:

$$f = \varepsilon - Ts = \mu n - p, \tag{4}$$

with the partial derivatives,

$$\mu = \frac{\partial f}{\partial n}, \qquad s = -\frac{\partial f}{\partial T}. \tag{5}$$

In the stringy QGP, defined by Equation (3), one obtains:

$$\begin{aligned} s &= s_{\text{id}}, & \mu &= \mu_{\text{id}} + \frac{2}{3}An^{-1/3}, \\ \varepsilon &= \varepsilon_{\text{id}} + An^{2/3}, & p &= p_{\text{id}} - \frac{1}{3}An^{2/3}. \end{aligned} \tag{6}$$

The negative contribution to the pressure indicates that strings pull, not push. The entropy density has only an ideal gas contribution, while the energy density receives the same correction term as the free energy density. It is a particular feature that there is a combination without interaction correction:

$$\varepsilon + 3p = \varepsilon_{\text{id}} + 3p_{\text{id}} = 6p_{\text{id}}. \tag{7}$$

It is noteworthy that the correction to the chemical potential is decreasing with increasing density [12]. The ideal part, in the Boltzmann approximation proportional to the logarithm of the density, is increasing on the other hand. The common effect of these two terms is a minimum at some n. Moreover at given temperatures, when this minimum is negative, the $\mu(n)$ curve crosses the zero axis, signalling changes in the chemical behavior of strings. Below such temperatures the minimum of $\mu(n)$ is positive and the density of string sources will be diminished indefinitely. The critical temperature, interfacing these two cases, proved to be proportional to $\sqrt{A}$. This agrees with the early lattice gauge theory calculation results [13,14].

More problematic is the border of mechanical stability. According to Equation (6), the pressure may become negative. The stringy interaction term, $An^{2/3}$, for massless sources with densities of $n \sim T^3$, represents an AT^2 order correction to the free QGP pressure. This is again in accordance with the lattice QCD findings, most noticeable in the studies of the interaction measure, $\Delta = (\varepsilon - 3p)/T^4$. We have analyzed such corrections among others in Refs. [1,12]. Assuming a thermal massless density of string sources, $n = \gamma T^3$, one concludes that $\Delta = 2\gamma A/T^2$. This is indeed an observed behaviour at above the color deconfinement temperature, $T > T_c$, in lattice QCD equation of state studies. This quantity drops to zero below this temperature, so the stringy interaction must be converted into masses in the hadron resonance gas.

The $p = 0$ mechanical stability limit line is an assumed point of rapid hadronization. Any interaction term in the equation of state which reduces the pressure while increasing the energy density modifies the expected E/N ratio.

$$\left.\frac{E}{N}\right|_{\text{hadronization}} = \left.\frac{\varepsilon}{n}\right|_{\text{p=0}} \tag{8}$$

can be expressed observing that for ideal gases, $n = p_{\text{id}}/T$ and $p = p_{\text{id}} - p_{\text{int}}$ along with $\varepsilon = \varepsilon_{\text{id}} + \varepsilon_{\text{int}}$. For a bag model type approach, $p_{\text{int}} = \varepsilon_{\text{int}} = B$, where B defines the bag pressure, and one obtains at $p = 0$:

$$\left.\frac{E}{N}\right|_{\text{bag}} = \frac{\varepsilon_{\text{id}} + p_{\text{id}}}{p_{\text{id}}/T} = 4T. \tag{9}$$

In view of the experimental results, this is indeed insufficient: a MIT (Massachusetts Institute of Technology) bag model equation of state for a quark-gluon plasma cannot match the measurements.

To the contrary, the stringy model just has the correct interaction terms. There, $\varepsilon_{\text{int}} = 3p_{\text{int}} = An^{2/3}$, and one arrives at the estimate (cf. Equation (7)):

$$\left.\frac{E}{N}\right|_{\text{string}} = \frac{\varepsilon_{\text{id}} + 3p_{\text{id}}}{p_{\text{id}}/T} = 6T. \tag{10}$$

This straight and remarkable result encouraged us for further studies. We have studied the mechanical instability border line, $p = 0$, also at a finite baryochemical potential [1]. Then the very formulas are more involved; however, the main message is the same: $E/N = 6T \approx 1$ GeV curve describes all other data as well, even that taken at higher baryon densities. Citing Berndt Müller [13]: "These results which are generally true for systems composed of massless particles, also remain valid when interactions are included." This is said about the relation $p_{\rm id} = \varepsilon_{\rm id}/3$. By that the $E/N = 1$ GeV value remains the same for all chemical potentials.

3. Tsallis-Fermi Problem

Another of our common projects with Jean was to explore the consequences ofTsallis-distributed hadrons and eventually quarks and gluons for the thermodynamics and, therefore, for the thermal model predictions too [15,16]. At a first glance a transverse momentum, p_T, distribution, which is not exponential in its tail, was predicted by perturbative QCD calculations quite early. On the other hand, Rolf Hagedorn had suggested to interpolate towards a Boltzmann exponential, typical in thermal equilibrium situations, by the so-called "cut power-law" distribution [17]. It turned out in the 1990s and, with higher momentum, after 2000 that such functions of $(1 + ax)^{-b}$ type can be viewed as a mathematical generalization of the exponential function and can be derived as canonical distributions from an altered entropy formula. Since then it is tagged as "Tsallis distribution", as the canonical energy distribution, associated with the q-entropy formula, promoted by Constantino Tsallis since 1988 [18,19].

The core of the use of Tsallis distribution is to replace the exponential function, $\exp(x)$, in statistical formulas by

$$e_q(x) \equiv (1 + (1-q)x)^{\frac{1}{1-q}}, \tag{11}$$

where q is the Tsallis (non-extensivity) parameter.

This approach has provided good agreement to the measured spectra in the Boltzmann approximation [20]. However, dealing with quarks and gluons on the one side or mesons and baryons on the other side, one is tempted to consider quantum statistics. In case of fermions it is even unavoidable at low temperatures, near the Fermi energy.

There are several ways to approach the quantum statistical pendants based on Tsallis distribution [4,5]. In particular, there is a symmetry between particles and holes (negative energy states) in the statistical approach to quantum field theory; referred to as the Kubo–Martin–Schwinger (KMS) relation, which explores the symmetry between negative and positive frequency waves in thermal equilibrium according to elementary commutation relations. In the case of the Bose–Einstein and Fermi–Dirac distribution, in the $q = 1$ case, the relation,

$$n(x) + n(-x) = \mp 1, \tag{12}$$

needs to be fulfilled for the Bose–Einstein distribution (with minus sign) and for the Fermi–Dirac distribution (with plus sign), respectively [12,14,16]. The resolution of this constraint leads to the form:

$$n(x) = \frac{1}{\exp(x) \mp 1}. \tag{13}$$

This fulfillment is based on the elementary identity,

$$\exp(x) \cdot \exp(-x) = 1. \tag{14}$$

Now, replacing the exponential function with another one, $e_q(x)$, brings a difficulty. For the Tsallis distribution, this product is not unity:

$$e_q(x) \cdot e_q(-x) \neq 1. \tag{15}$$

Instead, the basic formula reads:

$$e_q(x) \cdot e_{2-q}(-x) \;=\; 1. \tag{16}$$

So, either one shall not use the Tsallis' $e_q(x)$ (11) in the Bose–Einstein and Fermi–Dirac distributions instead of the exponential function or a more sophisticated relation is needed.

Both ways are possible. One may generalize the exponential function in another way, e.g., using the deformed exponential promoted by Giorgio Kaniadakis [21]:

$$e_k(x) \;=\; \left(\sqrt{1+k^2x^2}+kx\right)^{1/k}. \tag{17}$$

In this case,

$$e_k(x) \cdot e_k(-x) \;=\; 1. \tag{18}$$

Generalizing further, an even function, $b(x^2)$, can also be used, and by that extension, $e_k(x)$ can be related to a ratio of Tsallis expressions [5]. An expression like

$$e_b(x) \;=\; \left(\sqrt{1+k^2x^2b^2(x^2)}+kxb(x^2)\right)^{1/k}, \tag{19}$$

indeed satisfies $e_b(x)e_b(-x) = 1$. On the other hand,

$$f_q(x) \;=\; \frac{e_q(x/2)}{e_q(-x/2)} \tag{20}$$

demonstrates this feature too. One can then equate:

$$e_b(x) \;=\; f_q(x)^{1-q} \;=\; t, \tag{21}$$

and conclude that

$$b(x^2) \;=\; \frac{1}{2kx}\left[f_q(x)^{1-q} - f_q(-x)^{1-q}\right]. \tag{22}$$

In addition, as a further alternative, a combination of the naïve Bose–Einstein and Fermi–Dirac distributions,

$$n_q(x) \;=\; \frac{1}{e_q(x) \mp 1}, \tag{23}$$

can also be utilized to fulfill the KMS relation. It turns out that the linear combination,

$$n(x) \;=\; \frac{1}{2}\left[n_q(x) + n_{2-q}(x)\right], \tag{24}$$

also satisfies $n(x) + n(-x) = \mp 1$.

4. Conclusions

In conclusion, the phenomenology of heavy-ion collisions and the search for signals of quark-gluon plasma (QGP) formation sometimes meet with fundamental concepts. These concepts sometimes can be and indeed were handled in terms of simple enough models and elementary considerations. One of these models was the thermal model, which played an important role in the career of Jean Cleymans. It intrigued me to take steps towards understanding why and how the energy-per-particle ratio interrelation with temperature, $E/N = 6T$, is possible, in particular, with massless constituents, as high-T QCD (quantum chromodynamics) expected it.

Another vast streamline of the development of concepts is connected with statistical physics. While at the beginning, in the 1960s and 1970s the application of thermodynamics was almost unthinkable to high energy physics, the attitude had changed dramatically in the 1980s. Analogous to this, in the 2000s and 2010s the use of non-extensive statistics modified and extended the early Boltzmannian based concepts. Here, Jean could be convinced and joined to this enterprise to check the statistical consequences of an altered entropy formula together with an altered form of canonical distributions. Experiments namely supported this view much more strongly than the early fits of exponentials to a narrow window of spectra available to the date.

In relation to this advanced statistical approach, with Jean we have discussed what to do with the quantum statistics. He had chosen a cut-and-paste approach as a fast and practical cure to the particle–hole problem, inherent in field theory due to the Kubo–Martin–Schwinger (KMS) relation. We have investigated a more general class of possible solutions, all smooth at the Fermi surface. I believe that even not being a co-author, he deserves to be acknowledged and late but clearly mentioned connection with this issue too.

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Abbreviations

The following abbreviations are used in this manuscript:

CERN	Conseil (Organisation) Européenne pour la Recherche Nucléaire (European Organization for Nuclear Research)
GSI	Gesellschaft für Schwerionenforschung (German Society for Heavy Ion Research)
JINR	Joint Institute for Nuclear Research
QGP	quark-gluon plasma
QCD	quantum chromodynamics
KMS	Kubo–Martin–Schwinger
LHC	Large Hadron Collider
MIT	Massachusetts Institute of Technology
RHIC	Relativistic Heavy Ion Collider

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